HODGE THEORY AND THE TOPOLOGY OF KÄHLER AND PROJECTIVE MANIFOLDS

Claire Voisin

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I. HODGE THEORY
BASIC NOTIONS

II. PROJECTIVE / KÄHLER MANIFOLDS:
THE KODAIRA PROBLEM

III CONSTRUCTING EXAMPLES OF KÄHLER MANIFOLDS WHICH DO NOT HAVE THE TOPOLOGY OF PROJECTIVE ONES

Kähler manifolds

X = complex manifold

$\Rightarrow$ T_X has an almost complex structure.

$\Rightarrow$ Can speak of Hermitian metric on X.

In local holomorphic coordinates $z_1, \ldots, z_n$

$$h = \sum_{i,j} h_{ij} \, dz_i \otimes d\bar{z}_j$$

where $h_{ij} = \overline{h_{ji}}$

Consider the <u>real</u> $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{i,j} h_{ij} \, dz_i \wedge d\bar{z}_j$$

defn. h ~~[illegible]~~ is a Kähler metric if ω is closed

A very useful Lemma:

If h is Kähler, $\forall x \in X$ $\exists$ $z_1, , z_n$ holomorphic coord centered at x, such that

$$h_{ij} = \delta_{ij} + O(|z|)^2$$

[and conversely!]

This is used to prove

The Kähler identities = relations between operators naturally defined on Kähler manifolds.

Operators:

∂, $\bar\partial$, $d = \partial + \bar\partial$

1st order, depending only on the cx structure.

Hodge operator $*$:

$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, Vol$

0-th order, depends on the Riemannian metric $g = \operatorname{Re} h$

Lefschetz operator

$L = \omega \wedge$, 0-th order.

The Kähler identities are commutation relations between these operators.

As they involve the metric

only to first order, they can be checked for the flat Kähler metric

$$\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$$

Consequences : (of the Kähler identities)

1) Consider the Laplacian

$$\Delta_d = d^* d + d d^*$$

where $d^* = \pm * d *$ is the formal adjoint of d w.r.t. L^2 - metric on forms.

Then $\Delta_d : A^{p,q}(X) \to A^{p,q}(X)$ that is, Δ_d preserves the complex type of forms.

2) Δ_d commutes with L (5

Corollaries: (X,ω) Kähler

1) If $\alpha \in A^k(X)$ is harmonic, ie $\Delta_d \alpha = 0$ its (p,q)-components $\alpha^{p,q}$ are harmonic: $\Delta_d \alpha^{p,q} = 0$

2) If $\alpha \in A^k(X)$ is harmonic, then $L\alpha = \omega \wedge \alpha$ is harmonic.

Hodge theory says (6

That X compact diff.

g = metric on X

$\Rightarrow$ any cohomology class

$\alpha \in H^k(X, \mathbb{C})$ admits

a unique harmonic representa[tive]

$\tilde{\alpha}$, st $\Delta_d \tilde{\alpha} = 0$

(or equivalently: $\begin{cases} d\alpha = 0 \\ d(*\alpha) = 0 \end{cases}$

Corollary: (The Hodge decomposition) X = cpct Kähler

then $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, where

$$H^{p,q}(X) = \{ \alpha \in H^{k}(X, \mathbb{C}) \; / \; \alpha \text{ representable by a closed form of type } (p,q) \}$$

Note: by the definition above

Hodge symmetry

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$

Another corollary is the <u>hard Lefschetz theorem</u>: $L = [\omega] \cup$ on H^*

$$L^{n-k} : H^{k}(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism, for $k \leq n = \dim X$.

(a) $\wedge\, \omega^{n-k} : \mathcal{H}^k \longrightarrow \mathcal{H}^{2n-k}$

(Indeed, suffices to check injectivity of (*) on harmonic forms, hence on forms: this is a pointwise computation.) (8

$\rightsquigarrow$ 8 bis

From the hard Lefschetz theorem, one deduces easily the Lefschetz decomposition.

Introduce the primitive cohomology:

$$H^k(X, \mathbb{R})_{prim} := \mathrm{Ker}\; L^{n-k+1},$$

for $k \leq n$.

what is algebraic and what is not? 8bis

Assume X = complex projecti[ve]

$H^k(X, \mathbb{C})$ is algebraic

because $H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet)$

+ GAGA.

But $H^k(X, \mathbb{R})$ <u>is not</u>.

Thus Hodge decomp with Hodge symmetry is not an algebraic statement.

<u>weak version</u>: deg. at E_1 of Frölicher spectral sequence $\rightsquigarrow$

$\exists$ filtration $F^p H^k(X, \mathbb{C})$

with graded piece $Gr_p^F = H^{p,q} = H^q(X, \Omega_X^p)$

proved by Deligne. Illusie. 8 te

NB: Degeneracy is true for all compact cx surfaces, ~~[illegible]~~ ~~topology~~ but not Hodge decomp.

Hard Lefschetz theorem is an algebraic statement.

No alg. proof is known.

To do proof

weak version: Lefschetz hyperplane restriction thm

Cease $[\omega] = [H]$ $H \subset X$,

Then for $k \leq n$, the map (9

$$\bigoplus_{k-2r\geq 0} H^{k-2r}(X,\mathbb{R})_{prim} \xrightarrow{\Sigma L^r} H^k(X,\mathbb{R})$$

is an isomorphism.

<u>Note</u>: Each $L^r H^{k-2r}(X,\mathbb{R})_{prim}$ is stable under Hodge decomp

Polarization:

On $H^k(X,\mathbb{R})$ ~~prim~~,

consider the intersection form

$$q_L(\alpha,\beta) = \int_X L^{n-k}\alpha \cup \beta$$

non degenerate by hard Lefschetz

symmetric, k even

skew-symmetric, k odd.

Easy: The Lefschetz decomposition is $\perp$ wrt q_L.

Next, introduce the ~~sesquilinear~~ hermitian pairing on $H^k(X,\mathbb{C})$

$$h_L(\alpha,\beta) = i^k q_L(\alpha,\bar{\beta})$$

Easy: • The Hodge decomposition is $\perp$ wrt h_L.

("The first Hodge-Riemann bilinear relations")

2d Hodge-Riemann bilinear relations:

hard: •• The restriction of h_L to $H^{p,q}(X)_{prim}$, $p+q=k$ is definite, of sign $\varepsilon_k(-1)^p$

This is obtained by (1)

comparing h_L with

the L^2-metric on harmonic

forms representing cohomology

classes.

Hodge structures

This is the structure

one gets on the cohomology

mod. torsion of a

cpct Kähler mfd, by

the Hodge decomposition

thm:

L a lattice: a weight k Hodge structure on L = a decomposition

$$L_{\mathbb{C}} := L \otimes \mathbb{C} = \bigoplus_{p+q=k} L^{p,q},$$

$$\overline{L^{p,q}} = L^{q,p}$$

<u>NB</u>. • Effective if only $p \geq 0$, $q \geq 0$.

•• Above is the definition of an integral Hodge structure. Rational or even real Hodge structures can be considered.

Exples:

- X compact Kähler

$$L = H^k(X, \mathbb{Z}) / \text{torsion}$$

Then $L_{\mathbb{C}} = H^k(X, \mathbb{C})$ + Hodge decomposition = integral Hodge structure of weight k.

Dfn: Sub-Hodge structure: $L' \subset L$, $L'_{\mathbb{C}}$ stable under Hodge decomposition

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Exple: Primitive cohomology

$$H^k(X, \mathbb{R})_{prim} \subset H^k(X, \mathbb{R})$$

is a real sub-Hodge structure

(A rather weak notion) (14

When the class $[\omega]$ of the Kähler form is rational, $[\omega] \in H^2(X, \mathbb{Q})$, then

$$H^k(X, \mathbb{Q})_{prim} \subset H^k(X, \mathbb{Q})$$

is defined as $\operatorname{Ker} L^{n-k+1}$,

$$L : H^*(X, \mathbb{Q}) \to H^{*+2}(X, \mathbb{Q})$$

and it is then a rational sub-Hodge structure.

Polarization:

L = Hodge structure of weight k

(integral : L = lattice
rational : L = $\mathbb{Q}$-v. space
real : L = $\mathbb{R}$-v. space)

<u>defn</u>: A polarization on L is q_L = intersection form on L, skew for k odd, symmetric for k even

s.t. $h_L(\alpha, \beta) = i^k q_L(\alpha, \bar{\beta})$ on $L_{\mathbb{C}}$ satisfies

① Hodge decomp is $\perp$ wrt h_L

② $h_L \mid L^{p,q}$ is definite of sign $\varepsilon_k (-1)^p$.

Thus, the primitive cohomology

$$H^k(X, \mathbb{R})_{prim} \subset H^k(X, \mathbb{R})$$

(rel. to ω = Kähler form on X)
endowed with the rest. of q_L
is a real polarized Hodge structure of weight k.

If $[\omega] \in H^2(X, \mathbb{Q})$
$H^k(X, \mathbb{Q})_{prim}$ is
a rational polarized Hodge structure of weight k
as q_L is rational in this case.

We will be mostly interested in $k = 1$, $k = 2$.

$\underline{\underline{k=1}}$ Integral, effective, weight 1 Hodge structure =

lattice L of rank $2n$

$+ \; L^{1,0} \subset L_{\mathbb{C}}$, such that

$L^{1,0}$ = rk n cx v. space

$$L_{\mathbb{C}} = L^{1,0} \oplus \overline{L^{1,0}}$$

$\updownarrow$ eq. of categories

Complex tori $T = \mathbb{C}^n / L$, L = lattice

Indeed $L_{\mathbb{C}} = L^{1,0} \oplus \overline{L^{1,0}} \Rightarrow$

L projects to a lattice in $L_{\mathbb{C}} / L^{1,0}$

Thus $(L, L_{\mathbb{C}} = L^{1,0} \oplus \overline{L^{1,0}})$

$$\updownarrow$$

$$T = L_{\mathbb{C}} / (L^{1,0} \oplus L)$$

Next $\{$ integral, weight 1, effective polarized Hodge structures$\}$

$$\updownarrow$$

$\{$ complex tori with integral Kähler cohomology class $\}$

(= polarized abelian varieties)

Indeed: L, q_L, $L^{1,0}$,

with $q_L : \Lambda^2 L \to \mathbb{Z}$

Then $q_L \in \Lambda^2 L^* = H^2(T, \mathbb{Z})$,

$T = L_{\mathbb{C}} / L^{1,0} \oplus L$. Denote

α_L the coh. class on T so

obtained

Fact: • 1st Hodge-Riemann bilinear relns for $q_L \Leftrightarrow \alpha_L$ is of type $(1,1)$ on T. 19

•• 2d Hodge-Riemann bilinear relns for $q_L \Leftrightarrow$ α_L is Kähler on T.

Rmk: To understand de Rham representative of α_L on T:

Note $T \underset{\text{diffeo}}{\simeq} L_{\mathbb{R}}/L$

$q_L \in (\Lambda^2 L)^*$: extend $\mathbb{R}$-linearl[y]

$\to$ get 2-form on $L_{\mathbb{R}}$ = closed 2-form on T.

<u>Geometrically</u> :

X = cpct Kähler

Hodge structure on $H^1(X,\mathbb{Z})$

~~$H^2(X,\mathbb{Z})$~~

$\downarrow$

complex torus $\dfrac{H^1(X,\mathbb{C})}{H^{1,0}(X)\oplus H^1(X,\mathbb{Z})}$

This is $\mathrm{Pic}^0 X$ by

exp. exact sequence.

<u>Next</u> : $H^1(X,\mathbb{R}) = H^1(X,\mathbb{R})_{prim}$

If X admits an integral

Kähler class ($\Leftarrow$ X is projective)

This torus admits an

integral Kähler class,

that is $\mathrm{Pic}^0 X$ is projective

~~NB: I used $H^2(\mathbb{R})_{prim} \subset H^2(\mathbb{R})$~~.

(27)

Weight $k=2$

geometric case: Now

$$H^2(X)_{prim} = \left([\omega]^{n-1}\right)^{\perp}$$

Lefschetz decomp: $H^2(X) = \langle[\omega]\rangle \oplus H^2(X)_{prim}$

$$\cap \quad H^{2n-2}(X, \mathbb{R})$$

The 1st Hodge-Riemann relations

say
$$\int_X \alpha \cup \beta \cup [\omega]^{n-2} = 0$$
$\alpha \in H^{2,0}$, $\beta \in H^{2,0} \oplus H^{1,1}$

The 2d Hodge-Riemann relations

Hodge index

say
$$\int_X \alpha^2 \cup [\omega]^{n-2} > 0$$
for $0 \neq \alpha$ real in $H^{2,0} \oplus H^{0,2} \oplus [\omega]\mathbb{R}$

and
$$\int_X \alpha^2 [\omega]^{n-2} < 0, \quad 0 \neq \alpha \in H^{1,1}(X, \mathbb{R})_{pr}$$

A polarized Hodge structure of weight 2 is determined by L, q_L

+ $L^{2,0} \subset L_{\mathbb{C}}$,

which is subject to:

period domain

- $L^{2,0}$ = tot. isot. for q_L
- and $q_L\big|_{(L^{2,0} \oplus L^{0,2})_{\mathbb{R}}} > 0$

Indeed: $L^{0,2} = \overline{L^{2,0}}$ and

$(L^{2,0} \oplus L^{0,2})^{\perp} = L^{1,1}$