Lie Algebra Cohomology and the Borel-Weil-Bott Theorem

Math G4344, Spring 2012

We have seen that irreducible finite dimensional representations of a complex simple Lie algebra $\mathfrak{g}$ or corresponding compact Lie group are classified and can be constructed starting from an integral dominant weight. The dominance condition depends upon a choice of positive roots (or equivalently, a choice of invariant complex structure on the flag manifold.) An obvious question is that of what happens if we make a different choice of positive roots, or start with a non-dominant highest weight. The Weyl group permutes possible choices of positive roots, at the same time permuting highest weights.

It turns out that there is a generalization of the Borel-Weil theorem which describes the effect of these Weyl group permutations. This is the Borel-Weil-Bott theorem, which realizes representations in other cohomology degrees, not just the degree-zero case of holomorphic sections. This phenomenon is best understood in terms of the Lie algebra cohomology of the nilpotent radical subalgebra $\mathfrak{n}^+ \subset \mathfrak{g}$.

1 Lie algebra cohomology and cohomology of G/T with coefficients in a line bundle

Recall that one way to motivate Lie algebra cohomology is by starting with de Rham cohomology of a group. For G a compact, simple Lie group we have the de Rham complex

$$(\Omega^*(G), d)$$

of differential forms $\Omega^*(G)$ with the de Rham differential d satisfying $d^2 = 0$. By the de Rham theorem the cohomology of this complex gives the topological cohomology of the group.

One way to write these differential forms is as

$$\Omega^i(G) = Hom_{\mathbf{C}}(\Lambda^i(\mathfrak{g}), C^\infty(G))$$

To get Lie algebra cohomology one simply replaces $C^\infty(G)$ by an arbitrary representation V of the Lie algebra of G, so co-chains are

$$C^i(\mathfrak{g}, V) = Hom_{\mathbf{C}}(\Lambda^i(\mathfrak{g}), V) \simeq \Lambda^i(\mathfrak{g}^*) \otimes_{\mathbf{C}} V$$

For more details and the formula for d, see [1] or [4]. As usual, one defines cocycles as

$$Z^i(\mathfrak{g}, V) = ker\ d_{|C^i(\mathfrak{g},V)}$$

coboundaries as

$$B^i(\mathfrak{g}, V) = Im\ d_{|C^{i-1}(\mathfrak{g},V)}$$

and the cohomology as

$$H^i(\mathfrak{g}, V)$$

More abstractly, one can get this definition as the derived functors of the invariants functor in the category of $U(\mathfrak{g})$ modules. The invariants functor is

$$V \to Hom_{U(\mathfrak{g})}(\mathbf{C}, V) = V^{\mathfrak{g}}$$

Replacing the trivial representation $\mathbf{C}$ here by a certain free $U(\mathfrak{g})$ resolution called the Koszul resolution gives precisely the complex defined above.

Note that for a compact Lie group the invariants functor is exact, and the complexification doesn't change this so for semi-simple complex Lie algebras one has

$$H^i(\mathfrak{g}, V) = H^i(\mathfrak{g}, \mathbf{C}) \otimes V^{\mathfrak{g}}$$

Here Lie algebra cohomology carries no more information about the representation V than the dimension of its invariant subspace. For non-semi-simple Lie algebras the invariants functor is no longer exact, and the Lie algebra cohomology of a representation is a more interesting invariant than just its degree-zero piece, the invariants. We will be interested here in such a case, taking the Lie algebra cohomology with respect to the nilpotent radical $\mathfrak{n}^+$ of a semi-simple Lie algebra.

Recall that in our discussion of the Borel-Weil theorem we were using a complex line bundle L_λ over the flag manifold G/T (G is a compact simple Lie group, T a maximal torus). The integral weight λ labels a T representation ρ_λ on $\mathbf{C}$. Sections of this line bundle are explicitly

$$\begin{aligned}\Gamma(L_\lambda) &= \{f : G \to \mathbf{C},\ f(gt) = \rho_\lambda(t^{-1})f(g)\}\\ &= (C^\infty(G) \otimes \mathbf{C}_\lambda)^T\\ &= (C^\infty(G))_{-\lambda}\end{aligned}$$

and holomorphic sections are the subspace of this invariant under the right action of $\mathfrak{n}^+$.

We are interested now in using the structure of G/T as a complex manifold (which depends on the choice of positive roots) to define a holomorphic version of cohomology. The usual topological cohomology computes the derived functor of the functor of taking global sections of the sheaf of locally constant functions. For a complex manifold, we instead use the sheaf of local holomorphic functions, or more generally the sheaf of local holomorphic sections of a holomorphic line bundle such as L_λ. Just as the de Rham theorem allows computation of topological cohomology using differential forms, the Dolbeault theorem says we can compute holomorphic cohomology using the the bi-graded complex

$$(\Omega^{0,i}(G/T, L_\lambda), \overline{\partial})$$

of differential forms with coefficients in line bundle L_λ, of degree i in local variables $d\overline{z}$ (and degree 0 in the dz). In degree 0 we just get

$$H^0(G/T, L_\lambda) = \Gamma_{hol}(L_\lambda)$$

the holomorphic sections, but we can also get higher cohomology, in degrees up to the complex dimension of G/T.

If one works out explicitly what the Dolbeault complex is in this case, generalizing the case of holomorphic sections, one finds

$$(\Omega^{0,i}(G/T, L_\lambda), \overline{\partial}) = ((Hom(\Lambda^i(\mathfrak{n}^+), C^\infty(G) \otimes \mathbf{C}_\lambda))^T, d)$$

where T acts on $\mathfrak{n}^+$ by the adjoint representation, and the d is the d of Lie algebra cohomology for $\mathfrak{n}^+$, with $\mathfrak{n}^+$ acting on $C^\infty(G)$ by infinitesimal right translation.

Note that one has a commuting action of G on this complex, coming from the left G action on functions on G, so we will get G representations on the cohomology spaces

$$H^i(G/T, L_\lambda)$$

Recall that the way Borel-Weil works is that one uses Peter-Weyl to see that

$$\begin{aligned}
\Gamma(L_\lambda) &= (C^\infty(G) \otimes \mathbf{C}_\lambda)^T \\
&= (C^\infty(G))_{-\lambda} \\
&= \bigoplus_{\mu\ dominant} (V^\mu)^* \otimes V^\mu_{-\lambda}
\end{aligned}$$

and thus that

$$\Gamma_{hol}(L_{-\lambda} = (V^\lambda)^*$$

For higher cohomology, one has

$$\begin{aligned}
H(\Omega^{0,i}(G/T, L_\lambda), \overline{\partial}) &= H((Hom(\Lambda^i(\mathfrak{n}^+), C^\infty(G) \otimes \mathbf{C}_\lambda))^T, d) \\
&= H(\bigoplus_{\mu\ dominant} (V^\mu)^* \otimes (Hom(\Lambda^i(\mathfrak{n}^+), V^\mu \otimes \mathbf{C}_\lambda))^T, d) \\
&= \bigoplus_{\mu\ dominant} (V^\mu)^* \otimes (H^i(\mathfrak{n}^+, V^\mu \otimes \mathbf{C}_\lambda))^T \\
&= \bigoplus_{\mu\ dominant} (V^\mu)^* \otimes H^i(\mathfrak{n}^+, V^\mu)_{-\lambda}
\end{aligned}$$

so

$$H(\Omega^{0,i}(G/T, L_{-\lambda}), \overline{\partial}) = \bigoplus_{\mu\ dominant} (V^\mu)^* \otimes H^i(\mathfrak{n}^+, V^\mu)_\lambda$$

This show that in this case computing holomorphic cohomology comes down to computing $\mathfrak{n}^+$ Lie algebra cohomology. For some more details of this argument, see for instance [2].

2 Kostant's Theorem

The computation of the Lie algebra cohomology of the nilpotent radical was done by Kostant in 1961, with the result

Theorem 1 (Kostant's Theorem)**.** *For a finite dimensional highest-weight representation V^λ of a complex semi-simple Lie algebra $\mathfrak{g}$*

$$H^i(\mathfrak{n}^+, V^\lambda) = \bigoplus_{w\in W:l(w)=i} \mathbf{C}_{w(\lambda+\rho)-\rho}$$

There are at least four possible approaches to proving this:

- One can use the BGG resolution and the fact that for Verma modules $H^i(\mathfrak{g}, V(\mu))$ is $\mathbf{C}_\mu$ for $i = 0$, 0 for $i > 0$. This requires knowing the BGG resolution, which is a stronger result since it carries information about homomorphisms between Verma modules.

- One can prove Borel-Weil-Bott by other (e.g. topological) methods, then use this to prove Kostant's theorem. For an example of such a proof of Borel-Weil-Bott, see Jacob Lurie's notes[3].

- One can find explicit elements in $H^*(\mathfrak{n}^+, V^\lambda)$ that represent the cohomology classes in Kostant's theorem. One way to do this is to look for elements in
$$C^i(\mathfrak{n}^+, V^\lambda) = \Lambda^i(\mathfrak{n}^+)^* \otimes V^\lambda$$
that represent these cohomology classes. Note that the weights of $(\mathfrak{n}^+)^*$ are multiples of $-\alpha$ where $\alpha \in R^+$, the positive roots. A choice that gives the right element in degree i for each Weyl group element w such that $l(w) = i$ is:
$$\omega_{-\beta_1} \wedge \omega_{-\beta_2} \wedge \cdots \wedge \omega_{-\beta_i} \otimes V^\lambda(w\lambda)$$
where
$$\omega_{-\beta_j} \in (\mathfrak{n}^+)^*_{-\beta_j}$$
for β_j a positive root such that $w\beta_j$ is a negative root. $V^\lambda(w\lambda)$ is the transform by w of the highest weight space. Th more difficult part of this sort of proof is showing that only these elements can occur. One way to do this is to construct an analog of the Laplacian, and show that it acts like the Casimir on cohomology (this was Kostant's original method). A generalization of this uses the full center of the enveloping algebra, and the Casselman-Osborne lemma, which says that the center much act on the higher cohomology in just the way that the Harish-Chandra ismorphism says it acts in degree zero cohomology (the highest weight space). For more details on this argument see Goodman-Wallach[4].

- One can replace the use of the exterior algebra and a Laplacian by closely related spinors, and a "square-root" of the Laplacian known as the Dirac operator. We'll try and come back to this argument after developing the technology of spinors and Clifford algebras in the next couple weeks.

3 Borel-Weil-Bott and the Weyl Character Formula

Kostant's theorem gives the Borel-Weil-Bott theorem very directly. Recall that

$$H^i(G/T, \mathcal{O}(L_{-\lambda})) = \bigoplus_{\mu} (V^\mu)^* \otimes H^i(\mathfrak{n}^+, V^\mu)_\lambda$$

where the sum is over dominant integral weights μ. By Kostant's theorem we have

$$H^i(\mathfrak{n}^+, V^\mu)_\lambda = (\bigoplus_{w\in W: l(w)=i} \mathbf{C}_{w(\mu+\rho)-\rho})_\lambda$$

and this has a one-dimensional contribution iff

$$w(\mu+\rho) - \rho = \lambda$$

or equivalently

$$w(\mu+\rho) = \lambda + \rho$$

Note that the set of weights of the form $\mu + \rho$ for μ dominant integral are in the interior of the dominant Weyl chamber, and acting on these by Weyl group elements gives us sets of weights in the interiors of the other Weyl chambers. Weights λ such that $\lambda+\rho$ is on the boundary of a Weyl chamber will not occur. In summary, we have

Theorem 2 (Borel-Weil-Bott)**.** *If $\lambda + \rho$ is a singular weight then for all i we have*

$$H^i(G/T, \mathcal{O}(L_{-\lambda})) = 0$$

If $\lambda+\rho$ is a non-singular weight, there will be an i such that $w(\lambda+\rho) = \mu+\rho$ is in the interior of the dominant Weyl chamber for a $w : l(w) = i$ and

$$H^i(G/T, \mathcal{O}(L_{-\lambda})) = (V^\mu)^*$$

As usual, the simplest example is $G = SU(2)$, $G/T = \mathbf{C}P^1$, and the Borel-Weil-Bott theorem can be proved via Serre duality, which says that for line bundles L on a curve C one has

$$H^1(C, L) = H^0(C, L^* \otimes \omega_C)$$

where ω_C is the canonical bundle on C. In our case $C = \mathbf{C}P^1$, and line bundles L_n are labeled by an integer n with ρ corresponding to $n = 1$. The canonical bundle is L_2.

For $n \geq 0$ we have, as in the Borel-Weil theorem

$$H^0(\mathbf{C}P^1, L_{-n}) = (V^n)^*$$

where V^n is the irreducible $SU(2)$ representation of dimension $n+1$. By Serre duality

$$H^1(\mathbf{C}P^1, L_{-n}) = H^0(\mathbf{C}P^1, L_{n+2})$$

which is consistent with Borel-Weil-Bott which tells us that

$$H^1(\mathbf{C}P^1, L_{-n}) = (V^{-n-2})^*$$

when $n < -1$ and, in the singular $n = -1$ case

$$H^1(\mathbf{C}P^1, L_1) = H^0(\mathbf{C}P^1, L_1) = 0$$

So, for $n > 0$ one gets all irreducibles as holomorphic sections, whereas for $n < -1$ one gets all irreducibles again, but in higher cohomology (H^1).

Working out what happens in the $SU(3)$ case will be on the current problem set.

Another quick corollary of Kostant's theorem is the Weyl character formula. Recall that this says that the character $ch(V^\lambda)$ of a finite-dim irreducible of highest weight λ is given by

$$ch(V^\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W}(-1)^{l(w)}e^{w(\rho)-\rho}}$$

This follows from an application of the Euler-Poincaré principle, which says that in the case of an Abelian invariant like the character, its value on the alternating sum of the cohomology groups (the Euler characteristic) is the same as its value on the alternating sum of whatever co-chains ones uses to define cohomology, i.e. here we have

$$\sum_i (-1)^i ch(H^i(\mathfrak{n}^+, V)) = \sum_i (-1)^i ch(C^i(\mathfrak{n}^+, V))$$

This follows from two facts: the first is that

$$ch(C^i(\mathfrak{n}^+, V)) = ch(Z^i(\mathfrak{n}^+, V)) + ch(B^{i+1}(\mathfrak{n}^+, V))$$

since we have an exact sequence

$$0 \to Z^i(\mathfrak{n}^+, V) \longrightarrow C^i(\mathfrak{n}^+, V) \xrightarrow{d} B^{i+1}(\mathfrak{n}^+, V) \to 0$$

(here $Z^i(\mathfrak{n}^+, V)$ are the co-cycles on which $d = 0$, $B^{i+1}(\mathfrak{n}^+, V)$ are the co-boundaries which are in the image of d. Since

$$H^i(\mathfrak{n}^+, V) = Z^i(\mathfrak{n}^+, V)/B^i(\mathfrak{n}^+, V)$$

we also have a second fact

$$ch(H^i(\mathfrak{n}^+, V)) = ch(Z^i(\mathfrak{n}^+, V)) - ch(B^i(\mathfrak{n}^+, V))$$

and this together with our first fact gives the Euler-Poincaré principle.

Recall that

$$C^i(\mathfrak{n}^+, V) = Hom(\Lambda^i(\mathfrak{n}^+), V) = \Lambda^i(\mathfrak{n}^+)^* \otimes V$$

so we have

$$\sum_i (-1)^i ch(C^i(\mathfrak{n}^+, V^\lambda)) = \sum_i (-1)^i ch(\Lambda^i(\mathfrak{n}^+)^*) ch(V^\lambda)$$

whereas Kostant's theorem tells us that the Euler characteristic is

$$\sum_i (-1)^i ch(H^i(\mathfrak{n}^+, V^\lambda)) = \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}$$

Applying the Euler-Poincaré principle in the case $\lambda = 0$ gives

$$\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)-\rho} = \sum_i (-1)^i ch(\Lambda^i(\mathfrak{n}^+)^*)$$

and thus in the general case the Weyl character formula as

$$\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho} = (\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)-\rho}) ch(V^\lambda)$$

References

[1] Knapp, A., *Lie Groups, Lie Algebras, and Cohomology,* Princeton, 1988.

[2] Barchini, L., *Unitary Representations Attached to Elliptic Orbits. A Geometric Approach* (Lecture 3) *in*1998 PCMI lectures on Representation Theory of Lie Groups.

[3] Lurie, J., *A Proof of the Borel-Weil-Bott Theorem,* http://www.math.harvard.edu/~/lurie/papers/bwb.pdf

[4] Goodman, R. and Wallach, N., *Symmetry, Representations and Invariants, GTM 255* Springer, 2009. Appendix E. http://www.math.rutgers.edu/~goodman/pub/symmetry/appe.pdf