Eisenstein series

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Brazil vs. Australia

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Let

$$\mathcal{H} = \{x + \mathrm{iy} : \mathrm{y} > 0\}$$

be the upper half-plane. It is a symmetric space. The group $G = SL_2(\mathbb{R})$ acts by Möbius transformations

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az + b}{cz + d}$$

The measure $\frac{dx\ dy}{y^2}$ is invariant under G. The Laplacian is the second order differential operator given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It is positive-definite and commutes with the action of G; any other differential operator which commutes with the G-action is a polynomial in Δ.

Spectral analysis on $\mathcal{H}$.

Let $W_s(z) = \sqrt{y}K_s(2\pi y)e^{2\pi \mathrm{i}x}$, $s \in \mathbb{C}$. This is an eigenfunction for Δ with eigenvalue $\frac{1}{4} - s^2$. Similarly, $W_s(rz)$, $r > 0$.

Any $f \in \mathbb{C}_c^\infty(\mathcal{H})$ can be expanded as

$$f(z) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty (f, W_{\mathrm{it}}(r\cdot))_{\mathcal{H}} W_{\mathrm{it}}(rz) t \sinh \pi t \, dt \, \frac{dr}{r}$$

(corresponding to $G = NAK$) or, as

$$f(z) = \sum_{m\in\mathbb{Z}} \int_0^\infty (f, U_S^m) U_s^m(z) t \tanh \pi t \ dt$$

U_s^m - given in terms of Legendre function (corresponding to $G = KAK$; especially useful for K-invariant f's, i.e. those depending on $\rho(z, \mathrm{i})$, where ρ is the hyperbolic distance).

The subgroup $\Gamma = SL_2(\mathbb{Z})$ is discrete in G with $\mathrm{vol}(\Gamma\backslash G) < \infty$, i.e. it is a *lattice*.

Eisenstein series

$$E(z;s) = \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}$$
$$= \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma z)^{s+\frac{1}{2}} \quad z \in \mathcal{H}$$

where $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$.

Sometimes it is also convenient to use the normalized Eisenstein series

$$E^*(z;s) = \zeta^*(2s+1)E(z;s)$$
$$= \pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2}) \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}$$

(by pulling out $\gcd(m,n)$) where

$$\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \zeta^*(1-s)$$

Properties

1. converges for $\mathrm{Re}\, s > \frac{1}{2}$.

2. $E(\gamma z; s) = E(z; s)$ for all $\gamma \in \Gamma$.

3. $\Delta E(\cdot; s) = (\frac{1}{4} - s^2) E(\cdot; s)$

4. Analytic continuation to $s \in \mathbb{C}$ (except for a simple pole at $s = \pm\frac{1}{2}$) and a functional equation

$$E^*(z; -s) = E^*(z; s)$$

5. The residue at $s = \frac{1}{2}$ is identically 1.

6. The Fourier expansion at the cusp

$$\sum_{r \in \mathbb{Z}} a_r(y, s) (= \int_0^1 E^*(x + \mathrm{i}y; \mathrm{s}) \mathrm{e}^{-2\pi \mathrm{i} r x} \, \mathrm{d}x) \mathrm{e}^{2\pi \mathrm{i} r x}$$

is given by

$$a_r(y,s) = 4\,|r|^s\,\sigma_{-2s}(|r|)\sqrt{y}K_s(2\pi\,|r|\,y) \quad r \neq 0$$
$$a_0(y,s) = 2\zeta^*(2s+1)y^{s+\frac{1}{2}} + 2\zeta^*(1-2s)y^{-s+\frac{1}{2}}$$

where

$$\sigma_t(n) = \sum_{d|n} d^t$$

is the divisor function and

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\ \frac{dt}{t} = K_{-s}(y)$$

is the K-Bessel function

We can write the Eisenstein series an Epstein zeta function

$$E^*(z; s-\frac{1}{2}) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0,0\}} Q_z(m,n)^s$$

w.r.t. the binary quadratic form $Q_z(x_1,x_2) = |zx_1 + x_2|^2$

The holomorphy of $E(z;s)$ for $\mathrm{Re}(s) = 0$ and the formula for the first Fourier coefficient already imply that $\zeta(1 + \mathrm{i}t) \neq 0$ for all $t \in \mathbb{R}$, i.e. the Prime Number Theorem!

Special values:

$$E^*(\mathrm{i};\mathrm{s}) = 2^{\mathrm{s}}\zeta^*_{\mathrm{Q}(\sqrt{-1})}(\mathrm{s} + \frac{1}{2})$$

More generally, let $z \in \mathcal{H}$ be a CM point of discriminant $d < 0$, i.e.

$$az^2 + bz + c = 0, \quad a, b, c \in \mathbb{Z}, b^2 - 4ac = d$$

Assume that d is fundamental, that is d is square-free except for 4. Then Γz corresponds to the ideal class $\mathfrak{a}$ of $(a, (b+\sqrt{d})/2)$ in the ring of integers of $Q(\sqrt{d})$ and

$$E^*(z;s) = (\frac{2\pi}{\sqrt{|d|}})^{-(s+\frac{1}{2})}\Gamma(s + \frac{1}{2})\zeta_{\mathfrak{a}}(s + \frac{1}{2})$$

Thus,

$$\sum_{z\in\Lambda_d} E^*(z;s)\chi(z) = \frac{1}{2}\sqrt{|d|}^{s+\frac{1}{2}}L^*(\chi, s + \frac{1}{2})$$

where Λ_d is the set of Γ-orbits of CM points of discriminant d and χ is an ideal class character.

Bernstein's proof of the analytic continuation of Eisenstein series

Lemma 1. *For* $\mathrm{Re}(s) > \frac{1}{2}$ $E(z;s)$ *is the unique automorphic form* F *satisfying*

1. $\Delta(F) = (\frac{1}{4} - s^2)F$

2. $F_U = y^{s+\frac{1}{2}} + *y^{-s+\frac{1}{2}}$ *for some constant* $*$ *where* $F_U(y) = \int_0^1 F(x + \mathrm{iy})\ \mathrm{dx}$*. Alternatively,*

$$y\frac{d}{dy}(F_U - y^{s+\frac{1}{2}}) = (-s + \frac{1}{2})(F_U - y^{s+\frac{1}{2}}).$$

Proof. Consider $f = F - E(z;s)$. Then $f_U = *y^{-s+\frac{1}{2}}$ and therefore f is square-integrable. Since Δ is positive-definite, this implies that $\frac{1}{4} - s^2 \geq 0$, which contradicts the assumption that $\mathrm{Re}(s) > \frac{1}{2}$. $\square$

General principle: Suppose that S is a connected complex manifold and V a topological vector space. Let $\Xi = \Xi(s)_{s\in S}$ be a family of systems of linear equations in V depending holomorphically on S. That is, there exist analytic functions $c_i : S \to \mathbb{C}$ and $\mu_i : S \to V'$, $i \in I$ such that the system $\Xi(s)$ has the form

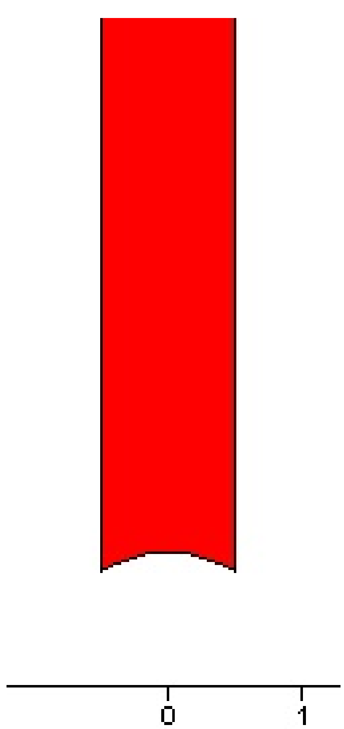
$$(\mu_i(s), v) = c_i(s).$$

Denote by $\mathrm{Sol}(\Xi(s))$ the set of solutions of the system $\Xi(s)$ in V. Suppose that for some open $U \subset S$ (in the complex topology) the system $\Xi(s)$ has a unique solution $v(s) \in V$. Suppose further that Ξ is of locally finite type, i.e., for every $s \in S$ there exists a neighborhood W, a finite-dimensional vector space L and an analytic family of linear maps $\lambda(s) : L \to V$ such that $\mathrm{Sol}(\Xi(s)) \subset \mathrm{Im}\,\lambda(s)$ for all $s \in W$. Then $\Xi(s)$ has a unique solution $v(s)$ on a dense open subset of S and $v(s)$ extends to a meromorphic function on S.

Proof. Let S_0 be the set of points $s \in S$ for which there exists a neighborhood on which $\Xi(s)$ has a unique solution. We will show that $\overline{S_0}$ is open and that $v(s)$ is meromorphic on $\overline{S_0}$. By connectedness, this will imply the statement. Now, let $s \in \overline{S_0}$ and W, L, λ as above. We show that W (or alternatively, a dense open subset of W) is contained in $\overline{S_0}$. Upon passing to a subspace of L, we may assume that $\lambda(s)$ is monomorphic for all $s \in W$. The system $\Xi(s)$ induces a system $\Xi'(s)$ on L which has a unique solution $v'(s)$ on the non-zero open subset $W \cap S_0$. Then some $k \times k$-determinant $D(s)$ of coefficients of $\Xi'(s)$ does not vanish on W where $k = \dim L$. On the dense open set $U = \{s \in W : D(s) \neq 0\}$ there is a unique solution $v'(s)$ for the $k \times k$ sub-system and by Cramer's rule $v'(s)$ is meromorphic on W. Clearly, $\lambda(s)(v'(s))$ is the unique solution of $\Xi(s)$ on U and in particular $\lambda(s)(v'(s)) = v(s)$ on $S_0 \cap W$. $\square$

It remains to show that the system defined by $\Delta f = (1/4 - s^2)f$ is of locally finite type. This is a technical strengthening of Harish-Chandra's finiteness theorem. It can be proved along the same lines.

Applications: Computation of the volume of the fundamental domain (Langlands, Boulder '65)



Naively, we can try to compute $\mathrm{vol}(\Gamma\backslash\mathcal{H})$ by computing

$$I(s) = \int_{\Gamma\backslash\mathcal{H}} E(z;s)\ dz$$

and taking residue at $s = \frac{1}{2}$. The problem is that $E(z;s) \notin L^1(\Gamma\backslash\mathcal{H})$ in the range of convergence. On the other hand $E(z;s) \in L^1(\Gamma\backslash\mathcal{H})$ if $|\mathrm{Re}(s)| < \frac{1}{2}$. However, we will soon see that $I(s) \equiv 0$. (We cannot take the limit inside the integral because of the non-compactness of the domain.)

Instead we take for any $f \in C_c^\infty(\mathbb{R}_{>0})$ the wave packet

$$\theta_f(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\,\gamma z)$$

This is a finite sum, and θ_f is compactly supported in $\Gamma\backslash\mathcal{H}$. By Mellin inversion,

$$f(y) = \int_{\mathrm{Re}(s)=s_0} \hat{f}(s) y^s\ ds$$

for any s_0 where $\widehat{f}$ is the Mellin transform of f

$$\widehat{f}(s) = \int_{\mathbb{R}_{>0}} f(y) y^{-s} \frac{dy}{y}.$$

(It is an entire function of Paley-Wiener type) Thus,

$$\begin{aligned}\theta_f(z) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathrm{Re}\, s = s_0} \widehat{f}(s) (\mathrm{Im}\, \gamma z)^s \\ &= \int_{\mathrm{Re}\, s = s_0} \widehat{f}(s) E(z; s - \frac{1}{2})\end{aligned}$$

provided that $s_0 > 1$. We can compute

$$I = \int_{\Gamma \backslash \mathcal{H}} \theta_f(z) \, dz$$

in two different ways. On the one hand we can shift the contour to $\mathrm{Re}\, s = \frac{1}{2}$ acquiring a residue at $s = 1$ to get

$$I = \mathrm{vol}(\Gamma \backslash H) \frac{6}{\pi} \widehat{f}(1) + \int_{\mathrm{Re}(s) = 0} \widehat{f}(s + \frac{1}{2}) I(s) \, dz$$

On the other hand we can compute I directly using the definition of θ_f. Unfolding the integral and the sum we get

$$I = \int_{\Gamma_\infty \backslash \mathcal{H}} f(\mathrm{Im}\, z)\ dz = \int_{\mathbb{R}_{>0}} f(y)\ \frac{dy}{y^2} = \widehat{f}(1)$$

Comparing the two formulae (as distributions in $\widehat{f}$) we infer that

$$\mathrm{vol}(\Gamma\backslash\mathcal{H}) = 2\,\mathrm{vol}(\overline{\Gamma}\backslash\mathcal{H}) = \frac{\pi}{6}$$

and $I(s) \equiv 0$. We used the following Lemma

Lemma 2. *Suppose that $I(t)$ is bounded and*

$$\int \widehat{f}(\mathrm{it})\mathrm{I}(\mathrm{t})\ \mathrm{dt} = \mathrm{a}\hat{\mathrm{f}}(1)$$

for all $f \in \mathbb{C}_c^\infty(\mathbb{R}_{>0})$. Then $I \equiv a = 0$.

Proof. By taking $f_1 = yf' - f$ we have $\widehat{f}_1 = (s-1)\widehat{f}$ and therefore $\int \widehat{f}(\mathrm{it})\mathrm{I}_1(\mathrm{t})\ \mathrm{dt} = 0$ for $I_1 = (\mathrm{it}-1)\mathrm{I}$. Since this is true for all f, $I_1 \equiv 0$. Therefore $I \equiv 0$. $\square$

Remark: Using this method Langlands computed $\mathrm{vol}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ for any semisimple Chevalley group. For non-split groups this was completed by Kottwitz using the trace formula, leading to the solution of a conjecture of Weil's.

Prime Number Theorem (with remainder) (Sarnak, Shalika 60th birthday volume)

Truncated Eisenstein series: for z in the Siegel domain set

$$\Lambda^T E^*(z;s) = \begin{cases} E^*(z;s) & y \leq T, \\ E^*(z;s) - a_0(y,s) & y > T. \end{cases}$$

It is rapidly decreasing at the cusp. Maass-Selberg relations:

$$\|\Lambda^T E^*(z;\mathrm{it})\|_2^2 = \\ 2\log T - \frac{\phi'}{\phi}(\mathrm{it}) + \frac{\overline{\phi}(\mathrm{it})\mathsf{T}^{2\mathrm{it}} - \phi(\mathrm{it})\mathsf{T}^{-2\mathrm{it}}}{2\mathrm{it}}.$$

where $\phi(s) = \frac{\zeta^*(2s)}{\zeta^*(2s+1)}$. Note that $|\phi(\mathrm{it})| = 1$ and

$$\frac{\phi'}{\phi}(\mathrm{it}) = \mathrm{Re}\,\frac{\zeta^{*\prime}}{\zeta^*}(1+2\mathrm{it})$$
$$= \frac{\zeta'}{\zeta}(1+2\mathrm{it}) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}+\mathrm{it}) + \frac{1}{2}\log\pi.$$

Thus, for T fixed, and $t \geq 2$

$$\|\zeta(1+2\mathrm{it})\Lambda^{\mathsf{T}}\mathsf{E}^*(\mathrm{z};\mathrm{it})\|_2^2 \leq$$
$$|\zeta(1+2\mathrm{it})|\left(|\zeta(1+2\mathrm{it})| + \left|\zeta'(1+2\mathrm{it})\right| + \frac{\Gamma'}{\Gamma}(\frac{1}{2}+\mathrm{it}) + 3\right)$$

By standard *upper* bounds for $\zeta(1+it)$ and $\zeta'(1+it)$ this is majorized by

$$|\zeta(1+2\mathrm{it})|\,(\log t)^2$$

OTOH

$$\|\zeta(1+2\mathrm{it})\Lambda^{\mathsf{T}}\mathsf{E}^*(\mathrm{z};\mathrm{it})\|_2^2 \geq$$
$$\int_1^\infty \int_0^1 \left|\zeta(1+2\mathrm{it})\Lambda^{\mathsf{T}}\mathsf{E}(\mathrm{x}+\mathrm{iy};\mathrm{it})\right|^2 \;\frac{dx\;dy}{y^2}.$$

By Bessel's inequality

$$\geq \sum_{m=1}^{\infty} \int_1^{\infty} \left| \frac{K_{\mathrm{it}}(2\pi \left|m\right| y)\sigma_{-2\mathrm{it}}(m)}{\Gamma(\frac{1}{2} + \mathrm{it})} \right|^2 \ \frac{dy}{y}$$

Taking only $m = 1$ and comparing the two inequalities we get

$$\int_1^{\infty} \left| \frac{K_{\mathrm{it}}(2\pi y)}{\Gamma(\frac{1}{2} + \mathrm{it})} \right|^2 \ \frac{dy}{y} \ll \left|\zeta(1 + 2\mathrm{it})\right| (\log t)^2$$

Using the precise asymptotic for the Bessel function in the regime $t/8 < y < t/4$, LHS $\gg \frac{1}{t}$ and therefore

$$\left|\zeta(1 + 2\mathrm{it})\right| \gg \frac{1}{t(\log t)^2}$$

In fact, we would have more precisely

$$\frac{1}{t} \sum_{m \leq t/8} \left|\sigma_{-2\mathrm{it}}(m)\right|^2 \ll \left|\zeta(1 + 2\mathrm{it})\right| (\log t)^2.$$

The fact that for p prime

$$\left|\sigma_{-2\mathrm{it}}(p) - \sigma_{-2\mathrm{it}}(p^2)\right| = 1$$

guarantees that

$$|\sigma_{-2\mathrm{it}}(p)|^2 + \left|\sigma_{-2\mathrm{it}}(p^2)\right|^2 \geq \frac{1}{2}$$

so that at least

$$\sum_{m \leq t/8} |\sigma_{-2\mathrm{it}}(m)|^2 \geq \frac{1}{2} \sum_{p \leq \sqrt{t/8} : p \text{ prime}} \gg \sqrt{t}/\log t$$

by Chebyshev. This gives

$$|\zeta(1+2\mathrm{it})| \gg \frac{1}{\sqrt{t}(\log t)^3}$$

By refining the argument one can get

$$|\zeta(1+2\mathrm{it})| \gg \frac{1}{(\log t)^3}$$

which gives a zero-free region which is almost as good as the standard one (à la de la Vallée Poussin).

Gauss class number problem

Gauss conjectured that $h(D) \to \infty$ as $D \to -\infty$ and gave a table for the D's with small class number.

It was known to Hecke and Landau in the 1920's that under GRH, $h(D) \gg \sqrt{D}/\log D$.

Deuring ('33) If RH is false, then $h(D) = 1$ for only finitely many $D < 0$.

$$\sum_{z \in \Lambda_D} E(z; s) = \zeta_{Q(\sqrt{D})}(s+\frac{1}{2}) = \zeta(s+\frac{1}{2})L(s+\frac{1}{2}, \chi_D)$$

Suppose that $\zeta(s_0 + \frac{1}{2}) = 0$ with $\text{Re}(s_0) > 0$. Then LHS is zero at s_0 for all D. However, if $h(D) = 1$ then LHS is just $E(\frac{\delta+\sqrt{D}}{2}; s_0)$ with $\delta = 0, 1$, $\delta \equiv D \pmod 4$. OTOH,

$$\frac{E(\frac{\delta+\sqrt{D}}{2}; s_0)}{\sqrt{|D|}} = |D|^{s_0} + \phi(s_0)\,|D|^{-s_0} + O(|D|^{-N})$$

for all $N > 0$. Clearly the first term on the RHS is dominant since $\text{Re}(s_0) > 0$, and therefore LHS cannot vanish.

Remark: Duering's idea was quickly generalized by Heilbronn and Siegel to show Gauss' conjecture under $\neg GRH$, (and therefore solving it, albeit non-effectively). The best effective lower bound is roughly $\log D$ (Goldfeld, Gross-Zagier). Interestingly enough it relies on a high order zero for an L-function (which is "not very far" from Deuring's point of departure).

Spectral decomposition.

Let

$$L^2(\Gamma\backslash\mathcal{H}) = L^2_{disc}(\Gamma\backslash\mathcal{H}) \oplus L^2_{cont}(\Gamma\backslash\mathcal{H})$$

be the spectral decomposition of Δ into a discrete and continuous part respectively. Let $L^2_{cusp}(\Gamma\backslash\mathcal{H})$ be the space of cusp forms, i.e. those f such that

$$\int_0^1 f(x+\mathrm{iy})\ \mathrm{dx} = 0 \text{ for almost all y.}$$

A-priori, it is not clear that $L^2_{cusp}(\Gamma\backslash\mathcal{H}) \neq 0$! At any rate, it is a fact that Δ decomposes discretely on $L^2_{cusp}(\Gamma\backslash\mathcal{H})$.

Theorem 1. $L^2_{disc}(\Gamma\backslash\mathcal{H}) = L^2_{cusp}(\Gamma\backslash\mathcal{H}) \oplus \mathbb{C}\cdot 1$

The map $L^2(\mathbb{R}_{\geq 0}) \to L^2(\Gamma\backslash\mathcal{H})$ *given by*

$$f \mapsto Ef = \int f(\mathrm{it})\mathrm{E}(\mathrm{z};\mathrm{it})\ \mathrm{dt}$$

is an isometry onto $L^2_{cont}(\Gamma\backslash\mathcal{H})$ *and*

$$\Delta(Ef) = E((\frac{1}{4} - t^2)f)$$

Alternatively, any $f \in L^2(\Gamma\backslash\mathcal{H})$ *has a decomposition*

$$f(z) = \sum_j (f, u_j)u_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty}(f, E(\cdot;\mathrm{it}))\mathrm{E}(\mathrm{z};\mathrm{it})\ \mathrm{dt}$$

in terms of eigenfunctions of Δ*. The first sum is taken over an orthonormal basis of the discrete part. Equivalently,*

$$\|f\|_2^2 = \sum_j \left|(f, u_j)\right|^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty}|(f, E(\cdot;\mathrm{it}))|^2\ dt$$

Connection with the holomorphic Eisenstein series pass to group setup: consider

$$E(\varphi, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\gamma g) y(\gamma g \mathrm{i})^{\mathrm{s}+\frac{1}{2}}$$

where $\varphi : B\backslash G \to \mathbb{C}$ where $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$. Previously we used $\varphi \equiv 1$ which gives rise to function $E(g\mathrm{i};\mathrm{s})$.

Now we get an intertwining map from $I(s) = \mathrm{Ind}_B^G(\left(\begin{smallmatrix} t_1 & * \\ 0 & t_2 \end{smallmatrix}\right) \mapsto \left|\frac{t_1}{t_2}\right|^s$ to the space of automorphic forms on $\Gamma\backslash G$.

For example, taking

$$\varphi_k(\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)) = e^{\mathrm{i}\theta\mathrm{k}}$$

and $s = k - \frac{1}{2}$. Then for $z = g\mathrm{i}$

$$\begin{aligned} G_{2k}(z) = \zeta(2k)\left(\frac{c\mathrm{i}+\mathrm{d}}{|c\mathrm{i}+\mathrm{d}|}\right)^k E(g, \varphi_k, k - \frac{1}{2}) \\ = \sum_{(m,n)\neq 0} (mz+n)^{-2k} \end{aligned}$$

is the holomorphic Eisenstein series. It has Fourier expansion

$$2\zeta(2k)(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n) \quad q=e^{2\pi \mathrm{iz}}$$

Note that $I(k-\frac{1}{2})$ is reducible:

$$0 \to F_{2k-1} \to I(k-\frac{1}{2}) \to D_{2k-1} \to 0$$

where F_l is the l-dimensional irreducible representation of $SL_2(\mathbb{R})$ and D_l is the discrete series representation. φ_k is the lowest K-type in D_{2k-1}.

Kronecker limit formula

$$E(z;s)=\frac{c_0}{s-\frac{1}{2}}+c_1\log(y^6\,|\Delta(z)|)+c_2+O(s-\frac{1}{2})$$

for certain constants c_0, c_1, c_2 where

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^24 \quad q=e^{2\pi iz}$$

Spectral theory for GL_2 - adelic version.

Let R be the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$, i.e. $R(g)\varphi(x) = \varphi(xg)$ for $\varphi \in L^2(G(F)\backslash G(\mathbb{A})$. For any $f \in C_c^\infty(G(\mathbb{A}))$ let $R(f)$ be the operator $\int_{G(\mathbb{A})} f(g)R(g)\, dg$, that is

$$R(f)\varphi(x) = \int_{G(\mathbb{A})} f(g)\varphi(xg)\ dg.$$

Then

$$R(f)\varphi(x) = \int_{G(\mathbb{A})} f(x^{-1}g)\varphi(g)\ dg = \int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\varphi(\gamma y)\ dg$$
$$= \int_{G(F)\backslash G(\mathbb{A})} K_f(x,y)\varphi(y)\ dy$$

i.e., $R(f)$ is an integral operator on $L^2(G(F)\backslash G(\mathbb{A}))$ with kernel

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$$

The spectral theory for PGL_2 gives

$$K_f(x,y) = K_f^{cusp}(x,y) + K_f^{res}(x,y) + K_f^{cont}(x,y)$$

where

$$K_f^{cusp} = \sum_{\{\varphi\}} R(f)\varphi(x)\overline{\varphi(y)}$$

the sum is taken over an orthonormal basis of cusp forms;

$$K_f^{res}(x,y) = \sum_{\substack{\chi: F^*\backslash \mathbb{I}_F \to \mathbb{C}^* \\ \chi^2=1}} \mathrm{vol}(G(F)\backslash G(\mathbb{A}))^{-1}$$

$$\int_{G(\mathbb{A})} f(g)\chi(\det g)\ dg \cdot \chi(\det xy^{-1})$$

and

$$K_f^{cont}(x,y) = \sum_{\chi: F^*\backslash \mathbb{I}_F^1 \to \mathbb{C}^*}$$

$$\sum_{\{\varphi\}} \int_{-\infty}^{\infty} E(x, I(f,\chi,\mathrm{it})\varphi, \mathrm{it})\overline{\mathrm{E(y,\varphi,it)}}\ \mathrm{dt}$$

where $\{\varphi\}$ is an orthonormal basis of the space

$$I(\chi) = \{\varphi : G(\mathbb{A}) \to \mathbb{C} | \varphi(\left(\begin{smallmatrix} t & * \\ 0 & 1 \end{smallmatrix}\right) g) = \chi(t)\,|t|^{\frac{1}{2}}\,\varphi(g)\}$$

with

$$(\varphi_1, \varphi_2) = \int_{\mathbb{R}_{>0}T(F)U(\mathbb{A})\backslash G(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\ dg$$

and for

$$\varphi_s(\left(\begin{smallmatrix} t & * \\ 0 & 1 \end{smallmatrix}\right)k) = |t|^s \varphi(k),$$

$$I(g,\chi,s)\varphi(\cdot) = (\varphi_s(\cdot g))_{-s}$$

Mirabolic Eisenstein series for GL_n. Let V be an n-dimensional space over $\mathbb{Q}$ and let $\tilde{V}$ be the dual space. For $\Phi \in \mathcal{S}(V(\mathbb{A}))$ set

$$E^V_\Phi(g,s) = |\det g|^{\frac{s}{n}+\frac{1}{2}} \int_0^\infty \sum_{v\in V(\mathbb{Q})\backslash\{0\}} \Phi_g(tv)\, |t|^{s+n/2} \frac{dt}{t}$$

where $\Phi_g(\cdot) = \Phi(\cdot g)$, $g \in GL(V(\mathbb{A}))$ acting on the right on $V(\mathbb{A})$. This is the Mellin transform of $\Theta^*_{\Phi_g} = \Theta_{\Phi_g} - \Phi(0)$ where

$$\Theta_\Phi(t) = \sum_{v\in V(\mathbb{Q})} \Phi(tv) \quad t \in \mathbb{R}_{>0}.$$

By Poisson summation formula

$$\Theta_\Phi(t) = t^{-n}\Theta_{\hat{\Phi}}(t^{-1})$$

where $\hat{\Phi} \in \mathcal{S}(\tilde{V}(\mathbb{A}))$ is given by

$$\hat{\Phi}(\tilde{v}) = \int_{V(\mathbb{A})} \Phi(v)\psi((\tilde{v}, v))\ dv \quad \tilde{v} \in \tilde{V}(\mathbb{A})$$

where ψ is a fixed non-trivial character of $\mathbb{Q}\backslash\mathbb{A}$. Also,

$$\widehat{\Phi_g} = |\det g|^{-1}\, \hat{\Phi}_{g^*}$$

where $(\tilde{v}g^*, v) = (\tilde{v}, vg^{-1})$. By Tate's thesis,

$$\begin{aligned} &E_\Phi^V(g,s) = \\ &\quad |\det g|^{\frac{s}{n}+\frac{1}{2}}\,(\int_1^\infty \Theta^*_{\Phi_g}(t) t^{s+n/2}\frac{dt}{t} - \frac{\Phi(0)}{s+n/2}) + \\ &\quad |\det g^*|^{\frac{1}{2}-\frac{s}{n}}\,(\int_1^\infty \Theta^*_{\hat{\Phi}_{g^*}}(t) t^{n/2-s}\frac{dt}{t} + \frac{\hat{\Phi}(0)}{s-n/2}) \\ &\qquad\qquad = E_{\hat{\Phi}}^{\tilde{V}}(g^*, -s). \end{aligned}$$

Note: For any field extension K of degree n, K^* is a torus in GL_n. We have

$$\int_{K^*\backslash\mathbb{I}_K^1} E_\Phi^V(k,s)\chi(k)\ dk = (*)L(s,\chi)$$

for any Hecke character χ of $\mathbb{I}_K$.

More generally, starting with a cusp form ϕ on $GL_n(F)\backslash GL_n(\mathbb{A})$ we can construct following Jacquet-Shalika, for each $\Phi \in \mathcal{S}(M_{n\times(n+1)}(\mathbb{A}))$

$$E(g;\Phi,\phi,s) = |\det g|^{ns} \int_{GL_n(F)\backslash GL_n(\mathbb{A})} \sum_{\substack{\eta\in M_{n\times(n+1)}(F)\\ \mathrm{rk}\,\eta=n}} \Phi(x^{-1}\eta g)\phi(x)\,|\det x|^{-(n+1)s}\;dx$$

As in Godement-Jacquet, this can be written as

$$\begin{aligned} E(g;\Phi,\phi,s) = &\\ &|\det g|^{ns} \int_{x\in GL_n(F)\backslash GL_n(\mathbb{A}):|\det x|\geq 1} \theta[{}_{x^{-1}}\Phi_g]\phi(x)\,|\det x|^{-(n+1)s}\;dx + \\ &|\det g^*|^{n(1-s)} \int_{x\in GL_n(F)\backslash GL_n(\mathbb{A}):|\det x|\geq 1} \theta[{}_{x^{-1}}\widehat{\Phi}_{g^*}]\phi^*(x)\,|\det x|^{-(n+1)(1-s)}\;dx \\ &= E(g^*;\widehat{\Phi},\phi^*,1-s) \end{aligned}$$

where $g \in GL_{n+1}(\mathbb{A})$, $\phi^*(x^*) = \phi(x)$, ${}_x\Phi_g(y) =$

$\Phi(xyg)$

$$\widehat{\Phi}(x) = \int_{M_{n\times(n+1)}(\mathbb{A})} \Phi(y)\psi(\mathrm{tr}(y\cdot {}^t x))\ dy$$

and

$$\theta[\Phi] = \sum_{\substack{\xi\in M_{n\times(n+1)}(F)\\ \mathrm{rk}\,\xi=n}} \Phi(\xi).$$