Tight Closure & Positivity

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X variety char $p>0$

Frobenius map $X \xrightarrow{F} X$ $\quad \mathcal{O}_X \longrightarrow F_* \mathcal{O}_X$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \longmapsto s^p$

Iterate:
$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rrrr, bend right, "F^e"'] & X \arrow[r, "F"] & X \arrow[r] & \cdots \arrow[r, "F"] & X \end{tikzcd} \qquad s \longmapsto s^{p^e}$$

Notation: $\mathcal{L}$ line bundle $\quad F^{e*}\mathcal{L} = \mathcal{L}^{p^e}$

η section $\quad F^{e*}(\eta) = \eta^{p^e}$

$\eta \in H^i(X, \mathcal{L}) \quad F^{e*}(\eta) \in H^i(X, F^{e*}\mathcal{L})$

$\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{\eta^{p^e}} \in \overset{\shortparallel}{H^i(X, \mathcal{L}^{p^e})}$

DEFINITION: A cohomology class $\eta \in H^i(X, \mathcal{L})$ is **PHANTOM** if there exists a non-zero section c of some $\mathcal{O}_X(D)$ such that

$$c\eta^{p^e} \in H^i(X, \mathcal{L}^{p^e}(D)) \text{ is \underline{zero} } \forall e \gg 0.$$

Example: $\mathcal{L}$ ample $\Rightarrow$ Every $\eta \in H^i(X, \mathcal{L})$ $i>0$ is Phantom.

Proposition - Definition: (s)

A <u>globally</u> <u>F-regular</u> variety is an irreducible projective variety of char $p>0$ satisfying the following equivalent properties:

1. All phantom classes vanish;
2. The (non-zero) $\eta \in H^d(X, \omega_X)$ is <u>NOT</u> phant
3. All ideals are tightly closed in some (equiv. every) section ring of X.

<u>Recall</u>: The section ring of X w.r.t. $\mathcal{L}$ ample is $S(X,\mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$.

<u>Rmk</u>: • Lots of cohomology vanishing on globally F-reg variet
• testable condition

Proposition-Definition: (s)

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1. All phantom classes vanish;
2. The (non-zero) $\eta \in H^d(X, \omega_X)$ is <u>NOT</u> phantom
3. All ideals are tightly closed in some (equiv. every) section ring of X.
4. There exists an ample $D = \{s=0\} \subseteq X$ whose compliment is smooth and s.t. the natural map

$$\begin{aligned} \mathcal{O}_X &\to F_*^e \mathcal{O}_X(D) \quad \text{split for } e \gg 0. \\ 1 &\longmapsto s \end{aligned}$$

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Rmk: • Lots of cohomology vanishing on globally F-reg variety

• testable condition

Examples of Globally F-regular varieties

① (S-) Smooth <u>Fano</u> varieties

($-K_X$ ample)

② Schubert varieties

(Lauritzen, Raben-Pedersen, Thomsen)

<u>Application</u>: Complete characterization of the simple objects in the category of equivariant, holonomic D_X-modules on a flag variety X. They are precisely $\mathcal{H}^c_Y(\mathcal{O}_X)$ where $Y \subseteq X$ is a Schubert variety of codim c. (uses work of Blickle)

(CHAR $p > 0$)

③ Projective Toric varieties

④ Other varieties with group action, including "large Schubert varieties"

(Brion-Thomsen)

Properties of Globally F-regular varieties

- normal, CM, mildly singular (eg rationally sing
- "Positive" in sense that $-K_X$ is Big
- Vanishing of cohomology

$$\{-K_X \text{ ample}\} \Longrightarrow \{\text{Globally F-reg}\} \underset{[\text{Brenner},-]}{\Longrightarrow} \{-K_X \text{ Big}$$

OPEN QUESTION:

Is globally F-regular equivalent to $-K_X$ Big (or $-K_X$ big & NEF)?

With Brenner, we have found nice, checkable criteria for globally F-regular varieties.

Sketch of Proof that Globally F-reg $\Rightarrow$ $-K_X$ Big:

1. Need to show $-K_X$ is in interior of effective cone.

   Fix D effective.

   Suffices to find $q = p^e$ s.t. $-qK_X + D$ is effective.

2. Choose M very very ample so $M - K_X + D$ is effective.

   Then $\exists$ a map $\mathcal{O}_X \to \mathcal{O}(M - K_X + D)$

   and hence a map $\omega_X \otimes \mathcal{O}(-M) \xrightarrow{f} \mathcal{O}(D)$.

3. For any section c of $\mathcal{O}(M)$, we know $c\eta^q \in H^d(X, \omega_X^q(M))$ is non-zero, where $\eta \in H^d(X, \omega_X) \setminus \{0\}$.

   By Serre duality, there is a corresponding n.z. map

   $$\omega_X^q(M) \to \omega_X$$

   and hence a n.z. map

   $$\omega_X^q \to \omega_X \otimes \mathcal{O}(-M).$$

4. Composing with f, we get a n.z. map

   $$\omega_X^q \to \omega_X \otimes \mathcal{O}(-M) \to \mathcal{O}(D)$$ as needed.

Tight Closure (Hochster Huneke)

$I \subseteq R$ domain char $p>0$

$I^* =$ the tight closure of I

Def:

$$z \in I^* \iff \exists\, c \neq 0 \text{ s.t. } cz^{p^e} \in I^{[p^e]} \quad \forall e \gg 0$$

$$I^{[p^e]} = \{x^{p^e} \mid x \in I\} = F^{e}$$

Note: $cz^{p^e} \in I^{[p^e]}$

$c^{1/p^e} z \in I$ in R^{1/p^e}

"Take limit as $e \to \infty$": $c^{1/p^e} \to 1$

$\therefore$ $z \in I^* \iff$ "z is in I up to Frobenius, in a limiting sense."

Cf. Phantom classes in "tight closure of 0."

DEFINITION: A ring is **F-regular** if all ideals are tightly closed.

II. Local Progress

"Singularities of $X/\mathbb{C}$ are reflected in tight clos[ure]
properties of X mod p for $p \gg 0$."

Theorem (—, Hara, Watanabe)

X normal$/\mathbb{C}$, K_X $\mathbb{Q}$-Cartier. THEN

- X has log terminal sings $\iff$ X mod p is F-regu[lar] for $p \gg 0$.
- X has rat. sings $\iff$ all parameter ideals are t.c. for X mod p $p \gg 0$.
- X has log canonical sings $\Longleftarrow$ X is locally Frobenius split mod $p \gg 0$.
  ($\overset{?}{\Longrightarrow}$ CONJ.)

II. Local Progress

"Singularities of $X/\mathbb{C}$ are reflected in tight closure properties of $X \bmod p$ for $p \gg 0$."

Theorem (—, Hara, Watanabe)

X normal$/\mathbb{C}$, K_X $\mathbb{Q}$-Cartier. THEN

- X has log terminal sings $\Leftrightarrow$ $X \bmod p$ is F-regular for $p \gg 0$.
- X has rat. sings $\Leftrightarrow$ all parameter ideals are t.c. for $X \bmod p$ $p \gg 0$.
- X has log canonical sings $\Leftarrow$ X is locally Frobenius split mod $p \gg 0$.
  ($\overset{?}{\Rightarrow}$ CONJ.)

In fact, the connection between log terminal & F-regular varieties is deeper. There are natural scheme structures on the NON-log terminal and NON-F-reg loci, and these agree!

TEST IDEAL (char $p>0$)

DEF: The test ideal $\tau \subseteq \mathcal{O}_X$ is the ideal $\{c \in \mathcal{O}_X \mid cI^* \subseteq I \text{ for all } I \subseteq \mathcal{O}_X\}$.

Note: $\tau = \mathcal{O}_X \iff$ all ideals are tightly closed

In fact, τ defines the non-F-regular locus of X

MULTIPLIER IDEAL: (char 0)

X normal, K_X $\mathbb{Q}$-Cartier

DEF. The multiplier ideal $\mathcal{J}(X, \quad)$ is $\pi_* \mathcal{O}_Y(\lceil K_{Y/X} \qquad \rceil)$ where $\pi: Y \to X$ is a resolution of singularities of X

" $K_{Y/X} = K_Y - \pi^* K_X$ "

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$\pi_* \mathcal{O}_Y(\lceil K_{Y/X}$

MULTIPLIER IDEAL: (char 0)

X normal, K_X $\mathbb{Q}$-Cartier $\mathfrak{a} \subseteq \mathcal{O}_X$ any ideal, $t \in \mathbb{Q}$

DEF. The multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ is $\pi_* \mathcal{O}_Y(\lceil K_{Y/X} - tA \rceil)$ where $\pi : Y \to X$ is a log resolution of singularities of $(X, \mathfrak{a})$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-A)$.

Theorem: (–, Hara)

X normal, $\mathbb{Q}$-Gorenstein $/\mathbb{C}$

Then $\mathcal{J}(X)$ reduces mod $p \gg 0$ to $\tau(X)$.

This implies the result linking F-regularity to log terminal singularities.

"Scheme structures on non log terminal and non F-regular loci are the same."

Tight Closure for Pairs

Hara, Watanabe, Takagi, Yoshida

Fix a pair $(R, \mathfrak{a})$ $\quad \mathfrak{a} \subseteq R$ char p domai[n]
$t \in \mathbb{Q}_{>0}$

DEF:
$$z \in I^{*\mathfrak{a}^t} \iff \exists\, c \neq 0 \text{ s.t. } c z^{p^e} \mathfrak{a}^{\lceil t p^e \rceil} \subseteq I^{[p^e]} \quad \forall e \gg 0.$$

- Case $\mathfrak{a} = R$ is the "classical tight closure."
- $I \subseteq I^{*\mathfrak{a}^t}$
- Many similar properties hold

TEST IDEAL FOR A PAIR $(R, \mathfrak{a})$

DEF:
$$\tau(\mathfrak{a}^t) = \left\{ c \;\middle|\; c I^{*\mathfrak{a}^t} \subseteq I \quad \forall I \right\}$$

"All same results regarding sings and tight closure are valid also for pairs."

Eg: Theorem (Hara - Yoshida)

X normal K_X $\mathbb{Q}$-Cartier over $\mathbb{C}$

Then $\mathcal{J}(X, \mathfrak{a}^t)$ reduces mod $p \gg 0$ to $\tau(\mathfrak{a}^t)$.

Moreover, independent proofs of basic properties like restriction theorem for multiplier ideals and subadditivity are given for test ideals.

Theorem (Subaddivity): X smooth.

- $\mathcal{J}(\mathfrak{a}^t \mathfrak{b}^s) \subseteq \mathcal{J}(\mathfrak{a}^t)\, \mathcal{J}(\mathfrak{b}^s)$ char 0 Demailly, Ein, La

- $\tau(\mathfrak{a}^t \mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t)\, \tau(\mathfrak{b}^s)$ char p Hara, Yoshida Takagi

Subadditivity on Singular Varieties

Theorem (Takagi): R f.g. domain $/k$ perfe

THEN

- $\mathrm{Jac}(R/k) \cdot \tau(a^t b^s) \subseteq \tau(a^t) \cdot \tau(b^s)$ char $p>0$
- $\mathrm{Jac}(R/k) \cdot \mathcal{J}(a^t b^s) \subseteq \mathcal{J}(a^t) \cdot \mathcal{J}(b^s)$ char 0

Application to symbolic powers:

Theorem (Takagi) IF $P \subseteq R$ is a prime ideal ht c in a f.g. domain R, then

$$\mathrm{Jac}(R/k) \cdot P^{(cN)} \subseteq P^N \quad \forall N.$$

Generalizes the corresponding result of Ein, Laz, – for R smooth of char 0.

Inversion of Adjunction

Theorem (Takagi):

X smooth$/\mathbb{C}$ $Y \subseteq X$ closed subscheme

If $(Z, Y|_Z)$ is k.l.t. (l.c.)

for some normal $\mathbb{Q}$-Gorenstein subvariety $Z \subseteq X$,

then $(X, Y+Z)$ is p.l.t. (l.c.) near Z.

- uses t.c. theory of pairs, gets corresponding char p analogs.
- generalizes results of Kollár & Shokurov for Z complete intersection,

Ambro, Ein-Mustata-Yasuda

III.

"Geometric meaning fairly well understood for properties that all (or certain) ideals are tightly closed, and for annihilators of tight closure."

BUT:

What is the meaning of tight closure itself? Is there a geometric interpretation for $f \in (f_1, \dots, f_n)^*$?

One answer:

X projective char p $\mathcal{L}$ ample

Consider a class $\eta \in H^i(X, \mathcal{L}^N)$ for some N

Then

η is phantom $\underset{\text{Hara}}{\Longleftrightarrow}$ $N \geq 0$

SUGGESTS: Cohomology classes are phantom iff they come from "positive" bundles.

Also: If we fix a system of parameters $x_0, \dots x_d$ for the section ring $S(X, \mathcal{L})$, then a NZ class η in $H^i(X, \mathcal{L}^N)$ is represented by some element $z \in S(X, \mathcal{L})$ NOT in $(x_0, \dots, x_i)$.

The class η is phantom $\Longleftrightarrow$ $z \in (x_0, \dots, x_i)^*$

Holger Brenner:

Fix R normal standard graded domain char p, homogeneous coordinate ring of X smooth, dim d.

Fix $\{f_1, \dots, f_n\}$ generating m-primary ideal, homog.
When is $f \in (f_1, \dots, f_n)^*$?

First: f determines a class $\delta(f) \in H^d(X, S)$ where
$S = Syz_d(f_1, \dots, f_n)(m)$ where $m = \deg f$.

Indeed: there are s.e.s.

$$0 \to Syz_1 \to \bigoplus_{i=1}^{n} \mathcal{O}_X(-\deg f_i) \twoheadrightarrow \mathcal{O}_X \to 0$$

$$0 \to Syz_2 \to \left(\text{locally free module}\right) \to Syz_1 \to 0$$

$$\vdots$$

AND

$$R_m \to H^0(X, \mathcal{O}(m)) \xrightarrow{\delta_1} H^1(X, Syz_1(m)) \xrightarrow{\delta_2} H^2(X, Syz_2(m)) \to \cdots$$

so finally: $R_m \xrightarrow{\delta} H^d(Syz_d(m))$

Now:
$f \in (f_1, \dots, f_n)^* \iff$ the class $\delta(f) \in H^d(Syz_d(m))$ is phantom.

Curve Case: X smooth dim 1, S locally free $\mathcal{O}_X$-mod

Brenner: "The phantom classes of $H^1(X,S)$ are precisely those coming from the Positive part of S."

Recall: $\mu(S) = \deg S / \operatorname{rk} S$

S has a unique Harder-Narasimhan filtration

$$S = S_t \supseteq S_{t-1} \supseteq \cdots\ S_1 \supseteq S_0 = 0$$ with semi-stable quotients

$$\mu_{min} = \mu\left(S/S_{t-1}\right) < \mu\left(S_{t-1}/S_{t-2}\right) < \cdots < \mu\left(S_2/S_1\right) < \mu(S_1) = \mu_{max}$$

Theorem (Brenner) X smooth curve, S v.b.

If $\mu_{min} \geq 0$, all of $H^1(X,S)$ is phantom

If $\mu_{max} < 0$, then $H^1(X,S)$ has NO non-zero phantom classes.

In general, let i be such that

$$\mu\left(S_i/S_{i-1}\right) \geq 0 \quad \text{BUT} \quad \mu\left(S_{i+1}/S_i\right) < 0, \text{ THEN}$$

the phantom classes in $H^1(X,S)$ are precisely those in the image of $H^1(X,S_i) \longrightarrow H^1(X,S)$.

CAUTION: I lied!

Semi-stability is NOT preserved under Frobenius pullba

Langer: For $e \gg 0$, $F^{e*}(S)$ has a strong HN filtra

so $F^{e*}(S) = S_t \supseteq \cdots S_i \supseteq \cdots \supseteq S_1 \supseteq 0$ each S_i/S_{i-1} s.s.
and this remains a HN filtration after further pullback

Theorem (Brenner):

Let X be a smooth curve of char $p > 0$, S a v.b. on X.
Fix e such that $F^{e*}(S)$ has a strong HN filtration
$F^{e*}(S) \supseteq S_{t-1} \supseteq S_{t-2} \supseteq \cdots \supseteq S_1 \supseteq 0.$
Let i be such that $\mu(S_i/S_{i-1}) \geq 0$ But $\mu(S_{i+1}/S_i) < 0$.
Then $\eta \in H^1(X, S)$ is phantom $\Longleftrightarrow$
η^{p^e} is in the image of $H^1(X, S_i) \longrightarrow H^1(X, F^{e*}(S))$.

Some ideas in the proof:

1. Tight closure = solid closure (Hochster)

$$f \in (f_1, \cdots f_n)^* \iff H^1_{m_R}\left(\frac{R[x_1, \cdots, x_n]}{f - x_1 f_1 - \cdots - x_n f_n}\right) \neq 0$$

2. Let $S = Syz_1(f_1, \cdots, f_n)$. Given $f \in R_m$, the class $\delta(f) \in H^1(Syz_1(m)) = Ext^1(\mathcal{O}_X, Syz_1(m))$ determines an extension $Syz_1(m) = S \hookrightarrow S'$.

The cohomological condition in 1 turns out to say that

$\mathbb{P}(S') - \mathbb{P}(S)$ has cohomological dim > 0,

ie, is NOT affine.

3. Idea to show a scheme is affine is to show it is the compliment of an ample divisor (positive) in a projective scheme.

(use results of Hartshorne, Gieseker)