1. Find all solutions to \( x^2 + 2y^2 = 3 \) with \( x, y \in \mathbb{Q} \), and prove that \( x^2 + 3y^2 = 2 \) has no solutions in \( \mathbb{Q} \). State and prove a generalization for \( ax^2 + by^2 = c \) with \( a, b, c \in k^\times \) and \( k \) an arbitrary field not of characteristic 2 (why is characteristic 2 more difficult?). Draw pictures.

2. Let \( k \) be an algebraically closed field. Give an example of affine algebraic sets \( Z_1, Z_2 \) in \( k^2 \) with \( \mathbb{I}(Z_1 \cap Z_2) \neq \mathbb{I}(Z_1) + \mathbb{I}(Z_2) \). What is the geometric significance? Draw a picture.

3. This exercise develops basic facts for manipulating polynomials in several variables.
   (i) Let \( R \) be a ring. Define \( R[X_1, \ldots, X_n] \) in terms of ‘sequences of coefficients’, define on it a structure of commutative \( R \)-algebra, and prove that it has the following universal mapping property: for any \( R \)-algebra \( A \) and any \( a_1, \ldots, a_n \in A \), there is a unique map of \( R \)-algebras \( R[X_1, \ldots, X_n] \rightarrow A \) which sends \( X_i \) to \( a_i \). The image of \( f \) under this map is called the value of \( f \) at \( (a_1, \ldots, a_n) \). Note that when \( R = 0 \), the only \( R \)-algebra is \( R \) itself (e.g., \( R[X] = R \) if \( R = 0 \)).
   (ii) If \( I \) is the ideal in \( R[X_1, \ldots, X_n] \) generated by elements \( f_n \), then state and prove a universal mapping property for the \( R \)-algebra \( R[X_1, \ldots, X_n]/I \). Interpret this in the special case \( I = (X_1 - r_1, \ldots, X_n - r_n) \) for \( r_j \in R \). Conclude that \( R[X] \) is not isomorphic to \( R \) as an \( R \)-algebra if \( R \neq 0 \), but give an example of a non-zero ring \( R \) for which there is an isomorphism \( R[X] \simeq R \) as abstract rings.
   (iii) For \( f \in R[X], g \in R[Y] \), prove that there are unique isomorphisms of \( R \)-algebras
   \[
   (R[Y]/g)[X]/(f) \simeq R[X,Y]/(f,g) \simeq (R[X]/f)[Y]/(g)
   \]
determined by “\( X \mapsto X \)” and “\( Y \mapsto Y \)”. Generalize for any finite number of variables, with \( (f) \) and \( (g) \) replaced by any ideals in the corresponding polynomial rings.

4. (i) If \( A \) is a UFD, prove that \( A[X_1, \ldots, X_n] \) is a UFD (e.g., \( A = \mathbb{Z} \) or \( A \) a field). Prove rigorously that \( k[X,Y,Z,W]/(XY -ZW) \) is a domain but is not a UFD, where \( k \) is an algebraically closed field.
   (ii) Prove that if \( k \) is a field and \( f \in k[X] \) with positive degree is a product of distinct irreducible polynomials, then \( Y^2 - f \in k[X,Y] \) is irreducible. For \( n > 1 \), prove that \( X^n + Y^n - 1 \in k[X,Y] \) is irreducible if the characteristic of \( k \) does not divide \( n \), but is reducible otherwise.

5. It is a basic fact that the ‘symmetric function’ polynomials \( S_1, \ldots, S_n \in \mathbb{Z}[T_1, \ldots, T_n] \) with
   \[
   S_i := \sum \sum_{\{a_1, \ldots, a_i\}} \prod_{k=1}^i T_{a_k}
   \]
   (ex: if \( n = 3 \), \( S_1 = T_1 + T_2 + T_3 \), \( S_2 = T_1 T_2 + T_1 T_3 + T_2 T_3 \) and \( S_3 = T_1 T_2 T_3 \)) are algebraically independent over \( \mathbb{Q} \) (i.e., the canonical map \( \mathbb{Q}[X_1, \ldots, X_n] \rightarrow \mathbb{Q}[T_1, \ldots, T_n] \) sending \( X_i \) to \( S_i \) is injective) and \( \mathbb{Q}[S_1, \ldots, S_n] \) is the subring of \( \mathfrak{S}_n \)-invariants in \( \mathbb{Q}[T_1, \ldots, T_n] \) (consult Lang’s Algebra, 3rd ed., Ch IV, §6 for a self-contained proof).
   (i) Let \( d \geq 1 \). Prove the existence of a ‘universal discriminant’ polynomial \( \Delta_d \in \mathbb{Z}[a_0, \ldots, a_{d-1}] \), unique up to sign, with the property that if \( k \) is any algebraically closed field and \( f = \sum a_i T^i \) is a monic polynomial of degree \( d \), then \( f \) is a product of \( d \) distinct linear factors if and only if \( \Delta_d(a_0, \ldots, a_{d-1}) \neq 0 \).
   (ii) Let \( f \in k[X,Y] \) be a non-constant polynomial and \( k \) an algebraically closed field. If \( f \) has distinct irreducible factors \( f_1, \ldots, f_s \), prove that \( Z(f) \) is the union of the \( Z(f_i) \)'s, with each \( Z(f_i) \) infinite and all \( Z(f_i) \cap Z(f_j) \) finite for \( i \neq j \). Prove that for any irreducible \( f \) of degree \( d > 1 \), all lines in \( k^2 \) meet \( f \) in \( \leq d \) points (what if \( d = 1 \)?), and the only lines \( y = ax + b \) in \( k^2 \) which fail to meet \( f \) in exactly \( d \) distinct points are those for which \( (a, b) \in k^2 \) satisfy a certain non-trivial polynomial relation (depending on \( f \)). In particular, there are infinitely many such exceptional lines. For \( f = Y^2 - X^2 \), what is the geometric meaning of this exceptional set of lines? How about \( f = Y^2 - X^3 \)? Draw pictures.