Here is a proof that a continuous function is uniformly continuous on a closed interval:

Thm ① Suppose f is continuous on $[a,b]$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that, for any $x, y \in [a,b]$

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon .$$

To prove this we need:

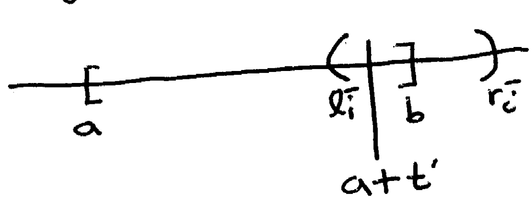
Lemma ① (Compactness of the closed interval). Suppose $\{(l_i, r_i) : i \in I\}$ is a set of open intervals $(l_i < r_i)$ so that $[a,b] \subseteq \bigcup_{i \in I} (l_i, r_i)$. Then there is a finite subset $I' \subseteq I$ such that

$$[a,b] \subseteq \bigcup_{i \in I'} (l_i, r_i).$$

Proof of Lemma ①: Consider the set

$$A := \{ t \in [0, b-a] : [a, a+\bar{t}] \subseteq \bigcup_{i \in I'} (l_i, r_i) \text{ for each } \bar{t} \le t \text{ for some finite } I' \subseteq I \}.$$

Clearly $0 \in A$ since $a \in (l_i, r_i)$ for some $i \in I$. Let $t_0 := \sup\{t \in A\}$ so $0 \le t_0 \le b-a$. First suppose $t_0 = b-a$. There is some $\bar{i}$ such that $b \in (l_{\bar{i}}, r_{\bar{i}})$



so there is some $t' < t_0 = b-a$ so that $a+t' > l_{\bar{i}}$.

Now, since $t' < t_0$, we have $t' \in A$ so $[a, a+t'] \subseteq \bigcup_{i \in I'} (l_i, r_i)$ for a finite set $I' \subseteq I$ hence

$$[a,b] \subseteq \bigcup_{i \in I' \cup \{\bar{i}\}} (l_i, r_i)$$

and $I' \cup \{\bar{i}\}$ is finite, so we are done.

A similar argument shows that we cannot have $t_0 < b-a$. (Why? Think about this!) ∎

<u>Proof of Thm ①</u>

f cont. on $[a,b] \Rightarrow$ for every $x \in [a,b]$ there is $\delta_x > 0$ such that $|y-x| < \delta_x \Rightarrow |f(x)-f(y)| < \varepsilon/2$.

Clearly $[a,b] \subseteq \bigcup_{x \in [a,b]} (x - \frac{1}{2}\delta_x, x + \frac{1}{2}\delta_x)$ so by Lemma ① there are $x_1, \dots, x_n$ such that

$$[a,b] \subseteq \bigcup_{i=1}^{n} (x_i - \tfrac{1}{2}\delta_{x_i}, x_i + \tfrac{1}{2}\delta_{x_i}). \qquad (2)$$

I claim $\delta := \dfrac{\min\{\delta_{x_1}, \dots, \delta_{x_n}\}}{2} > 0$ is as desired.

Indeed, suppose $x, y \in [a,b]$ and $|x-y| < \delta$.

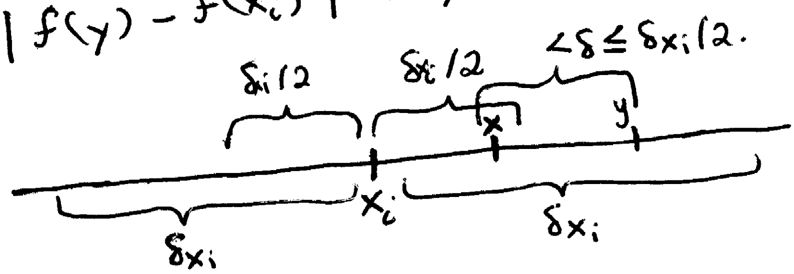
Because of (2), there is some i such that $x \in (x_i - \frac{1}{2}\delta_{x_i}, x_i + \frac{1}{2}\delta_{x_i}) \Rightarrow |x - x_i| < \delta_{x_i}$

so, since $\delta \le \frac{1}{2}\delta_{x_i}$ and $|x-y| < \delta$ we also have $y \in (x_i - \delta_{x_i}, x_i + \delta_{x_i}) \Rightarrow |y - x_i| < \delta_{x_i}$

so by definition of δ_{x_i} we have

$$\left.\begin{array}{l} |f(x) - f(x_i)| < \varepsilon/2 \\ \text{and } |f(y) - f(x_i)| < \varepsilon/2 \end{array}\right\} \Rightarrow |f(x) - f(y)| < \varepsilon.$$

∎



Thm ② If f is continuous on $[a,b]$ and $\varepsilon > 0$, then there is a $\delta > 0$ such that if P is any partition of $[a,b]$ with $\Delta(P) < \delta$ then $A_P^+ - A_P^- < \varepsilon$.

Proof By Theorem ① there is $\delta > 0$ so that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$$

for every x, y in $[a,b]$. If $\Delta(P) < \delta$ then on any subinterval $[x_i, x_{i+1}]$ of P we have

$$\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f < \frac{\varepsilon}{(b-a)}$$

so

$$\begin{aligned} A_P^+ - A_P^- &= \sum_{i=0}^{n-1} (x_{i+1} - x_i)\left(\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f\right) \\ &< \sum_{i=0}^{n-1} (x_{i+1} - x_i)\left(\frac{\varepsilon}{b-a}\right) \\ &= \left(\frac{\varepsilon}{b-a}\right) \sum_{i=0}^{n-1} (x_{i+1} - x_i) = \frac{\varepsilon}{(b-a)}(b-a) = \varepsilon. \end{aligned}$$

$\blacksquare$

Lemma ② If $P' = (x_0' = a < x_1' < \cdots < x_{N'}' = b)$ refines $P = (x_0 = a < \cdots < x_n = b)$ (i.e. $\{x_0, \ldots, x_n\} \subseteq \{x_0', \ldots, x_{N'}'\}$) then $A_{P'}^+ \le A_P^+$ and $A_P^- \le A_{P'}^-$.

Proof This comes down to noticing that if $[x_k', x_{k+1}'] \subseteq [x_i, x_{i+1}]$ then $\sup_{[x_k', x_{k+1}']} f \le \sup_{[x_i, x_{i+1}]} f$ and $\inf_{[x_i,x_{i+1}]} f \le \inf_{[x_k', x_{k+1}']} f$. $\blacksquare$

Thm ③ If f is continuous on $[a,b]$, then f is integrable on $[a,b]$: $\lim_{\Delta(P)\to 0} A_P^+ = \lim_{\Delta(P) \to 0} A_P^-$

(and both exist).