LECTURES ON INTEGRAL REPRESENTATIONS OF L-FUNCTIONS

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These are the typed version of my notes for the lectures. Being such, you should take everything with a small grain of salt and check the original sources for subtleties. I have separate references at the end of each lecture. Any comments or corrections are welcome.

1. L-functions for GL_n

The purpose of this lecture is to explicate the theory of automorphic L-functions for $GL_n \times GL_m$. If one is interested in the theory of standard L-functions for GL_n, $n \geq 2$, by this method, one considers the theory for $GL_n \times GL_1$ and places the trivial character on GL_1.

The majority of the results are by either Jacquet and Shalika or Jacquet, Piatetski-Shapiro, and Shalika. I have written two longer surveys on this material, [1] and [2], which are also available on my web page (*www.math.ohio-state.edu/∼cogdell*). References to the original papers can be found in the bibliographies of these works.

Let k be a number field, $\mathbb{A}$ its ring of adeles, and $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$ a non-trivial additive character.

1.1. Eulerian integrals.

The first step in the method of integral representations is to write down a family of global adelic integrals that have nice analytic properties and possess an Euler factorization. For GL_n these integrals follow either the paradigm of Hecke or of Rankin and Selberg.

1.1.1. $GL_2 \times GL_1$. We review *Hecke* in this context.

Let (π, V_π) be a cuspidal representation of $GL_2(\mathbb{A})$, which we take to be irreducible, smooth, and unitary. Let $\chi : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ an idele class character, i.e., an automorphic form on $GL_1(\mathbb{A})$.

For $\varphi \in V_\pi$ a cusp form, set

$$I(s, \varphi, \chi) = \int_{k^\times\backslash\mathbb{A}^\times} \varphi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}}\, d^\times x.$$

- *Analytic properties.* By the rapid decrease of cusp forms, these integrals are *nice*, i.e.,

– entire as a functions of s
– bounded in vertical strips (BVS)
– satisfy a functional equation

$$I(s,\varphi,\chi) = I(1-s,\tilde{\varphi},\chi^{-1}),$$

where $\tilde{\varphi}(g) = \varphi({}^t g^{-1})$, coming from the change of variables $a \mapsto 1/a$.

• *Eulerian property.* Cuspidal representations themselves already have a Eulerian factorization $\pi \simeq \otimes' \pi_v$ into a restricted tensor product over the places v of k of local representations π_v of $GL_2(k_v)$. So we could hope to see an Eulerian factorization already at the level of our global integrals. However, due to their invariance under the diagonally embedded $GL_2(k)$, cusp forms themselves *do not factor*.

Cusp forms on GL_2 have a natural *Fourier expansion*: if we let

$$W_\varphi(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi^{-1}(x)\, dx \quad \in \mathcal{W}(\pi,\psi)$$

then

$$\varphi(g) = \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

If we substitute this Fourier expansion onto the integral and unfold we find

$$I(s,\varphi,\chi) = \int_{\mathbb{A}^\times} W_\varphi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}}\, d^\times a \qquad Re(s) > 1.$$

The Whittaker functions satisfy a strong uniqueness property that does give them an Eulerian factorization:

Uniqueness of Whittaker Models: Let v be a place of k, (π_v, V_{π_v}) an irreducible admissible smooth representation of $GL_2(k_v)$. Then there exists at most one (up to scalar multiples) continuous linear functional $\Lambda_v : V_{\pi_v} \longrightarrow \mathbb{C}$ such that

$$\Lambda_v\left(\pi_v\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\xi_v\right) = \psi_v(x)\Lambda_v(\xi_v)$$

for all $\xi \in V_{\pi_v}$ and $x \in k_v$.

Such functionals are called *Whittaker functionals* and they correspond to *Whittaker models* via

$$W_{\xi_v}(g) = \Lambda_v(\pi_v(g)\xi_v) \in \mathcal{W}(\pi_v,\psi_v).$$

Now, the factorization of π plus the local uniqueness of Whittaker models implies the global uniqueness of Whittaker models, which in turn implies the Eulerian factorization of global Whittaker functions: if $V_\pi \simeq \otimes' V_{\pi_v}$ and φ is factorisable in the sense that under this isomorphism we have $\varphi \simeq \otimes \xi_v$ then

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v).$$

Using this factorization, our global integrals become

$$\begin{aligned} I(s,\varphi,\chi) &= \int_{\mathbb{A}^\times} W_\varphi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}}\ d^\times \\ &= \prod_v \int_{k_v^\times} W_{\xi_v} \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} \chi_v(a_v)|a_v|_v^{s-\frac{1}{2}}\ d^\times a_v \\ &= \prod_v I_v(s, W_{\xi_v}, \chi_v) \qquad Re(s) > 1. \end{aligned}$$

This is our Eulerian factorization of our global integrals in this context.

1.1.2. $GL_n \times GL_m$ *with* $m < n$. The paradigm here is still that of Hecke outlined above. The modifications are as follow. Now we let (π, V_π) be a cuspidal representation of $GL_n(\mathbb{A})$ and $\varphi \in V_\pi$ be a cusp form.

Fourier expansion: We let

$$W_\varphi(g) = \int_{N(k)\backslash N(\mathbb{A})} \varphi(\underline{x}g)\psi^{-1}(\underline{x})\ d\underline{x} \in \mathcal{W}(\pi,\psi)$$

where $N = N_n \subset GL_n$ is the maximal unipotent subgroup of upper-triangular matrices with ones on the diagonal,

$$\underline{x} = \begin{pmatrix} 1 & x_{1,2} & & & * \\ & 1 & x_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \in N$$

and $\psi(\underline{x}) = \psi(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n})$. Then

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)$$

as before.

Global integrals: Let Q be the parabolic subgroup of GL_n associated to the partition $(m+1, 1, \dots, 1)$ of n. Let $Q = MY$ with $M \simeq GL_{m+1} \times GL_1 \times \cdots \times GL_1$ its Levi subgroup and $Y \subset N$ its unipotent radical. Note that we can restrict the additive character ψ above from N to Y and the stabilizer of the pair (Y, ψ) in M is the mirabolic subgroup

$$P_{m+1} = \left\{ \begin{pmatrix} & * & * \\ & & \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset GL_{m+1} \subset M.$$

If $\varphi \in V_\pi$, we first project φ to a cuspidal function on $P_{m+1}(\mathbb{A}) \subset GL_{m+1}(\mathbb{A})$ by setting

$$\mathbb{P}\varphi(p) = |\det p|^{-\frac{n-m-1}{2}} \int_{Y(k)\backslash Y(\mathbb{A})} \varphi\left(y \begin{pmatrix} p & \\ & I_{n-m-1} \end{pmatrix}\right) \psi^{-1}(y)\ dy.$$

Then for $\varphi' \in V_{\pi'}$, with $(\pi', V_{\pi'})$ a cuspidal representation of $GL_m(\mathbb{A})$, our global integrals are

$$I(s,\varphi,\varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh.$$

These integrals are again *nice* analytically: they define entire functions of s which are bounded in vertical strips and satisfy a functional equation of the form

$$I(s,\varphi,\varphi') = \tilde{I}(1-s,\tilde{\varphi},\tilde{\varphi}')$$

coming from the change of variables $h \mapsto {}^t h^{-1}$. The new integral $\tilde{I}$ comes from the modification of the projection $\mathbb{P}$ under this change of variables.

They are also *Eulerian* as before:

$$\begin{aligned} I(s,\varphi,\varphi') &= \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'}(h) |\det h|^{s-\frac{n-m}{2}} \, dh \\ &= \prod_v \int_{N_m(k_v)\backslash GL_m(k_v)} W_{\xi_v} \begin{pmatrix} h_v & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\xi'_v}(h_v) |\det h_v|_v^{s-\frac{n-m}{2}} \, dh_v \\ &= \prod_v I_v(s, W_{\xi_v}, W'_{\xi'_v}) \qquad Re(s) > 1 \end{aligned}$$

where now $W'_{\varphi'} \in \mathcal{W}(\pi', \psi^{-1})$. Again the factorization is based on the local uniqueness of Whittaker models for GL_n.

1.1.3. $GL_n \times GL_n$. The paradigm is now due to Rankin and Selberg. It involves integrating a pair of cusp form against an Eisenstein series.

• *The Eisenstein series*: We will consider the needed Eisenstein series from two points of view: one to establish the analytic properties and one needed to prove our global integrals are Eulerian.

First, let $\Phi \in \mathcal{S}(\mathbb{A}^n)$ be a Schwartz-Bruhat function on $\mathbb{A}^n$. To Φ we can associate a theta series in the classical sense:

$$\Theta_\Phi(a,g) = \sum_{\xi \in k^n} \Phi(a\xi g) \qquad \text{for } a \in \mathbb{A}^\times, \; g \in GL_n(\mathbb{A}).$$

Then our first realization of our Eisenstein series is as the Mellin transform of this theta series:

$$E(g,s;\Phi,\eta) = |\det g|^s \int_{k^\times\backslash\mathbb{A}^\times} \Theta'_\Phi(a,g) |a|^{ns} \eta(a) \, d^\times a \quad \text{ for } Re(s) > 1.$$

Here η is a unitary idele class character and as usual the notation Θ'_Φ indicates that one removes the term for $\xi = 0$ from the sum defining Θ_Φ. We can now use Poisson summation on Θ_Φ to show:

- $E(g,s;\Phi,\eta)$ extends to a meromorphic function of s, still automorphic in g,
- $E(g,s;\Phi,\eta)$ has possible simple poles at $s = i\sigma, 1+i\sigma$ with $\sigma \in \mathbb{R}$ such that $\eta(a) = |a|^{-in\sigma}$ (so entire if η is ramified),

– it satisfies the functional equation $E(g,s;\Phi,\eta) = E({}^tg^{-1}, 1-s; \hat{\Phi}, \eta^{-1})$, where $\hat{\Phi}$ is the Fourier transform of Φ.

On the other hand, if we set

$$F(g,a;\Phi,\eta) = |\det g|^s \int_{\mathbb{A}^\times} \Phi(ae_n g)|a|^{ns}\eta(a)\ d^\times a$$

with $e_n = (0,\dots,0,1)$ then F is a section of the induced representation

$$F(g,s;\Phi,\eta) \in \mathrm{Ind}_{P'(\mathbb{A})}^{GL_n(\mathbb{A})}(\delta_{P'}^{s-\frac{1}{2}}\eta^{-1})$$

where P' is the standard parabolic associated to the partition $(n-1,1)$ of n and

$$E(g,s;\Phi,\eta) = \sum_{\gamma \in P'(k)\backslash GL_n(k)} F(\gamma g, s; \Phi, \eta) \qquad \text{for } Re(s) > 1$$

à la Langlands.

• *The global integrals.* Now let both π and π' be unitary cuspidal representations of $GL_n(\mathbb{A})$. Then for cusp forms $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ we set

$$I(s,\varphi,\varphi',\Phi) = \int_{Z_n(\mathbb{A})GL_n(k)\backslash GL_n(\mathbb{A})} \varphi(g)\varphi'(g)E(g,s;\Phi,\omega_\pi\omega_{\pi'})\ dg$$

with ω_π and $\omega_{\pi'}$ the central characters of π and π' respectively. These integrals inherit the analytic properties of the Eisenstein series, i.e.,

– each $I(s,\varphi,\varphi',\Phi)$ extends to a meromorphic function of s with simple poles at $s = i\sigma, 1+i\sigma$ such that $\tilde{\pi} \simeq \pi' \otimes |\det|^{i\sigma}$,
– they are bounded in vertical strips away from their poles,
– they satisfy the functional equation $I(s,\varphi,\varphi',\Phi) = I(1-s,\tilde{\varphi},\tilde{\varphi}',\hat{\Phi})$.

If we replace the Eisenstein series by its representation as a sum as above and unfold, we find that our integrals are Eulerian

$$\begin{aligned} I(s,\varphi,\varphi',\Phi) &= \int_{N_n(\mathbb{A})\backslash GL_n(\mathbb{A})} W_\varphi(g)W'_{\varphi'}(g)\Phi(e_n g)|\det g|^s\ dg \\ &= \prod_v \int_{N_n(k_v)\backslash GL_n(k_v)} W_{\xi_v}(g_v)W'_{\xi_v}(g_v)\Phi_v(e_n g_v)|\det g_v|^s\ dg_v \\ &= \prod_v I_v(s, W_{\xi_v}, W'_{\xi'_v}, \Phi_v) \qquad \text{for } Re(s) > 1 \end{aligned}$$

again using the Fourier expansion of cusp forms and the uniqueness of the Whitaker models.

1.2. **Unramified calculation.** Once one has a family of Eulerian integrals, the next step is to identify which L-function the integral represents in terms of Langlands' prescription. One does this by explicitly computing the local integrals at the finite places where the local representations are unramified. This of course is almost all finite places.

So let v be a finite place of k where π_v,π'_v, and ψ_v are all unramified, i.e., have a vector fixed under the relevant maximal compact subgroup. Such a vector is unique up to scalar multiples.

The unramified generic (i.e., having a Whittaker model) representations of $GL_n(k_v)$ have been classified. If π_v is such, then there are unramified characters $\mu_{v,1},\dots,\mu_{v,n}$ of $k_v^\times$ such that

$$\pi_v \simeq \mathrm{Ind}_{B_n(k_v)}^{GL_n(k_v)}(\mu_{v,1}\otimes\cdots\otimes\mu_{v,n})$$

where B_n is the standard Borel subgroup of GL_n. Since these characters are unramified, they are completely determined by their values on the uniformizer ϖ_v of k_v and so π_v is completely determined by the n complex numbers $\mu_{v,1}(\varpi_v),\dots,\mu_{v,n}(\varpi_v)$. These are the Satake parameters of π_v and they are usually used encoded in a semisimple matrix or conjugacy class

$$A_{\pi_v} = \begin{pmatrix} \mu_{v,1}(\varpi_v) & & \\ & \ddots & \\ & & \mu_{v,n}(\varpi_v) \end{pmatrix} \in GL_n(\mathbb{C}).$$

Now let $W_v^\circ \in \mathcal{W}(\pi_v,\psi_v)$ and $W_v'^\circ \in \mathcal{W}(\pi'_v,\psi_v^{-1})$ be the normalized K_v-fixed vectors such that $W_v^\circ(e)=1$, etc. If $m=n$ we also take Φ_v° to be the characteristic function of the integral lattice $\mathcal{O}_v^n \subset k_v^n$. Then one explicitly computes that

$$\begin{aligned} I(s,W_v^\circ,W_v'^\circ) &= \det(I_n - q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1} \\ &= \prod_{i,j}(1-q_v^{-s}\mu_{v,i}(\varpi_v)\mu'_{v,j}(\varpi))^{-1} \\ &= L(s,\pi_v\times\pi'_v) \end{aligned}$$

where one replaces $I(s,W_v^\circ,W_v'^\circ)$ by $I(s,W_v^\circ,W_v'^\circ,\Phi^\circ)$ in case $m=n$. This is a standard Euler factor of degree mn associated to the tensor product mapping of Langlands dual groups $\otimes : GL_n(\mathbb{C})\times GL_m(\mathbb{C}) \to GL_{mn}(\mathbb{C})$. The proof uses the usual Hecke recursion relations to express

$$W_v^\circ(\varpi_v^J) = \delta_{B_n}^{1/2}(\varpi_v^J)\chi_J(A_{\pi_v})$$

where $J=(j_1,\dots,j_n)$ with $j_1\geq\cdots\geq j_n$, $\varpi^J = \mathrm{diag}(\varpi_v^{j_1},\dots,\varpi_v^{j_n}) \in GL_n(k_v)$ and χ_J is the character of the finite dimensional representation of $GL_n(\mathbb{C})$ of highest weight J. Then one uses results in the finite dimensional representation theory of $GL_n(\mathbb{C})$ to evaluate the integral. Unramified calculations invariably reduce to computations in invariant theory.

1.3. **The non-archimedean local theory.** The next step in the method is to analyze the families of local integrals appearing in the Euler factorization of our global integrals at each place, beginning with the general non-archimedean place. We will indicate how this works in the case of $m=n$ for notational convenience. The statements are all true for $m<n$ as well.

Let v be a non-archimedean place of k, k_v the completion, $\mathfrak{o}_v \supset \mathfrak{p}_v = (\varpi_v)$ its ring of integers and maximal ideal. Let $q_v = |\varpi_v|^{-1} = |\mathfrak{o}_v/\mathfrak{p}_v|$. Recall that our family of local integrals is

$$I_v(s, W_v, W'_v, \Phi_v) = \int_{N_n(k_v)\backslash GL_n(k_v)} W_v(g) W'_v(g) \Phi_v(e_n g) |\det g|^s \, dg$$

for $Re(s) > 1$, where $W_v \in \mathcal{W}(\pi_v, \psi_v)$, $W'_v \in \mathcal{W}(\pi'_v, \psi_v^{-1})$, and $\Phi_v \in \mathcal{S}(K_v^n)$. Then one proceeds to establish the following facts.

1. Each local integral is a rational function $I(s, W_v, W'_v, \Phi_v) \in \mathbb{C}(q_v^{-s})$. This gives meromorphic continuation of the local integrals.

The denominator of these rational functions come from the asymptotics of the Whittaker functions on the diagonal torus $A_n = \{\mathrm{diag}(a_1, \ldots, a_n)\}$ as the simple roots a_i/a_{i+1} approach 0. These asymptotics depend only on π_v and π'_v through their Jacquet modules and not on the specific Whittaker functions. Hence these rational functions all have *bounded denominators*. Also, by making appropriate choices of W_v, W'_v and particularly Φ_v, one can find a local integral which is independent of s and then in fact equal to the constant 1. So if we let

$$\mathcal{I}(\pi_v \times \pi'_v) = \langle I(s, W_v, W'_v, \Phi_v) \mid W_v \in \mathcal{W}(\pi_v, \psi_v), W'_v \in \mathcal{W}(\pi'_v, \psi_v^{-1}), \Phi_v \in \mathcal{S}(K_v^n) \rangle$$

then we can show

2. $\mathcal{I}(\pi_v \times \pi'_v)$ is a $\mathbb{C}[q_v^s, q_v^{-s}]$-fractional ideal in $\mathbb{C}(q_v^{-s})$ containing 1. So it has a normalized generator

$$\mathcal{I}(\pi_v \times \pi'_v) = \left(\frac{1}{P_v(q_v^{-s})} \right)$$

with $P_v(X) \in \mathbb{C}[X]$ with $P_v(0) = 1$.

One then *defines* the local L-function by

$$L(s, \pi_v \times \pi'_v) = P_v(q_v^{-s})^{-1}.$$

One of course must check that this is consistent with the unramified calculation at the unramified places. Since the L-function is realized as the generator of this fractional ideal, then one obtains

– There exist finite collections $\{W_{v,i}\}$, $\{W'_{v,i}\}$ and $\{\Phi_{v,i}\}$ such that

$$L(s, \pi \times \pi') = \sum_i I(s, W_{v,i}, W'_{v,i}, \Phi_{v,i}).$$

This generalizes the unramified computation to some extent, but it is just an existential statement in general.

– The ratios

$$e(s, W_v, W'_v, \Phi_v) = \frac{I(s, W_v, W'_v, \Phi_v)}{L(s, \pi_v \times \pi'_v)}$$

are entire functions of s and for any given $s_0 \in \mathbb{C}$ there exist choices of W_v, W'_v, and Φ_v such that $e(s_0, W_v, W'_v, \Phi_v) \neq 0$.

3. There is a *local functional equation* of the form

$$I(1-s,\widetilde{W}_v,\widetilde{W}'_v,\hat{\Phi}_v)=\omega_{\pi'_v}(-1)^{n-1}\gamma(s,\pi_v\times\pi'_v,\psi_v)I(s,W_v,W'_v,\Phi_v)$$

with $\gamma(s,\pi_v\times\pi'_v,\psi_v)\in\mathbb{C}(q_v^{-s})$ a rational function independent of the choices of W_v, W'_v, and Φ_v.

The local functional equation results from viewing $I(s,W_v,W'_v,\Phi_v)$ as a family of $GL_n(k_v)$-equivariant trilinear forms

$$I_s:\mathcal{W}(\pi_v,\psi_v)\times\mathcal{W}(\pi'_v,\psi_v^{-1})\times\mathcal{S}(k_v^n)\to\mathbb{C}$$

and then proving a uniqueness statement for such trilinear forms (for s in general position).

4. There is a *local ε–factor* defined through

$$\gamma(s,\pi_v\times\pi'_v,\psi_v)=\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\frac{L(1-s,\tilde{\pi}_v\times\tilde{\pi}'_v)}{L(s,\pi_v\times\pi'_v)}.$$

This factor satisfies

$$\varepsilon(1-s,\tilde{\pi}_v\times\tilde{\pi}'_v,\psi_v^{-1})\varepsilon(s,\pi_v\times\pi'_v,\psi_v)=1$$

and is then of the form

$$\varepsilon(s,\pi_v\times\pi'_v,\psi_v)=Wq_v^{-f(s-1/2)}.$$

1.4. **The archimedean local theory.** So now v is a place of k where $k_v=\mathbb{R}$ or $\mathbb{C}$.

The archimedean local theory has a slightly different paradigm. It is based on the prior existence of the local Langlands correspondence over archimedean fields, i.e., the arithmetic Langlands parameterization of local representations in terms of representations of the Weil group W_{k_v}. For GL_n it states that there is a (canonical) bijection between

$$\{n\text{-dimensional complex semisimple representations } \tau_v \text{ of } W_{k_v}\}$$

$$\updownarrow$$

$$\{\text{irreducible admissible smooth representations } \pi_v \text{ of } GL_n(k_v) \text{ of uniform moderate growth}\}.$$

If τ_v is a representation of W_{k_v} we will let $\pi_v(\tau_v)$ denote the corresponding representation of $GL_n(k_v)$.

If π_v and π'_v are representations of $GL_n(k_v)$ and $GL_m(k_v)$ respectively such that $\pi_v=\pi_v(\tau_v)$ and $\pi'_v=\pi_v(\tau'_v)$ then one begins by *defining*

$$L(s,\pi_v\times\pi'_v)=L(s,\tau_v\otimes\tau'_v)$$
$$\varepsilon(s,\pi_v\times\pi'_v,\psi_v)=\varepsilon(s,\tau_v\otimes\tau'_v,\psi)$$

where the factors on the right are the usual archimedean factors attached to local Galois representations by Artin or Weil. Now we must compare these with our family of local integrals. (We still discuss the $m=n$ case, the $m<n$ case being analogous.)

1. Each ratio $e(s,W_{\xi_v},W'_{\xi'_v},\Phi_v)=\dfrac{I(s,,W_{\xi_v},W'_{\xi'_v},\Phi_v)}{L(s,\pi_v\times\pi'_v)}$ is entire.

2. The desired local functional equation holds:

$$\frac{I(1-s,\widetilde{W}_{\xi_v},\widetilde{W}'_{\xi'_v},\hat{\Phi}_v)}{L(1-s,\tilde{\pi}_v\times\tilde{\pi}'_v)}=\omega_{\pi'_v}(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\frac{I(s,W_{\xi_v},W'_{\xi'_v},\Phi_v)}{L(s,\pi_v\times\pi'_v)}$$

3. Again viewing the local integral $I(s,W_{\xi_v},W'_{\xi'_v},\Phi_v)$ as a family of continuous linear forms

$$I_s:\mathcal{W}(\pi_v,\psi_v)\otimes\mathcal{W}(\pi'_v,\psi_v^{-1})\otimes\mathcal{S}(k_v^n)\to\mathbb{C}$$

these extend to the topological tensor product of the first two factors

$$I_s:[\mathcal{W}(\pi_v,\psi_v)\hat{\otimes}\mathcal{W}(\pi'_v,\psi_v^{-1})]\otimes\mathcal{S}(k_v^n)\to\mathbb{C}.$$

We can view $\mathcal{W}(\pi_v,\psi_v)\hat{\otimes}\mathcal{W}(\pi'_v,\psi_v^{-1})$ as the Whittaker model of the topological tensor product of the smooth u.m.g. representations $\mathcal{W}(\pi_v\hat{\otimes}\pi'_v)$, or , what is the same, the Casselman–Wallach canonical completion of the algebraic tensor product. Then what one can show is that one can represent the L-function *in this larger space*, i.e., there are finite collections $\{\underline{W}_{v,i}\}\subset\mathcal{W}(\pi_v\hat{\otimes}\pi'_v)$ and $\{\Phi_{v,i}\}\subset\mathcal{S}(k_v^n)$ such that

$$L(s,\pi_v\times\pi'_v)=\sum_i I(s,\underline{W}_{v,i},\Phi_{v,i}).$$

Remark: If in fact $m=n$ (as we are assuming here) or $m=n-1$ then we can find finite collections in the algebraic tensor product that suffice, i.e., collections $\{W_{\xi_{v,i}}\}\subset\mathcal{W}(\pi_v,\psi_v)$, $\{W'_{\xi'_{v,i}}\}\subset\mathcal{W}(\pi'_v,\psi_v^{-1})$ and $\{\Phi_{v,i}\}\subset\mathcal{S}(k_v^n)$ such that

$$L(s,\pi_v\times\pi'_v)=\sum_i I(s,W_{\xi_{v,i}},W'_{\xi'_{v,i}}\Phi_{v,i})$$

and if we are in the unramified case, then the unramified vectors $W_{\xi_v^\circ}$, $W'_{\xi_v'^\circ}$ and and Φ_v° alone work.

1.5. **Global theory.** In the end, we combine the analytic analysis of our global integrals and our local analysis to define and analyze the global L-functions.

Now once again $\pi\simeq\otimes'\pi_v$ and $\pi'\simeq\otimes'\pi'_v$ are cuspidal representations of $GL_n(\mathbb{A})$ and $GL_m(\mathbb{A})$ respectively. Having defined the local Euler factors $L(s,\pi_v\times\pi'_v)$ at all places we simply *define* the global L-function through an Euler product:

$$L(s,\pi\times\pi')=\prod_v L(s,\pi_v\times\pi'_v)$$
$$\varepsilon(s,\pi\times\pi')=\prod_v \varepsilon(s,\pi_v\times\pi'_v,\psi_v)$$

where implicit in the notation is that the global ε–factor is independent of the choice of additive character.

These are analyzed as follows. Let S be a finite set of places of k, containing all archimdean places, such that for $v\notin S$ we have π_v, π'_v, and ψ_v all unramified. We then consider

– decomposable cusp forms $\varphi \in V_\pi$ such that under the decomposition $\pi \simeq \otimes' \pi_v$ we have $\varphi \simeq \otimes \xi_v$ with $\xi_v = \xi_v^\circ$ the normalized $K_{n,v}$-fixed vector for $v \notin S$,
– decomposable cusp forms $\varphi' \in V_{\pi'}$ such that under the decomposition $\pi' \simeq \otimes' \pi'_v$ we have $\varphi' \simeq \otimes \xi'_v$ with $\xi'_v = {\xi'_v}^\circ$ the normalized $K_{m,v}$-fixed vector for $v \notin S$,
– if $m = n$, decomposable Schwartz functions $\Phi = \otimes \Phi_v$ with $\Phi_v = \Phi_v^\circ$ for $v \notin S$.

Then from our Euler factorization of the global integrals plus our unramified calculation we find, say for $m = n$,

$$\begin{aligned} I(s, \varphi, \varphi', \Phi) &= \prod_v I_v(s, W_{\xi_v}, W'_{\xi'_v}, \Phi_v) \\ &= \left(\prod_{v \in S} I_v(s, W_{\xi_v}, W'_{\xi'_v}, \Phi_v) \right) L^S(s, \pi \times \pi') \\ &= \left(\prod_{v \in S} \frac{I_v(s, W_{\xi_v}, W'_{\xi'_v}, \Phi_v)}{L(s, \pi_v \times \pi'_v)} \right) L(s, \pi \times \pi') \end{aligned}$$

where as is customary, $L^S(s, \pi \times \pi')$ is the partial Euler product

$$L^S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi'_v).$$

If we now use what we know about the analytic properties of the global integrals (including the functional equation) and combine this with the analysis of the local integrals (including the local functional equation), we find the following.

1. The Euler product $L(s, \pi \times \pi')$ is absolutely convergent in a right half plane $Re(s) >> 0$.

2. $L(s, \pi \times \pi')$ has a meromorphic continuation to all $\mathbb{C}$.

3. If $m < n$ then $L(s, \pi \times \pi')$ is entire. If $m = n$, then $L(s, \pi \times \pi')$ has simple poles at $s = i\sigma, 1 + i\sigma$ where $\tilde{\pi} \simeq \pi' \otimes |\det|^{i\sigma}$.

4. Global functional equation: $L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \tilde{\pi} \times \tilde{\pi}')$.

5. If $m = n$ or $m = n - 1$ there are finite collections $\{\varphi_i\}$, $\{\varphi'_i\}$, and if necessary $\{\Phi_i\}$ such that

$$L(s, \pi \times \pi') = \begin{cases} \sum_i I(s, \varphi_i, \varphi', \Phi_i) \\ \sum_i I(s, \varphi_i, \varphi'_i) \end{cases}$$

so that $L(s, \pi \times \pi')$ is bounded in vertical strips. (If $m < n - 1$ one can appeal to results of Gelbart and Shahidi to reach the same conclusion.)

1.6. **Some open questions.** Here are a couple of things that we would like to know and would probably be of use.

1. *Test vectors*: Given a non-archimedean place where π_v or π'_v is ramified, find *explicit* vectors ξ_v and ξ'_v (and Φ_v if necessary) such that

$$L(s, \pi_v \times \pi'_v) = I(s, W_{\xi_v}, W'_{\xi'_v}, \Phi_v)$$

or at least *explicit families* so that

$$L(s, \pi_v \times \pi'_v) = \sum_i I(s, W_{\xi_{v,i}}, W'_{\xi'_{v,i}}, \Phi_{v,i}).$$

One can also ask for an archimedean theory of test vectors and for $m < n - 1$ whether we can express the local L-function as a finite sum of local integrals *without* passing to the topological tensor product.

2. *Cohomological vectors*: Let v be an archimedean place, and let π_v and π'_v be representations with non-trivial $(\mathfrak{g}_v, K_v)$-cohomology, i.e., that contribute to cohomology. Suppose in addition that ξ_v and ξ'_v are cohomological vectors. Can we guarantee that the corresponding local integral is non-vanishing, i.e, is

$$I(s, W_{\xi_v}, W'_{\xi'_v}) \neq 0$$

and if so, do we have any hope of actually computing it?

References for GL_n

[1] J.W. Cogdell, L-functions and Converse Theorems for GL_n (Park City Lectures), in: *Automorphic Forms and Applications*, IAS/Park City Mathematics Series, to appear.

[2] J.W. Cogdell, Lectures on L-functions, Converse Theorems, and Functoriality for GL_n (Fields Institute Lectures), in: *Lectures on Automorphic L-functions.* Fields Institute Monographs No.20, AMS, Providence, 2004.

References to the original sources can be found in the bibliographies of these papers.

Both are available at *www.math.ohio-state.edu/∼cogdell*

2. The doubling method

For GL_n, all cuspidal representations are generic,i.e., have globally defined Whittaker models (thanks to the existence of the Fourier expansion). The uniqueness of the Whittaker models played an important role in the Eulerian analysis of integral representations of L-functions for GL_n. For other groups cuspidal representations are not necessarily globally generic, yet almost all integral representations of L-functions, whether they are of Rankin–Selberg type or arise in the Langlands–Shahidi method, rely on Whittaker models.

In the 1980's, Piatetski-Shapiro and Rallis discovered a family of Rankin–Selberg integrals for the classical groups that did not rely on Whittaker models. This is the so-called doubling method. (It grew out of Rallis' work on the inner products of theta lifts ... the Rallis inner product formula.)

While the method works for all classical groups, I will concentrate in this lecture on the unitary groups since there seems to be much current interest in this case. I will concentrate on the analytic aspects of the construction and leave the arithmetic applications to others.

2.1. **The setup.** Let k be a number field and K/k a quadratic extension. Let V be a vector space over K of dimension n equipped with a non-degenerate (skew)-Hermitian form $\langle\ ,\ \rangle$. Let $G = U(V) \subset GL_n(K)$ be the associated unitary group. This is an algebraic group defined over k.

Let $W = V \oplus (-V)$ be the doubled space, by which we mean that $W = V \oplus V$ as a vector space, equipped with the skew-Hermitian form

$$\langle\langle (v_1, v_2), (v_1', v_2') \rangle\rangle = \langle v_1, v_1' \rangle - \langle v_2, v_2' \rangle.$$

Let $H = U(W) \subset GL_{2n}(K)$. This is a quasi-split unitary group with $H \simeq U(n,n)$. Note that we have a natural embedding

$$G \times G \hookrightarrow H$$

and we identify $G \times G$ with this subgroup when convenient.

Let

$$V^\Delta = \{(v,v) \mid v \in V\} \subset W \text{ and } V^{-\Delta} = \{(v,-v) \mid v \in V\} \subset W.$$

These are complementary maximal isotropic subspaces of W, non-degenerately paired under $\langle\langle\ ,\ \rangle\rangle$. Let $P^\Delta \subset H$ be the parabolic subgroup preserving V^Δ, a "Siegel parabolic". Then $P^\Delta = M^\Delta N^\Delta$ with $M^\Delta \simeq GL_n(K)$ and $N^\Delta \simeq Herm_n(K)$, the space of $n \times n$ Hermitian matrices. Note that $M^\Delta \cap (G \times G) = G^\Delta = \{(g,g) \mid g \in G\} \subset G \times G$.

2.2. **The global integrals.** The construction is a Rankin–Selberg type construction involving the integration of a pair of cusp forms on $G(\mathbb{A})$ against the restriction of a degenerate Eisenstein series on $H(\mathbb{A})$. We begin with the Eisenstein series.

The Eisenstein series: Let $\chi : K^\times\backslash\mathbb{A}_K^\times \to C^\times$ be an idele class character for K. We form an induced representation of $H(\mathbb{A})$

$$I(s,\chi) = \mathrm{Ind}_{P^\Delta(\mathbb{A})}^{H(\mathbb{A})}(\chi(\det)|\det|^{s-\frac{1}{2}})$$

acting on a space of functions $V(s,\chi)$ by right translation. If $f_{s,\chi}(h) \in V(s,\chi)$ is a section of this induced representation, we can form from it an Eisenstein series

$$E(h, f_{s,\chi}) = \sum_{\gamma\in P^\Delta(k)\backslash H(k)} f_{s,\chi}(\gamma h) \qquad \text{for } Re(s) >> 0.$$

There is a standard intertwining operator

$$M(s,\chi) : V(s,\chi) \to V(1-s,\check{\chi}),$$

with $\check{\chi}(x) = \chi(\overline{x})^{-1}$, defined for $Re(s) > \frac{n}{2}$ by the integral

$$\begin{aligned}[M(s,\chi)f_{s,\chi}](h) &= \int_{N^\Delta(\mathbb{A})} f_{s,\chi}(wnh)\ dn \\ &= \int_{Herm_n(\mathbb{A})} f_{s,\chi}(wn(X)h)\ dX\end{aligned}$$

where w is the Weyl element interchanging V^Δ and $V^{-\Delta}$, given by $w = (I_n, -I_n) \in G\times G \subset H$. Then, at least if the section $f_{s,\chi}$ is K_H-finite, we know

– $E(h, f_{s,\chi})$ extends to a meromorphic function of s, always automorphic in h;
– $E(h, f_{s,\chi})$ satisfies the functional equation $E(h, f_{s,\chi}) = E(h, M(s,\chi)f_{s,\chi})$.

Global integrals: Now let (π, V_π) be a cuspidal representation, $(\tilde{\pi}, V_{\tilde{\pi}})$ its contragredient cuspidal representation (so $V_{\tilde{\pi}} = \overline{V_\pi}$), and $\varphi \in V_\pi$ and $\tilde{\varphi} \in V_{\tilde{\pi}}$ cusp forms on $G(\mathbb{A})$. (Note that, in contrast to Lecture 1, φ and $\tilde{\varphi}$ are now independent cusp forms.) Let $f_{s,\chi} \in V(s,\chi)$ define an Eisenstein series $E(h, f_{s,\chi})$ on $H(\mathbb{A})$, which we can pull back to $G(\mathbb{A}) \times G(\mathbb{A}) \subset H(\mathbb{A})$. Then the global integral for the doubling method is

$$I(\varphi, \tilde{\varphi}, f_{s,\chi}) = \int_{[G\times G](k)\backslash[G\times G](\mathbb{A})^{(1)}} \varphi(g_1)\tilde{\varphi}(g_2)E((g_1,g_2), f_{s,\chi})\chi^{-1}(\det g_2)\ dg_1dg_2$$

where $[G\times G](\mathbb{A})^{(1)} = \{(g_1,g_2) \in [G\times G](\mathbb{A}) \mid |\det(g_1g_2)| = 1\}$. This inherits the analytic properties of the Eisenstein series, so

– $I(\varphi, \tilde{\varphi}, f_{s,\chi})$ extends to a meromorphic function of s;
– $I(\varphi, \tilde{\varphi}, f_{s,\chi})$ satisfies the functional equation $I(\varphi, \tilde{\varphi}, , f_{s,\chi}) = I(\varphi, \tilde{\varphi}, M(s,\chi)f_{s,\chi})$.

Eulerian product: To see that the global integrals are Eulerian, one inserts the definition of the Eisenstein series and unfolds. One must analyze the orbits of $G\times G$ on $P^\Delta\backslash H$. Most are *negligible*, that is, the stabilizer in $G\times G$ contains the unipotent radical of a proper parabolic subgroup of one factor of G as a normal subgroup, leading to an contribution of 0 since φ and $\tilde{\varphi}$ are cusp forms. There is one non-negligible orbit, with stabilizer G^Δ, giving, for $Re(s) >> 0$,

$$\begin{aligned} I(\varphi, \tilde{\varphi}, f_{s,\chi}) &= \int_{G^{\Delta}(\mathbb{A})\backslash[G\times G](\mathbb{A})} f_{s,\chi}((g_1, g_2))\chi^{-1}(\det g_2)\langle \pi(g_1)\varphi | \tilde{\pi}(g_2)\tilde{\varphi}\rangle \, dg_1 dg_2 \\ &= \int_{G(\mathbb{A})} f_{s,\chi}((g, 1))\langle \pi(g)\varphi | \tilde{\varphi}\rangle \, dg \end{aligned}$$

where $\langle \varphi | \tilde{\varphi}\rangle$ is the invariant pairing

$$\langle \varphi | \tilde{\varphi}\rangle = \int_{G(k)\backslash G(\mathbb{A})^{(1)}} \varphi(g)\tilde{\varphi}(g) \, dg$$

with $G(\mathbb{A})^{(1)} = \{g \in G(\mathbb{A}) \mid |\det(g)| = 1\}$. We now assume that all functions correspond to factorisable vectors in their respective representations: $\varphi \simeq \otimes\xi_v \in V_\pi \simeq \otimes' V_{\pi_v}$, $\tilde{\varphi} \simeq \otimes\tilde{\xi}_v \in V_{\tilde{\pi}} \simeq \otimes' V_{\tilde{\pi}_v}$, and $f_{s,\chi} \simeq \otimes f_{s,\chi_v} \in V(s,\chi) \simeq \otimes' V(s,\chi_v)$. (We also assume that in the restricted tensor products, the K_{G_v}-fixed vectors ξ_v° and $\tilde{\xi}_v^\circ$ with respect to which the tensor product is restricted satisfy $\langle \xi_v^\circ | \tilde{\xi}_v^\circ\rangle = 1$.) Then by the uniqueness of the invariant pairing we have a factorization

$$I(\varphi, \tilde{\varphi}, f_{s,\chi}) = \prod_v I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v})$$

where the local integrals are given by

$$I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v}) = \int_{G(k_v)} f_{s,\chi_v}((g,1))\langle \pi_v(g)\xi_v | \tilde{\xi}_v\rangle \, dg \quad \text{for } Re(s) >> 0.$$

2.3. **The unramified calculation.** To determine which L-function we are representing by this family of Eulerian integrals, we proceed with the unramified calculation. The situation turns out to be more complicated than in the GL_n case. Note that there are two types of unramified non-archimedean places: those that split in K, so $K_v \simeq k_v \oplus k_v$ and for which $G(k_v) \simeq GL_n(k_v)$, and those that remain inert, so K_v is still a quadratic extension of k_v and for which $G(k_v)$ remains a unitary group. When the place splits in K, the theory reduces to the local theory of Godement and Jacquet which is well understood (see [2]). We will thus assume that v is inert in K.

So suppose v is an inert non-archimedean place such that π_v and χ_v are both unramified. Let ξ_v° and $\tilde{\xi}_v^\circ$ be the normalized K_{G_v}-fixed vectors as above and take f_{s,χ_v}° the standard K_{H_v}-invariant section whose restriction to K_{H_v} is $\equiv 1$. Then, as was computed by Jian-Shu Li in [8],

$$I(\xi_v^\circ, \tilde{\xi}_v^\circ, f_{s,\chi_v}^\circ) = \frac{L(s, \pi_v \times \chi_v)}{d_v(s,\chi_v)}.$$

The numerator is the standard Langlands L-function attached to π_v and χ_v, which in terms of L-function for $GL_n(K_v)$ is given by

$$L(s, \pi_v \times \chi_v) = L(s, BC_{K_v/k_v}(\pi_v) \otimes \chi_v)$$

where $BC_{K_v/k_v}(\pi_v)$ is the local base change of π_v from $G(k_v)$ to $GL_n(K_v)$. This is the L-function we are interested in understanding.

The denominator $d_v(s,\chi_v)$ is the so-called local normalizing factor for the Eisenstein series. It is given by

$$d_v(s,\chi_v)=\prod_{j=1}^{n} L(2s+j-1,\chi_v^\circ\eta_v^{n-j})$$

where we have set $\chi_v^\circ=\chi_v|_{k_v}$ and $\eta_v=\eta_{K_v/k_v}$ is the quadratic character of k_v attached to the extension K_v/k_v by local class field theory. This normalization factor, as the name suggests, should be included in the section f_{s,χ_v}, or globally in the Eisenstein series, so that we have only the L-function of interest. So we would really prefer to use a "non-standard" section of the form $f^*_{s,\chi_v}=d_v(s,\chi_v)f^\circ_{s,\chi_v}$ so that the unramified calculation takes the form

$$I(\xi_v^\circ,\tilde{\xi}_v^\circ,f^*_{s,\chi_v})=L(s,\pi_v\times\chi_v).$$

Note that the normalizing factor is a product of Tate L-functions and is completely understood.

2.4. **Normalization of the intertwining operator.** If we want to carry out a more complete local analysis of $L(s,\pi_v\times\chi_v)$ for all places v, we need a local normalization of the Eisenstein series for all v. The way to do this in a conceptual manner is to normalize the local intertwining operator $M(s,\chi_v):V(s,\chi_v)\to V(1-s,\check{\chi}_v)$. We continue work at places v which are inert or ramified in K, the local theory at the split places being better understood.

To normalize the intertwining operator, we use a local uniqueness principle, similar to the local uniqueness of the Whittaker model. Each $T\in Herm_n(K_v)$ defines a character of $N^\Delta(k_v)\simeq Herm_n(K_v)$ by $\psi_T(n(X))=\psi_v(tr(TX))$. If $\det T\neq 0$ then $V(s,\chi_v)$ carries a unique continuous ψ_T-quasi-invariant functional Λ_T such that:

$$\Lambda_T(I(n(X))f_{s,\chi_v})=\psi_T^{-1}(n(X))\Lambda_T(f_{s,\chi_v}).$$

Such a functional exists, for we can always take

$$\Lambda_T(f_{s,\chi_v})=\int_{N^\Delta(k_v)} f_{s,\chi_v}(wn(X))\psi_T(n(X))\,dX,$$

which converges for $Re(s)>>0$ and then analytically continues. Given this, we can define an analogue of Shahidi's local coefficients and use these to normalize the local intertwining operator. The diagram

$$\begin{tikzcd} V(s,\chi_v) \arrow[r, "M(s,\chi_v)"] \arrow[d, "\Lambda_T"'] & V(1-s,\check{\chi}_v) \arrow[d, "\Lambda_T"] \\ \mathbb{C}^\times & \mathbb{C}^\times \arrow[l, "c(s,\chi_v,T,\psi_v)"] \end{tikzcd}$$

defines $c(s,\chi_v,T,\psi_v)\in\mathbb{C}^\times$ such that $\Lambda_T=c(s,\chi_v,T,\psi_v)\Lambda_T\circ M(s,\chi_v)$.

The factor $c(s,\chi_v,T,\psi_v)$ was computed explicitly by Harris-Kudla-Sweet in [9] and independently analyzed by Lapid-Rallis in [6]. Following Harris-Kudla-Sweet, if we set

$$c(s,\chi_v,\psi_v)=\frac{c(s,\chi_v,T,\psi_v)}{\chi_v(\det T)|\det T|^{2s-1}\eta_v(\det T)^{n-1}}$$

then this factor is independent of T and is essentially a product of γ–factors for GL_1

$$c(s,\chi_v,\psi_v)\sim\prod_{j=1}^{n}\gamma(2s-j,\chi_v^\circ\eta_v^{n-j},\psi_v)$$

where as usual for a character χ_v' of $GL_1(k_v)$ we set

$$\gamma(s,\chi_v',\psi_v)=\varepsilon(s,\chi_v',\psi_v)\frac{L(1-s,{\chi_v'}^{-1})}{L(s,\chi_v')}.$$

(*Remark*: As Lapid pointed out at the Conference, he and Rallis define $c(s,\chi_v,\psi_v)$ slightly differently than Harris-Kudla-Sweet, the difference being a factor of $\omega_{\pi_v}(-1)$. I have chosen to follow Harris-Kudla-Sweet at this point, but will build in the $\omega_{\pi_v}(-1)$ below.)

We then normalize the intertwining operator by setting

$$M^*(s,\chi_v)=c(s,\chi_v,\psi_v)M(s,\chi_v).$$

This satisfies the functional equation $M^*(1-s,\check{\chi}_v)M^*(s,\chi_v)=1$.

2.5. **Non-archimedean local theory (via integrals).** We continue to follow Harris-Kudla-Sweet [9].

We first need a notion of *good sections* of $V(s,\chi_v)$ that incorporate the normalizations.

A section $f_{s,\chi_v}\in V(s,\chi_v)$ is called a *standard section* if its restriction to the maximal compact K_{H_v} is independent of s. Let $V_{std}(s,\chi_v)$ denote the space of standard sections.

In order to allow for the normalization, we need to allow these sections to vary arithmetically in s. To this end, Harris-Kudla-Sweet (following earlier work of Piatetski-Shapiro and Rallis) define the space $V_{good}(s,\chi_v)$ of *good sections* to consist of the following:

(i) $\mathbb{C}[q_v^s,q_v^{-s}]V_{std}(s,\chi_v)$,
(ii) $M^*(1-s,\check{\chi}_v)[\mathbb{C}[q_v^s,q_v^{-s}]V_{std}(s,\chi_v)]$,
(iii) if χ_v is unramified, the sections of the form

$$d_v(s,\chi_v)\cdot f^\circ_{s,\chi_v}*\beta(s)$$

with $\beta(s)\in\mathcal{H}(H(k_v)//K_{H_v})[q_v^s,q_v^{-s}]$.

(One can find sections of the form (iii) already in [2], where χ_v is trivial.) Then one checks that $M^*(s,\chi_v)V_{good}(s,\chi_v)\subset V_{good}(1-s,\check{\chi}_v)$, so this is indeed a good family of sections for the normalized intertwining operator, and they include the "normalized section" from Section 2.3.

Now one proceeds as in the GL_n situation (Section 1.3). We consider the family of integrals $I_v(\xi_v,\tilde{\xi}_v,f_{s,\chi_v})$ where as before $\xi_v\in V_{\pi_v}$, $\tilde{\xi}_v\in V_{\tilde{\pi}_v}$, but now $f_{s,\chi_v}\in V_{good}(s,\chi_v)$, i.e., we consider only good sections of the induced representation. Let $\mathcal{I}(\pi_v\times\chi_v)$ denote the span of these integrals.

1. Each integral $I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v}) \in \mathbb{C}(q_v^{-s})$ is a rational function of q_v^{-s}.

Moreover, for fixed ξ_v and $\tilde{\xi}_v$ there is a good section f_{s,χ_v} such that $I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v}) = 1$.

2. $\mathcal{I}(\pi_v \times \chi_v)$ is a $\mathbb{C}[q_v^s, q_v^{-s}]$-fractional ideal in $\mathbb{C}(q_v^{-s})$ containing 1. So it has a normalized generator

$$\mathcal{I}(\pi_v \times \chi_v) = \left(\frac{1}{P_v(q_v^{-s})}\right)$$

with $P_v(X) \in \mathbb{C}[X]$ such that $P_v(0) = 1$.

One then *defines* the local L-function by

$$L(s, \pi_v \times \chi_v) = P_v(q_v^{-s})^{-1}.$$

In this context, one can find vectors ξ_v, $\tilde{\xi}_v$, and good section f_{s,χ_v} such that

$$I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v}) = P_v(q_v^{-s})^{-1} = L(s, \pi_v \times \chi_v).$$

3. There is a local functional equation of the form

$$I_v(\xi_v, \tilde{\xi}_v, M^*(s, \chi_v) f_{s,\chi_v}) = \omega_{\pi_v}(-1)\gamma(s, \pi_v \times \chi_v, \psi_v) I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v})$$

with $\gamma(s, \pi_v \times \chi_v, \psi_v) \in \mathbb{C}(q_v^{-s})$ a rational function independent of ξ_v, $\tilde{\xi}_v$ and f_{s,χ_v}. (*Note*: By incorporating the $\omega_{\pi_v}(-1)$ explicitly in the local functional equation, this becomes consistent with the GL_n formulation and makes the local γ-factor agree with that of Lapid-Rallis.)

4. There is a *local ε–factor* defined through

$$\gamma(s, \pi_v \times \chi_v, \psi_v) = \varepsilon(s, \pi_v \times \chi_v, \psi_v) \frac{L(1-s, \pi_v \times \check{\chi}_v)}{L(s, \pi_v \times \chi_v)}.$$

This factor satisfies

$$\varepsilon(1-s, \pi_v \times \check{\chi}_v, \psi_v^{-1}) \varepsilon(s, \pi_v \times \chi_v, \psi_v) = 1$$

and is then of the form

$$\varepsilon(s, \pi_v \times \chi_v, \psi_v) = W q_v^{-f(s-1/2)}.$$

2.6. **The archimedean local theory (via integrals).** Kudla and Rallis carried this out in the orthogonal and symplectic groups in Section 3 of [4] with an eye to understanding the poles of standard L-functions for these groups. This has **not** been carried out for the unitary groups, but their results should carry over. Below I have included the statements that should follow from the techniques of Kudla and Rallis for our unitary integrals, *but I have not checked the details.*

Note: Garrett has some archimedean calculations for the unitary group, but they are primarily aimed at algebraicity results for special values of s and not the analytic theory.

So let v be an archimedean place of k, so $k_v = \mathbb{R}$ or $\mathbb{C}$. Kudla and Rallis consider smooth entire sections of $V(s,\chi_v)$, which we will denote by $V^\infty(s,\chi_v)$. Then for $\xi_v \in V_{\pi_v}$, $\tilde{\xi}_v \in V_{\pi_v}$ and $f_{s,\chi_v} \in V^\infty(s,\chi_v)$ we should have the following facts.

1. Each $I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v})$ has a meromorphic continuation to all $\mathbb{C}$.

2. For fixed ξ_v and $\tilde{\xi}_v$ there exists f_{s,χ_v} such that $I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v})$ is non-zero and independent of s.

3. For any fixed s_0, the order of the pole of the family

$$\mathcal{I}(\pi_v \times \chi_v) = \langle I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v}) \rangle$$

is bounded and depends only on π_v and s_0.

4. For any s_0 there exist ξ_v, $\tilde{\xi}_v$ and f_{s,χ_v} a K_{H_v}-finite section, such that $I_v(s, \xi_v, \tilde{\xi}_v, f_{s,\chi_v})$ has either a pole or is non-zero at s_0.

Note that there is no reference to the archimedean Langlands classification, no definition of an archimedean L-function and no local functional equation. However, the local functional equation remains valid (see below).

The control of the analytic properties at given s_0 is useful for understanding the location of the poles of a partial global L-function $L^S(s, \pi \times \chi)$ in terms of the those of the global integrals.

2.7. **Local theory via γ–factors.** This is carried out in the paper of Lapid and Rallis [6]. It is *independent* of the local non-archimedean theory and local archimedean theory in Sections 2.5 and 2.6 above.

They essentially begin with the *local functional equation* defining the local γ–factor, which they note is valid for all places of k:

$$I_v(\xi_v, \tilde{\xi}_v, M^*(s,\chi_v) f_{s,\chi_v}) = \omega_{\pi_v}(-1)\gamma(s, \pi_v \times \chi_v, \psi_v) I_v(\xi_v, \tilde{\xi}_v, f_{s,\chi_v}).$$

Note that once again we have included the factor of $\omega_{\pi_v}(-1)$ in our definition of γ. One *expects* that this is related to a theory of local L – and ε– factors by

$$\gamma(s, \pi_v \times \chi_v, \psi_v) = \varepsilon(s, \pi_v \times \chi_v, \psi_v) \frac{L(1-s, \pi_v \times \check{\chi}_v)}{L(s, \pi_v \times \chi_v)}.$$

Shahidi used this idea to successfully *define* local L– and ε–factors in the context of the Langlands–Shahidi method. Lapid and Rallis follow this paradigm here.

2.7.1. *The non-archimedean local theory.* Let v be a non-archimedean place of k. In this case one can follow Shahidi's paradigm and use the local γ–factor to *define* local L– and ε–factors. As before, we still know that $\gamma(s, \pi_v \times \chi_v, \psi_v) \in \mathbb{C}(q_v^{-s})$ is a rational function of q_v^{-s}.

1. If π_v is a *tempered* representation, one does not expect cancellation in the L-factors, so one defines $L(s, \pi_v \times \chi_v)$ by

$$L(s, \pi_v \times \chi_v)^{-1} = \text{ Numerator of } \gamma(s, \pi_v \times \chi_v),$$

normalized to be of the form $P_v(q_v^{-s})^{-1}$ as before. Once one has defined the L–functions, one can use the relation above to define $\varepsilon(s, \pi_v \times \chi_v, \psi_v)$, as one does when using the local integrals.

2. In general, use the classification of representations to write π_v as the Langlands quotient of an induced representation $\mathrm{Ind}_{Q(k_v)}^{G(k_v)}(\sigma_v)$ with σ_v a tempered representation of the Levi subgroup of $Q(k_v)$. The one inductively defines

$$L(s, \pi \times \chi_v) = L(s, \sigma_v \times \chi_v)$$
$$\varepsilon(s, \pi_v \times \chi_v, \psi_v) = \varepsilon(s, \sigma_v \times \chi_v, \psi_v).$$

This is an inductive definition and uses strongly the mulitplicativity of the γ–factors, established in [6].

3. Once one has local L-and ε defined in this way, one again has

$$\varepsilon(1 - s, \pi_v \times \check{\chi}_v, \psi_v^{-1})\varepsilon(s, \pi_v \times \chi_v, \psi_v) = 1$$

and is then of the form

$$\varepsilon(s, \pi_v \times \chi_v, \psi_v) = W q_v^{-f(s-1/2)}.$$

4. When π_v has a vector fixed by an Iwahori subgroup, one knows how to arithmetically parameterize π_v by a representation τ_v of the Weil-Deligne group. In this case, the L– and ε–factors defined in this way agree with those of the associated representation of the Weil-Deligne group. So these factors are *arithmetically correct.* When π_v is generic, these factors agree with those defined by Shahidi using the Langlands–Shahidi method. Also, since they agree with the normalized unramified calculation (unramified representations trivially having a Iwahori fixed vector), they are compatible with the local integrals at those places.

Note: At the ramified place, it is *not known* whether the local L- and ε-factors defined here and those defined in Section 2.5 agree.

2.7.2. *The archimedean local theory.* If v is an archimedean place of k, then for all π_v we know how to parameterize π_v through a representation τ_v of the local Weil group as in Section 1.4. In this case, what Lapid and Rallis establish is that $\gamma(s, \pi_v \times \chi_v, \psi_v)$ is compatible with the Langlands classification. Hence if we define the local L– and ε–factors through the arithmetic Langlands classification, the arithmetic γ-factor is consistent with the local functional equation of the archimedean integrals.

2.8. **Global theory.** We now find ourselves in the position of having two possible different local theories at the ramified places. So we possibly have two distinct global theories as well.

Let S be a finite set of places, containing the archimedean places, so that for all $v \notin S$ we have v is unramified in K and π_v, χ_v, and ψ_v are all unramified. For the places $v \notin S$ we have the local factor $L(s, \pi_v \times \chi_v)$ is the same by either method. Hence the partial L-function

$$L^S(s, \pi \times \chi) = \prod_{v \notin S} L(s, \pi_v \times \chi_v)$$

is unambiguously defined.

We then choose

- decomposable cusp forms $\varphi \in V_\pi$ such that under the decomposition $\pi \simeq \otimes' \pi_v$ we have $\varphi \simeq \otimes \xi_v$ with $\xi_v = \xi_v^\circ$ the normalized K_{G_v}-fixed vector for $v \notin S$,
- decomposable cusp forms $\tilde{\varphi} \in V_{\tilde{\pi}}$ such that under the decomposition $\tilde{\pi} \simeq \otimes' \tilde{\pi}_v$ we have $\tilde{\varphi} \simeq \otimes \tilde{\xi}_v$ with $\tilde{\xi}_v = \tilde{\xi}_v^\circ$ the normalized K_{G_v}-fixed vector for $v \notin S$,
- decomposable globally good section $f_{s,\chi} = \otimes f_{s,\chi_v}$ with $f_{s,\chi_v} = d_v(s, \chi_v) f^\circ_{s,\chi_v}$ for $v \notin S$.

Then we can write

$$I(\varphi, \tilde{\varphi}, f_{s,\chi}) = \left(\prod_{v \in S} I_v(s, \xi_v, \tilde{\xi}_v, f_{s,\chi_v}) \right) L^S(s, \pi \times \chi)$$

from which we can conclude the meromorphic continuation of the partial L-function.

If we combine the global functional equation for the global integrals with the normalized local functional equations, we can arrive at a global functional equation for the partial L-function of the form

$$L^S(s, \pi \times \chi) = \left(\prod_{v \in S} \gamma(s, \pi_v \times \chi_v, \psi_v) \right) L^S(1-s, \pi \times \check{\chi})$$

involving the local γ-factors.

For the non-archimedean places $v \in S$ we have two different definitions of local L– and ε–factors, both of which satisfy

$$\gamma(s, \pi_v \times \chi_v, \psi_v) = \varepsilon(s, \pi_v \times \chi_v, \psi_v) \frac{L(1-s, \pi_v \times \check{\chi}_v)}{L(s, \pi_v \times \chi_v)},$$

while at the archimedean places, we know that the arithmetically defined L– and ε–factors are consistent with the local γ-factor. Hence using *either* of the definitions of local factors at the ramified non-archimedean places and the arithmetic factors at the archimedean places, we can complete

$$L(s, \pi \times \chi) = \left(\prod_{v \in S} L(s, \pi_v \times \chi_v) \right) L^S(s, \pi \times \chi)$$
$$\varepsilon(s, \pi \times \chi) = \prod_{v \in S} \varepsilon(s, \pi_v \times \chi_v, \psi_v)$$

and conclude

1. The global L-function $L(s, \pi \times \chi)$ extends to a meromorphic function of s.

2. We have the global functional equation $L(s, \pi \times \chi) = \varepsilon(s, \pi \times \chi) L(1 - s, \pi \times \check{\chi})$.

On the other hand, if we complete the archimedean local theory for the unitary groups via the integrals as in Section 2.6 and use the local factors from the integrals at the unramified places as in Section 2.5, then following Kudla and Rallis [4] we *should* be able to show that $L(s, \pi \times \chi)$ has at most a finite number of simple poles at prescribed places on the real axis. In essence, using these local factors, any pole of the L–function must give a pole of the global integral and hence a pole of the normalized Eisenstein series. These should be finite in number with understood locations, as in [4].

2.9. **On "twisted doubling".** One would like to generalize the theory outlined above to a theory of L-functions for $L(s, \pi \times \sigma)$ for σ a cuspidal representation of $GL_m(\mathbb{A}_K)$. With such a theory, one should be able to establish base change from G to GL_n via the converse theorem, for example.

For orthogonal groups, there is the beginnings of such a theory in the Ginzburg-Piatetski-Shapiro–Rallis Memoir [10]. The method is based on the theory of *Gelfand-Graev models*. The method should work for other classical groups, such as the unitary and symplectic groups, and one can find at least the global integrals for these cases in Soudry's ICM talk [12].

To my mind, the construction looks like a variant of the doubling method (which is why I refer to it as "twisted doubling") even though it does not specialize to doubling (and so "twisted doubling" may well be a misnomer).

In terms of our paradigm for analyzing integral representations, Ginzburg–PS–Rallis work out the following:

1. Global integrals.

2. At least a partial Eulerian factorization (see the last section of [12] and [11])

3. The unramified calculation.

4. The local functional equation.

It is still in a rather primitive (i.e., complicated) phase, although Rallis and Soudry are involved in a project to establish a general functoriality from orthogonal groups to GL_n using these integrals and the converse theorem [12]. The required stability of the local γ–factor reduces to that of the γ-factor coming from the doubling method and can be found in [7] for example. (This local stability of γ for the doubling method has recently been established for the unitary groups by Eliot Brenner.)

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