Flips and finite generated algebras

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1. Flips and flops: typical flip, Kawamata's example, D-flip, divisorial D-contraction, singularities, log flip, flop, log flip conjecture, results (characteristic 0).

2. Functional algebras: graded algebra, Proj of a finite generated graded algebra, FGA conjecture.

   Examples: 1-dimensional, 2-dimensional local and global cases, toric situation.

3. Reductions (characteristic 0): divisorial, flipping, and restricted algebras, pl flip, reduction to FGA conjecture, 3-fold log flips.

   Instead of epilogue (Hacon, McKernan).

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1 Flips and flops

Typical flip is a commutative diagram:

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X^+, \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & Z & \end{array}$$

where

X, X^+ are normal irreducible algebraic varieties,

$\varphi\colon X \dashrightarrow X^+$ is a birational map, and

$f\colon X \to Z, f^+\colon X^+ \to Z$ are small birational contractions, that is, f, f^+ are birational proper, the exceptional loci

$$X \supset E = \operatorname{Exc} f = \bigcup_{X \supset C,\text{ a curve with } f(C)=\text{pt.}} C, \ E^+ = \operatorname{Exc} f^+ \subset X^+$$

have codimension ≥ 2, and $f_*\mathcal{O}_X = \mathcal{O}_Z = f^+_*\mathcal{O}_{X^+}$ (thus any Frobenius morphism is not a contraction, and not birational),
such that:

X, X^+ are $\mathbb{Q}$-Gorenstein, or equivalently, their canonical (Weil) divisors K_X, K_{X^+} are $\mathbb{Q}$-Cartier, that is, mK_X, mK_{X^+} are Cartier divisors for some natural number $m > 0$,

K_X is numerically negative, or equivalently, $-K_X$ is (numerically) ample over Z, but

K_{X^+} is ample over Z.

We say also that φ or the variety X^+, more precisely, the contraction f^+ is a *flip* of the variety X, more precisely, of the contraction f.

Optional assumptions:

f is *extremal* if (the relative Picard number) $\rho(X/Z) = 1$;

X is a $\mathbb{Q}$-factorial variety.

Example (Kawamata): If X is nonsingular, and a flip φ is nontrivial, then $\dim X \geq 4$. Moreover, in dimension 4 the only possible flip is as follows: every its connected component with respect to E is

$$\begin{array}{ccccccc}
\widetilde{E} = \mathbb{P}^2 \times \mathbb{P}^1 & \subset & & \widetilde{X} & & & \\
\text{blowup of } E & & \swarrow & & \searrow & & \text{blowdown of } \widetilde{E} \\
E = \mathbb{P}^2 \subset & X & & \overset{\varphi}{\dashrightarrow} & & X^+ & \supset E^+ = \mathbb{P}^1. \\
& {\scriptstyle f} & \searrow & & \swarrow & {\scriptstyle f^+} & \\
& & & Z & & &
\end{array}$$

In general, even for nonsingular X, its flip X^+ can be singular.

D-**flip** is a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+, \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

where again

X, X^+ are normal irreducible algebraic varieties,

$\varphi\colon X \dashrightarrow X^+$ is a birational map, and

$f\colon X \to Z, f^+\colon X^+ \to Z$ are small birational contractions

but now such that:

a (Weil) divisor D (instead of K_X) and its birational transform $D^+ = \varphi(D)$ (instead of K_{X^+}) are $\mathbb{Q}$-Cartier, that is, mD, mD^+ are Cartier divisors for some natural number $m > 0$,

D is numerically negative, or equivalently, $-D$ is (numerically) ample over Z, and

D^+ is ample over Z.

More generally, we can assume only the following conditions:

X, X^+ are normal irreducible algebraic varieties,

φ is a birational map, and

f^+ is a small birational contraction (then f is a birational contraction) such that:

$D^+ = \varphi(D)$ is $\mathbb{Q}$-Cartier, and

ample over Z.

Uniqueness of flips: For any birational contraction f and a divisor D, a D-flip is unique, that is, a flip diagram is unique up to isomorphism (see Flipping algebra below). Moreover, D-flips is the same for mD, where $m > 0$ is a natural number, and for $D' \sim_{\mathbb{Q}}$ or even $\sim_{\mathbb{R}} D$ where the $\mathbb{Q}$-linear or $\mathbb{R}$-linear equivalences can be taken locally over Z, and in most known cases even replaced by the numerical equivalence over Z.

In particular, flips, log flips, and directed flops are unique.

D-contraction is a contraction (not necessarily birational) $f\colon X \to Z$ with ample $-D$ over Z.

(Algebraically) divisorial contraction with respect to a divisor D on X is a birational contraction $f\colon X \to Z$ such that $f(D)$ is $\mathbb{R}$-Cartier, or even $\mathbb{Q}$-Cartier for a $\mathbb{Q}$-divisor D. A D-flip

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+, \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

of such a contraction exists, and $f^+\colon X^+ \to Z$ is an isomorphism, that is, we can take $X^+ = Z$.

If it is also D-contraction, it is called *divisorial D-contraction*.

Directed or D-flop is a D-flip such that K_X, K_{X^+} are numerically or $\mathbb{Q}$-linear trivial locally over Z. The last condition means that mK_X, mK_{X^+} are linear equivalent locally over Z to 0 (principal divisors) for some natural number $m > 0$. Under this condition someone says that X and X^+ (and Z too) *have the same K(canonical)-level*.

Those birational modifications preserve most of cohomological invariants according to Batyrev, Kontsevich, Bridgeland, Bondal, Orlov, Kawamata, etc.

Example (Atiyah): The simplest flop is a 3-fold nonsingular symmetric flop:

$$\begin{array}{ccccccc}
\widetilde{E} = \mathbb{P}^1 \times \mathbb{P}^1 & \subset & & \widetilde{X} & & & \\
\text{blowup of } E & & \swarrow & & \searrow & & \text{blowdown of } \widetilde{E} \\
E = \mathbb{P}^1 \subset & X & & \overset{\varphi}{\dashrightarrow} & & X^+ & \supset E^+ = \mathbb{P}^1. \\
& & f \searrow & & \swarrow f^+ & & \\
& & & Z & & &
\end{array}$$

It is directed with respect to any divisor D with $D.E < 0$.

Log philosophy: If someone would like to establish a certain property/result for a divisor D, and the property/result does not expected for all divisors then he/she should present D in the log canonical form

$$D = K_X + B,$$

and hereafter should find sufficient conditions in terms of B and the ambient variety X, in other words, in terms of the log variety (X, B).

Examples: 1. Kodaira and Kawamata-Viehweg vanishing theorems.

2. (Fujita's conjecture): It is expected that $D = K_X + B$ is (base point) free (respectively very ample) if

X is nonsingular, and

$B = mH$ where H is ample and $m \geq \dim X + 1$ (respectively $m > \dim X + 1$).

Established for $\dim X \leq 4$.

Log flip is a D-flip of a (small) D-contraction $f\colon X \to Z$ for $D = K_X + B$ where the log variety (X, B) has only log terminal (lt) singularities and B is a boundary. The latter means that $B = \sum b_i D_i$ where D_i are distinct prime divisors and $b_i \in [0, 1]$.

If B is a $\mathbb{Q}$-boundary, that is, in addition each $b_i \in \mathbb{Q}$, we can replace $\mathbb{Q}$-divisor $K_X + B$ by an *integral* Weil divisor $m(K_X + B)$ for some natural number $m > 0$ (see Uniqueness of flips above).

Singularities: Recursive approach:

X is nonsingular, then X itself and the log variety $(X, 0)$ with $B = 0$ has only terminal (trm) singularities;

if X is nonsingular and $\operatorname{Supp} B = \sum D_i$ has only simple normal crossing then (X, B) has only lt singularities. Such a log variety is called *log nonsingular*.

In both cases, X is $\mathbb{Q}$-factorial.

Any divisorial $D = K_X + B$-contraction and such a log flip of X preserve lt singularities, and trm singularities for $B = 0$. Such a birational modification of any extremal contraction of $\mathbb{Q}$-factorial X (thus *strict* trm or lt X) preserves the $\mathbb{Q}$-factorial property of X.

According to the Log Minimal Model Program (LMMP), it is expected that any $\mathbb{Q}$-factorial variety X with only trm singularities, or more generally, any log variety (X, B) with $\mathbb{Q}$-factorial X and only lt singularities, can be obtain from nonsingular X, respectively from log nonsingular (X, B), by a sequence of divisorial K_X-contractions and flips, respectively $D = K_X + B$-contractions and log flips.

In terms of discrepancies (see Discrepancies below):

X has only trm singularities if each exceptional discrepancy $a_i > 0$.

(X, B) has only lt singularities if each exceptional discrepancy $a_i > -1$ for *some* log desingularization.

Log flip conjecture: Any birational $D = K_X + B$-contraction of a log variety (X, B) with only lt, or even log canonical (lc), singularities has a log flip.

For $B = 0$, this gives flips if X has only trm (lt or lc) singularities and K_X is numerically negative over Z.

Existence of directed flops: Any flopping contraction $f\colon X \to Z$, that is,

f is birational,

X has only lt singularities, and

K_X is $\mathbb{Q}$-linear trivial over Z,

and with respect to any divisor D, numerically negative over Z, has a D-flop.

Follows from Log flip conjecture with $B = \varepsilon D$ for some sufficiently small rational $\varepsilon > 0$ if D is effective. Otherwise replace D by its effective linearly equivalent divisor locally over Z.

Results in characteristic 0: Log flip conjecture established up to dimension 4: [Sh00, Corollary 1.8]. History and other previous results mainly in dimension 3 (Mori, Kawamata, Sh–, Kollár, Takagi, Kachi) see in [Sh00, History 1.12].

Directed flops as in Existence of directed flops exist up to dimension 4 [Sh00, Corollary 1.9]. History and other previous results mainly in dimension 3 (Vic. Kulikov, Reid, Kawamata, Mori, Kollár, Sh–) see in [Sh00, History 1.12].

See also [ACFMcT] [ISh], and Instead of epilogue below.

Corollary (MMP for 4-folds): Minimal Model Program (MMP) hold up to dimension 4. (Termination due to Kawamata, Matsuda and Matsuki [KMM, Theorem 5-1-15].)

Thus each irreducible 4-dimensional variety has either a minimal model X, that is, X is birationally equivalent to a normal variety with only trm singularities and nef K_X, or

X is birationally equivalent to a *Mori-Fano fiber space*: $X \to Z$ is a fibred (nonbirational) K-contraction such that

X is $\mathbb{Q}$-factorial, and

(the relative Picard number) $\rho(X/Z) = 1$.

Open problems for 4-folds: Log termination, that is, any sequence of nontrivial log flips is finite. Positive characteristic.

For 4-folds, the termination follows from the conjectural ascending chain condition (acc) for minimal log discrepancies of 4-folds [Sh03, Corollary 5], and,

if the log Kodaira dimension $\kappa(X, B) \geq 0$, from the acc for log canonical thresholds of 4-folds [B]. In its turn, the latter follows from the Alexeev-Borisov brothers conjecture in dimension 3 [McP].

Abundance conjecture [KMM, Conjecture 6-1-14]: if X is a minimal model then

(numerical Kodaira dimension) $\nu(X) = \kappa(X)$ (Kodaira dimension), or equivalently,

K_X is semiample.

Still unknown cases for 4-folds: $\nu(X) = 1, 2$, and 3.

2 Functional algebras

Graded algebra is a graded commutative ring

$$\mathcal{R} = \bigoplus_{i \geq 0} \mathcal{R}_i$$

which is considered as $A = \mathcal{R}_0$-algebra.

Usually in algebraic geometry, those algebras are coherent functional, that is, each A-module $\mathcal{R}_i$ has a finite type and consists of rational functions of some fixed variety (see Functional algebra below).

Proj**:** Let $\mathcal{R}$ be a graded algebra of finite type over $\mathcal{R}_0 = A$, and A be a commutative k-algebra of finite type. Then it determines a projective contraction:

$$f\colon X = \operatorname{Proj} \mathcal{R} \to Z = \operatorname{Spec} A.$$

Moreover after truncation, the algebra $\mathcal{R}$ can be embedded (noncanonically) into the *complete algebra of rational functions* of X: for some natural number $I > 0$,

$$\mathcal{R}^{[I]} = \bigoplus_{I|i} \mathcal{R}_i T^i \subseteq k(X)[T],$$

and, for each i divided by I, the embedding determines a very ample linear system: a very ample subsystem in a complet one

$$|M_i|, M_i = \max \text{ divisor of poles of } s \in \mathcal{R}_i,$$

where the divisor $M_i = (i/I)M$ is also very ample, and M is an ample divisor; $A = \Gamma(Z, \mathcal{O}_Z)$ is the *coordinate ring* of the affine variety Z, and $k(X)$ is the *homogeneous algebra of fractions* of $\mathcal{R}$, the algebra of rational functions of X. The algebra of rational functions is a field if and only if X is irreducible and generically reduced.

If the algebra $\mathcal{R}$ is integral and integrally closed then each above subsystem is complete and X is a normal irreducible variety; Z is an affine irreducible variety.

Functional algebra: Let $f\colon X \to Z$ be a birational contraction of a normal irreducible variety on an affine variety Z with the coordinate ring $A = \Gamma(Z, \mathcal{O}_Z)$. A *functional algebra* on X/Z is a graded A-subalgebra

$$\mathcal{R} \subset k(X)[T],$$

with $A = \mathcal{R}_0$ where the multiplication is defined as the usual multiplication of functions. In addition, each A-module $\mathcal{R}_i$ is assumed to be finitely generated (f.g.) over A (coherent). A functional algebra is *bounded* if there exists a Weil divisor D on X such that $\mathcal{R}_i \subset \Gamma(X, iD)$ holds for each i.

Canonical algebra (as functional one): Let X be a nonsingular complete irreducible variety of dimension n over a field k. The *canonical* algebra

$$\mathcal{R}_{X/\mathrm{pt.}} = \bigoplus_{i\geq 0} \Gamma(X, (\Omega_X^n)^{\otimes i}) \subset k(X)[T]$$

is its birational invariant. The inclusion is given by a choice of a rational differential n-form $T = \omega \neq 0$:

$$s \in \Gamma(X, (\Omega_X^n)^{\otimes i}) \mapsto s/\omega^{\otimes i} \in k(X).$$

If X is of general type, that is, $\kappa(X) = n$, and the algebra is f.g. over k then X has a canonical model X_{can} (Reid), and

$$X_{\mathrm{can}} = \operatorname{Proj} \mathcal{R}_{X/\mathrm{pt.}}\,.$$

Mobile and characteristic systems [Sh00, Proposition 4.15] of a functional algebra $\mathcal{R}$ on X/Z are respectively sequences of *b-free* and *rational b-semiample* b-divisors:

$$\mathbb{M}_\bullet = (\mathbb{M}_i)_{i\geq 1}, \mathbb{D}_\bullet = (\mathbb{D}_i = \mathbb{M}_i/i)_{i\geq 1}$$

where each $\mathbb{M}_i$ is a free (Cartier) divisor on a resolution $f_i\colon X_i \to X$ (birational contraction) such that

$$\mathbb{M}_i = \max \text{ divisor of poles of } s \in \mathcal{R}_i.$$

Note that, for all $i, j \geq 1$,

$$\mathbb{M}_{i+j} \geq \mathbb{M}_i + \mathbb{M}_j, \text{ and } \mathbb{D}_{i+j} \geq \frac{i}{i+j}\mathbb{D}_i + \frac{j}{i+j}\mathbb{D}_j,$$

and both sequences are unique as b-divisors: for any higher model

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dd] \\ X_i \arrow[dr, "f_i"'] & \\ & Z, \end{tikzcd}$$

the corresponding $\mathbb{M}_{i,Y} = g^*\mathbb{M}_i$, and $\mathbb{D}_{i,Y} = g^*\mathbb{D}_i$. Thus above inequalities are meaningful on common model for each pair i, j.

Limiting criterion [Sh00, Theorem 4.28]: A functional algebra $\mathcal{R}$ is f.g. if and only if the limit $\mathbb{D} = \lim_{i\to\infty} \mathbb{D}_i$ of its characteristic system *stabilizes*, that is, $\mathbb{D} = \mathbb{D}_i$ *for some* $i \gg 1$, or equivalently, there exist a natural number $I > 0$ such that

$$\mathbb{D}_{Ii} = \mathbb{D}_I = \mathbb{D} \Leftrightarrow \mathbb{M}_{Ii} = i\mathbb{M}_I = Ii\mathbb{D} \text{ for all } i \geq 1.$$

Under either assumption, the limit $\mathbb{D}$ is a (b-)*semiample* $\mathbb{Q}$-*divisor*; the algebra and its characteristic system are bounded.

Discrepancies: Let (X, B) be a log variety such that
$K_X + B$ is $\mathbb{R}$-Cartier (or just $\mathbb{Q}$-Cartier if B is a $\mathbb{Q}$-divisor), and
$g\colon Y \to X$ be a birational contraction (extraction) with normal Y.
Then

$$K_Y + g^{-1}B = g^*(K_X + B) + \sum a_i E_i$$

where the multiplicities a_i are called (exceptional) discrepancies. The *discrepancy* divisor of (X, B) on Y is

$$A = g^{-1}B + \sum a_i E_i.$$

Note that (X, B) has only Kawamata lt (Klt) singularities if and only if $\lceil A \rceil \geq 0$ for any Y: all $b_i < 1$ and all $a_i > -1$.

FGA conjecture [Sh00, 4.39]: Suppose that a log pair $(S/T, B_S)$ satisfies (Klt log Fano contraction):

(i) $S \to T$ is a contraction;

(ii) (S, B_S) has only Klt singularities; and

(iii) $-(K_S + B_S)$ is ample over T.

Then any functional algebra $\mathcal{R}$ on S/T is f.g. or its characteristic system $\mathbb{D}_\bullet$ stabilizes if
the algebra or the system is bounded and
log canonical asymptotic (lca) saturated over $(S/T, B_S)$. The latter means up to truncation that, for all natural numbers i, j and any sufficiently high birational model $Y \to T$ of $S \to T$ (depending on i, j),

$$\operatorname{Mob} \lceil j\mathbb{D}_i + A(S, B_S) \rceil \leq j\mathbb{D}_j$$

where $A(S, B_S)$ is the discrepancy divisor of (S, B_S) on Y.

In Reductions below, the birational case of FGA, that is, when $S \to T$ is birational, is the most important.

Examples: 1. 1-dimensional [Sh00, Example 4.41]: Almost immediate by the following fact in Diophantine geometry: for any real number $a > -1$, any real number d satisfying

$$\lceil jd + a \rceil \leq jd$$

for all natural numbers $j \gg 1$ is *rational*.

2. 2-dimensional [Sh00, Example 6.25 and Proposition 6.26]: The birational case uses the rationality of a minimal resolution of the surface S: each Cartier divisor that is nef over T is free locally over T (essentially due to M. Artin).

General 2-dimensional case is more difficult and is crucial for 4-fold flips.

3. Toric version of FGA conjecture, established in [A, Theorem 1], gives a new construction of toric log flips.

3 Reductions

Divisorial algebra: Let $f\colon X \to Z$ be a contraction of a normal variety X onto an affine variety Z, and D be a Weil divisor on X. Then the functional algebra

$$\mathcal{R}_{X/Z}\, D = \bigoplus_{i\geq 0} \Gamma(X, \mathcal{O}_X(iD))$$

is called *divisorial* associated with D. Since f is a contraction to an affine variety, $A = \Gamma(X, \mathcal{O}_X) = \Gamma(Z, \mathcal{O}_Z)$.

In general, if f is not a contraction (just proper) and Z is not affine (just a variety), one can consider divisorial or functional algebras as graded sheaf of $\mathcal{O}_Z$-algebras [Sh00, Definition 3.10] with possibly $A \neq \Gamma(Z, \mathcal{O}_Z)$.

Note: $\mathcal{R}_{X/Z}\, D$ is determined uniquely up to a small modification (in codimension ≥ 2) of X (cf. Flipping algebra below).

For a divisorial algebra $\mathcal{R}_{X/Z}\, D$ as for a functional one, its mobile and characteristic systems $\mathbb{M}_\bullet$ and $\mathbb{D}_\bullet$ are defined whereas

$$f_{i*}\mathbb{M}_i = \operatorname{Mob} iD, \text{ and } \mathbb{M}_i = f_i^{-1} \operatorname{Mob}(iD) + \sum e_i E_i$$

with exceptional on X divisors E_i.

Ample algebra: Let $f\colon X \to Z$ be a projective morphism onto an affine variety Z, and M be its hyperplane section, that is, a very ample divisor over Z. Then $\mathcal{R} = \mathcal{R}_{X/Z}\, M$ is f.g., and

$$\begin{array}{ccc} X & \cong & \operatorname{Proj}_{X/Z} \mathcal{R} \\ & & \downarrow \\ f \downarrow & & \operatorname{Spec} A \\ & \swarrow & \\ Z & & \end{array}$$

is a Stein factorization. In this case, we can take $X_i = X$ and $\mathbb{M}_i = M_i = iM$, and $\mathbb{D}_i = M_i/i = M$.

Canonical algebra (as divisorial one): A canonical algebra is (isomorphic to) a divisorial one:

$$\mathcal{R}_{X/\mathrm{pt.}} \cong \mathcal{R}_{X/\mathrm{pt.}}\, K_X$$

where $K_X = (\omega)$ is a canonical divisor, the divisor of zeros and poles of a rational differential n-form $\omega \neq 0, n = \dim X$.

If X is a minimal model of general type, that is, X has only terminal singularities and K_X is nef and big, then the canonical algebra is f.g., and there exist a natural number I such that IK_X is free. For $I|i$, we can take $X_i = X$ and $\mathbb{M}_i = iK_X$ and $\mathbb{D}_i = K_X$.

Thus for varieties of general type, the existence of a canonical model follows from the existence of a minimal one X:

$$X \to X_{\mathrm{can}} = \operatorname{Proj} \mathcal{R}_{X/\mathrm{pt.}}\, K_X$$

is a birational canonical contraction.

Flipping algebra [Sh00, Example 3.15]: Let $f\colon X \to Z$ be a birational contraction, and D be a Weil divisor on X. The *flipping* algebra of the pair $(X/Z, D)$ is the divisorial algebra

$$\mathcal{FR}_{X/Z}\, D = \mathcal{R}_{Z/Z}\, f(D).$$

For any $\mathbb{Q}$-divisor D, the D-flip exists if and only if $\mathcal{FR}_{X/Z}\, D$ is f.g. [Sh00, Corollary 3.32]:

$$\begin{array}{rl} X^+ = & \operatorname{Proj} \mathcal{FR}_{X/Z}\, D \\ & f^+ \downarrow \\ & \quad Z. \end{array}$$

If f is small then

$$\mathcal{FR}_{X/Z}\, D = \mathcal{R}_{X/Z}\, D$$

(see Note in Divisorial algebra). In addition, for the flipping algebra, $f_{i*}\mathbb{M}_i = \operatorname{Mob}(iD) = iD$, and $f_{i*}\mathbb{D}_i = D$ stabilizes but it is still far away from stabilization in Limiting criterion above.

Restricted algebra $\mathcal{R}\,|_S$ on a normal divisorial subvariety $S \subset X$ for a divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}\, D$ is the image

$$\mathcal{R}_{X/Z}\, D = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{O}_X(iD)) \to \mathcal{R}_{S/T}\, D|_S = \bigoplus_{i \geq 0} \Gamma(S, \mathcal{O}_S(iD|_S))$$

under the restriction of functions. A correct definition requiers a general position: $S \not\subseteq \operatorname{Supp} D$ that an isomorphic algebra

$$\mathcal{R}_{X/Z}\, D' \text{ for } D' \sim D$$

satisfies; D, D' are possibly noneffective.

In general, the induced morphism $S \to T = f(S) \subseteq Z$ is not a contraction, and the algebra is not divisorial but it is functional.

Reduction to pl flips [Sh00, Theorems 1.2 and 2.3]: The existence of log flips in dimension $\leq n$ follows from LMMP in dimension $\leq n-1$ and the existence of prelimiting (pl) flips in dimension n: a log flip of an *elementary birational pl* contraction of a log pair $(X, S+B)$ with $\dim X \leq n$:

$S \subset X$ is an irreducible and normal divisorial subvariety,

$\lfloor B \rfloor = 0$,

$(X, S+B)$ has only lt (thus purely lt) singularities,

$X \to Z$ is an extremal $K_X + S + B$- and S-contraction simultaneously.

For a pl flip, $(S/T, B_S)$, where a boundary B_S is defined by adjunction, and the restricted algebra algebra

$$\mathcal{R}_S = \mathcal{R}\,|_S \text{ for } \mathcal{R} = \mathcal{R}_{X/Z}(K_X + S + B)$$

satisfy the assumptions of FGA conjecture, in particular, $S \to T = f(S)$ is a contraction, birational whenever f is small.

Main Lemma [Sh00, 3.43]: Pl flips exists if and only if the algebra $\mathcal{R} = \mathcal{R}_{X/Z}(K_X + S + B)$ is f.g., or equivalently, its restriction $\mathcal{R}\,|_S$ is f.g.

Reduction to FGA: The birational case of FGA conjecture in dimension $\le n-1$ implies the existence of pl flips in dimension $\le n$.

Corollary: The existence of 3-fold log flips and directed flops.

Open problems: The birational case of FGA and CCS (see [Sh00, Conjecture 6.14]) for 3-folds.

Reduction: CCS implies FGA.

Instead of epilogue (Hacon,McKernan): LMMP with $\mathbb{R}$-boundaries in dimension $\le n-1$ implies the existence of log flips in dimension n [HMc, Therem 1.1]. Essentially, generalizes the above 3-fold log flips case.

Remarks:

1. It is enough the big log termination as in [B]. Thus the existence of log flips in dimension 5 follows from the Alexeev-Borisov brothers conjecture in dimension 3 [B] [McP] (see Open problems for 4-folds).

2. Instead of the complete class of algebras in FGA conjecture Hacon and McKernan consider a subclass which satisfies CCS for each their algebra individually. To verify the CCS developed the technic of Extending sections due to Sui and Kawamata (see references in [HMc]).

If the birational case of FGA conjecture holds, algebras in the conjecture up to truncation are the same as in the subclass. However it is still difficult, unknown, and not interesting for today.