Recall from previous talk:

Γ: rank $2n$ lattice, $\varphi: \Gamma \to \Gamma$

endo, distinct eigenvalues
none real.

choose $\lambda_1, \ldots, \lambda_n$ cx eigenvalues
so that $\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n$

= all eigenvalues.

$\Gamma^{1,0}$ = eigenspace assoc. to

$\lambda_1, \ldots, \lambda_n$, and

$$T = \Gamma_{\mathbb{C}} / \Gamma^{1,0} + \Gamma$$

so $\varphi_T : T \to T$.

Propn: If $n \geq 2$ and $\mathcal{O}$bis

$\mathrm{Gal}(\mathbb{Q}(\lambda_1 \ldots \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n) : \mathbb{Q})$

acts as σ_{2n} on the

$2n$ eigenvalues of φ,

then T is not

projective.

(1

III Examples of compact Kähler manifolds which do not admit a complex projective structure.

1. The torus example:

T, $\varphi_T \circlearrowright T$ as before. Inside $T \times T$ consider the 4 sub-tori

$T_1 = T \times 0$, $T_2 = 0 \times T$ ($\subset$?)

$T_3 = \text{diag}(T) = \{(x, x), x \in T\}$

$T_4 = \text{graph}(\varphi_T) = \{(x, \varphi_T(x)), x \in T\}$

Because φ. has not the eigenvalue 1 or 0, these sub-tori meet pairwise transversally in finitely many points $x_1, \ldots, x_N$.

a) Blow-up $x_1, \ldots, x_N$

$\rightsquigarrow$ Proper transforms $\tilde{T}_i$ are smooth, do not meet

b) Blow-up the $\tilde{T}_i$

The resulting X is compact Kähler.

<u>Thm</u>: Y compact Kähler.

Assume $\exists\, \gamma : H^*(Y,\mathbb{Z}) \cong H^*(X,\mathbb{Z})$

iso of graded rings.

Then Y is not projective.

Thus X does not have the cohomology ring of a projective complex mfd.

Want to show:
for Y, γ as above,
the Hodge structure
on $H^1(Y, \mathbb{Z})$ cannot be
polarized.

On X, let $E_i :=$ excep.
divisor over $\tilde{T}_i$,

$$e_i := [E_i] \in H^2(X, \mathbb{Z})$$

<u>KEY LEMMA</u>: For Y, γ as
above

$$a_i := \gamma^{-1}(e_i) \quad \text{is}$$

a Hodge class on Y.

Assuming this : will (5

show that Hodge structure on $H^1(Y, \mathbb{Z})$ splits as $L \oplus L$, and there exists $\psi \circlearrowright L$ (endo of Hodge structure), ψ conjugate to $\pm\varphi$ acting on Γ^*.

(cf. p. 0bis)

Applying Propn to L, conclude that L cannot be polarized, and neither $H^1(Y, \mathbb{Z})$. $\boxed{}$

illustration of (ii)

To get this splitting of $H^1(Y,\mathbb{Z})$: use

$\cup a_i : H^1(Y,\mathbb{Z}) \to H^3(Y,\mathbb{Z})$

As a_i = Hodge class,

$\cup a_i$ = morphism of Hodge structures $\Rightarrow$

$L_i := \operatorname{Ker} \cup a_i$ is a sub-Hodge structure of $H^1(Y,\mathbb{Z})$.

Note $L_i = \gamma^{-1}(\operatorname{Ker} \cup e_i)$

$\Rightarrow$

Computing on X:

Find that: (7

$$H^1(Y_1, \mathbb{Z}) \underset{H.S}{\simeq} L_1 \oplus L_2$$

$L_3 \subset L_1 \oplus L_2$ is iso to L_1, L_2 by the two projections $\Rightarrow$

$L_1 \simeq L_2 \simeq: L$
as H.S.

Finally $L_4 \subset L_1 \oplus L_2 \simeq L \oplus L$

$L_4 \simeq L$ by first projection

$\Rightarrow L_4$ is graph(ψ) for some $\psi : L \to L$ morph. of Hodge structures

ψ conjugate to $t\varphi$, via γ. ☒

Remains to prove the Lemma:

<u>A geometric proof</u>: later on, I'll give a proof due to Deligne, ok for $\mathbb{Q}$-coeff.

1) Observe that $\mathrm{alb}_Y : Y \to \mathrm{Alb}\, Y$ is birational: Indeed, this is equivalent to

$$\Lambda^{4n} H^1(Y, \mathbb{Z}) \simeq H^{4n}(Y, \mathbb{Z})$$

$2n = \dim Y$. This is a property of the cohomology ring.

So true for $X \Rightarrow$ true for Y.

- Next verify that $\gamma^{-1}(e_i) = a_i$ belongs to $\operatorname{Ker}\left((\mathrm{alb}_Y)_* : H^2(Y, \mathbb{Z}) \to H^2(\mathrm{Alb}\, Y, \mathbb{Z}) \right)$

(In fact this kernel can be expressed using the cohomology ring only, so true for $X \Rightarrow$ true for Y).

- As alb_Y is birational, this kernel is of type $(1,1)$.

$\Rightarrow$ Lemma is proved $\square$

Further examples:

1) Simply connected:

T, φ_T as above.

Let K be Kummer variety of T = desingularization of $T/\pm 1$ obtained by blowing-up the 2-torsion points = fixed points of $-Id$.

φ_T induces

$\varphi_K : K \dashrightarrow K$ rational

Consider

Δ = diag $\subset K \times K$

Γ = Graph $(\varphi_K) \subset K \times K$

- Blow-up Δ, then proper transform of Γ.

Result is a smooth compact Kähler X. X is simply connected. Still:

Assume $n \geq 3$.

Thm: If Y is s.t.

$\exists \, \gamma : H^*(Y, \mathbb{Q}) \simeq H^*(X, \mathbb{Q})$

grad. algebras

Then Y is not projective.

Now, we consider the Hodge structure on $H^2(Y,\mathbb{Q})$, want to show it cannot be polarized by a $\omega \in H^2(Y,\mathbb{Q})$.

<u>KEY POINT</u>: Exhibit sub-Hodge structures of $H^2(Y,\mathbb{Q})$.

Recall

$$\begin{tikzcd}
X \arrow[r, "\tau"] & T/\pm 1 \times T/\pm 1 \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
T/\pm 1 & & T/\pm 1
\end{tikzcd}$$

$\rightsquigarrow$ get

$$A_i^2 := \tau^*\left(pr_i^*\left(H^2(T/\pm 1), \mathbb{Q}\right)\right)$$
$$\subset H^2(X, \mathbb{Q}).$$

One proves:

LEMMA: Y, γ as above.
Then $\gamma^{-1}(A_i^2)$ are sub. Hodge structures of $H^2(Y, \mathbb{Q})$.

• Next, show as in previous case the existence of interesting Hodge classes on Y: 13

Namely $a_i := \gamma^{-1}(e_i \cdot)$ are Hodge, where $e_1 = [E_\Delta]$, $e_2 = [E_P] \in H^2(X, \mathbb{Z})$.

•• Finally, use them to show that

$\gamma^{-1}(A_1^2) \overset{HS}{\simeq} \gamma^{-1}(A_2^2) =: L$

+ $\exists$ endo of Hodge structures

$L \xrightarrow{\varphi} L$, conjugate to $\Lambda^{2t}\varphi$.

Irreducibility of $\Lambda^{2t}\varphi$ $\Rightarrow$ either

L is trivial, or

L has no $\neq 0$ Hodge class.

Exclude L trivial by 2d Hodge-Riemann bilinear relations.

$\Rightarrow$ L has no Hodge class.

$\Rightarrow$ All Hodge classes of $H^2(Y, \mathbb{Q})$ lie in a certain complementary sub-Hodge structure of $L \oplus L$ $\Rightarrow$ Cannot polarize

Key point: exhibit (15

sub-Hodge structures

(if lucky, sub-Hodge structure => Hodge classes)

only from the cohomology ring

This is done using:

Lemma (Deligne) [Providing alternative proof of the fact that the $\gamma^{-1}(e_i)$ are Hodge on 1st example].

$A = \oplus A^*$ a f.d. $\mathbb{Q}$-algebra

Each A^k endowed with

rational Hodge structure

compatible with product:

$A^k \otimes A^l \to A^{k+l}$ is

a morphism of Hodge structures

$Z \subset A^k_{\mathbb{C}}$ alg. subset

defined by homogeneous equations expressed using only the product on A:

Examples: $Z = \{\alpha \in A^k_{\mathbb{C}} \mid \alpha^2 = 0\}$

$Z = \{\alpha \in A^k_{\mathbb{C}},\ \mathrm{rk}\ \alpha : A^l \to A^{k+l} \leq r\}$

....

Lemma: Z' irred. component of Z

Assume $\langle Z' \rangle \subset A^k_{\mathbb{C}}$ defined over $\mathbb{Q}$:

i.e.

($\langle Z' \rangle$ = cx v.-space generated by Z'.

$\langle Z' \rangle$ defined over $\mathbb{Q}$ means: $\langle Z' \rangle = B \otimes \mathbb{C}$, for some $B \subset A^k_{\mathbb{Q}}$.)

Then B is a sub-Hodge structure of $A^k_{\mathbb{Q}}$.

Pf: Hodge decomp on $A^k_{\mathbb{C}}$ provides $\mathbb{C}^*$-action on $A^k_{\mathbb{C}}$: z acts by $z^p \bar{z}^q$ on $H^{p,q}$.

(18)

B = sub Hodge structure of $A^h_{\mathbb{Q}}$

$\Leftrightarrow$ $B_{\mathbb{C}}$ = stable under Hodge decomp of $A^h_{\mathbb{C}}$ $\Leftrightarrow$

$B_{\mathbb{C}}$ = stable under $\mathbb{C}^*$ action.

But the product is compatible with Hodge decomp $\Leftrightarrow$ The product is equivariant / $\mathbb{C}^*$-action.

$\Rightarrow$ Z stable under $\mathbb{C}^*$-action

$\Rightarrow$ Z' ————————

$\Rightarrow$ $B_{\mathbb{C}} = \langle Z' \rangle$ ————————

$\blacksquare$

The birational example:

T, φ_T as before

(assume $n \geq 4$)

Introduce dual torus

$$\hat{T} = \mathrm{Pic}^0 T.$$

Poincaré line bundle

$\mathcal{L}$ on $T \times \hat{T}$

Also $\mathcal{L}_\varphi = (\varphi_T, \mathrm{Id})^* \mathcal{L}$

characterized by

1) $\mathcal{L}_{|T \times 0}$ trivial, $\mathcal{L}_{|0 \times \hat{T}}$ trivial.

2) $c_1(\mathcal{L}) \in H^1(T, \mathbb{Z}) \otimes H^1(\hat{T}, \mathbb{Z})$

is $Id \in \Gamma^* \otimes \Gamma$ (??

$$\Gamma^* = H^1(T, \mathbb{Z})$$
$$\Gamma = H^1(\hat{T}, \mathbb{Z})$$

Consider

$$E = \mathcal{L} \oplus \mathcal{L}^{-1}$$

$$E_\varphi = \mathcal{L}_\varphi \oplus \mathcal{L}_\varphi^{-1} \quad \text{on } T \times \hat{T}$$

The involutions
$$i = (-1, Id)$$
$$\hat{i} = (Id, -1)$$

A lift to E, E_φ

using
$$i^* \mathcal{L} = \mathcal{L}^{-1}$$
$$\hat{i}^* \mathcal{L} = \mathcal{L}^{-1}$$
$$i^* \mathcal{L}_\varphi = \mathcal{L}_\varphi^{-1}$$
$$\hat{i}^* \mathcal{L}_\varphi = \mathcal{L}_\varphi^{-1}$$

$\Rightarrow$ $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action

on $\mathbb{P}(\mathcal{E}) \times_{T \times \hat{T}} \mathbb{P}(\mathcal{E}_\varphi)$ lifting

$\langle i, \hat{i} \rangle$

The quotient is a singular

$\mathbb{P}^1 \times \mathbb{P}^1$ bundle over

$T/\pm 1 \times \hat{T}/\pm 1$

Choose a compact Kähler desingularization X of this quotient.

Thm: For any smooth (22

bimeromorphic model

X' of X, any Y

s.t. $\exists\ \gamma : H^*(Y,\mathbb{Q}) \cong H^*(X',\mathbb{Q})$

iso graded algebras

Then Y is not projective.

Key point: X contains

very few subvarieties

dominating $K \times \hat{K}$ $\Rightarrow$

few bimeromorphic transf.

In fact:

X admits

$\exists\ f : X \longrightarrow T/_{\pm 1} \times \hat{T}/_{\pm 1}$

holomorphic.

$\forall X'$: <u>smooth birat.</u>

Can show: $\exists\ \emptyset \neq U \subset T/_{\pm 1} \times \hat{T}/_{\pm 1}$

Zar. open, such that

$X' \dashrightarrow X$ is well

defined over $f^{-1}(U)$.

$\leadsto$ Keep control on

cohomology of X'.

<u>Final remark</u>:

The example in the birational case has Kodaira dimension $-\infty$. Thus the following would remain true:

Q. (Tsunoda 85, Campana)
X = Kähler cpct with $\kappa(X) \geq 0$. Does X admit a smooth birational model which deform to a proj. mfd?