Geometry and physics of localization sums

based on joint work with A. Iqbal, R. Kenyon, D. Maulik,
N. Nekrasov, R. Pandharipande, N. Reshetikhin, C. Vafa.

Equivariant localization (*Atiyah & Bott, Duistermaat & Heckman, Berline & Vergne, . . .*) and its generalization to virtual classes (*Graber & Pandharipande*) may be the single most powerful tool currently available in enumerative geometry (. . . , *Ellingsrud & Strømme, Kontsevich, . . .*).

$(\mathbb{C}^\times)^{\cdots}$

It reduces computations in T-equivariant cohomology to contributions of torus-fixed loci.

In the case when the torus-fixed points are isolated, the result is a finite sum.

Albeit finite, this sum may be complicated and extracting useful information from it may not be easy.

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Adopting a probabilistic viewpoint may help.

This will be illustrated by a single example, namely . . .

Localization in Donaldson-Thomas theory

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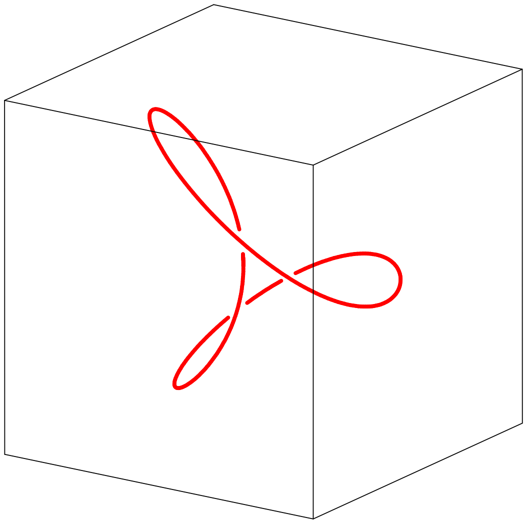
Localization sums arising in instanton counting (i.e. from moduli of framed sheaves on $\mathbb{P}^2$) form an important special case.

Donaldson-Thomas theory is an enumerative theory of curves in a nonsingular projective 3-fold X.

Its relationship to the Gromov-Witten theory of the same 3-fold X is the subject of conjectures proposed in [MNOP].

The double life of a curve

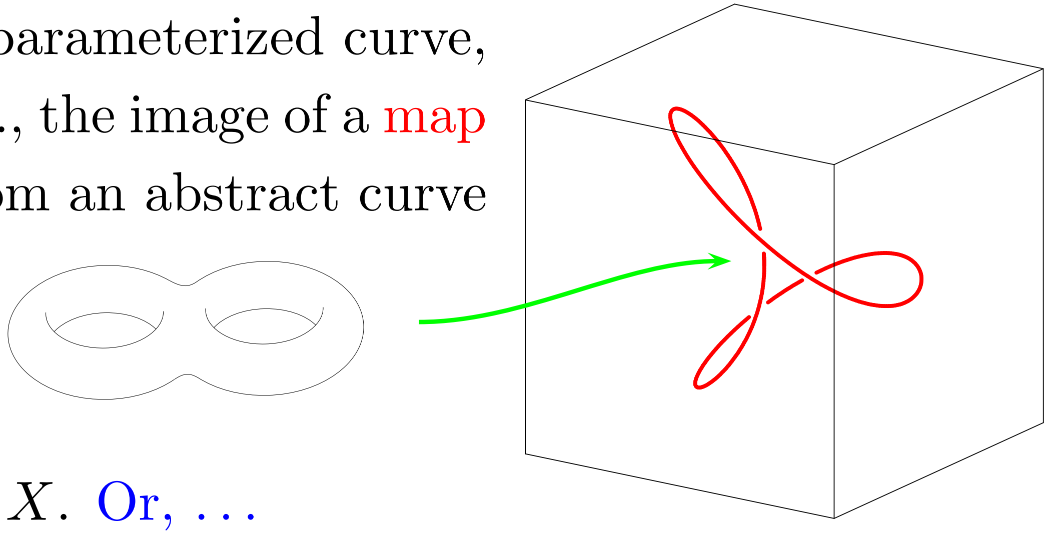
A curve in X can be viewed as either . . .



The double life of a curve

A curve in X can be viewed as either …

a parameterized curve, i.e., the image of a map from an abstract curve to X. Or, …



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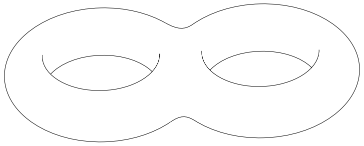
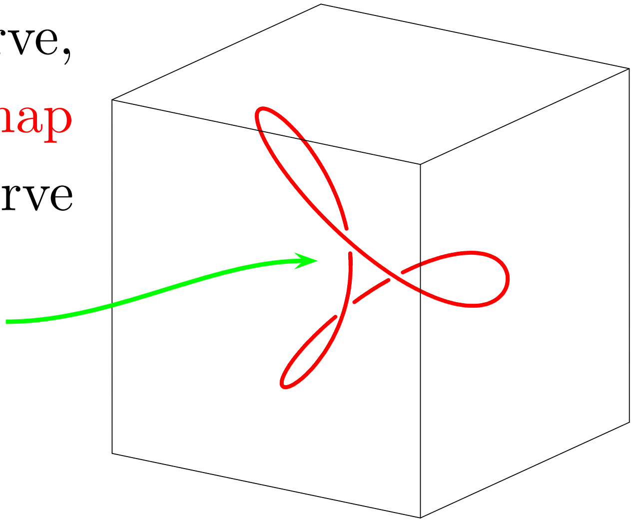
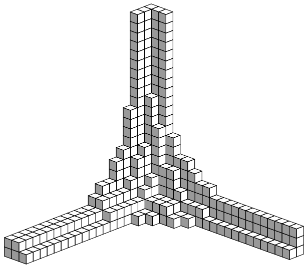




to X. Or, …

it may be described by its equations, i.e. by an ideal sheaf on X



`plot(...)` vs. `implicitplot(...)`

The natural moduli space on two sides are Kontsevich's moduli space of stable maps $\overline{\mathcal{M}}_g(X;\beta)$ and the Hilbert scheme of curves $\mathrm{Hilb}(X;\beta,\chi)$, respectively. Here

$$\beta \in H_2(X) \text{ is the degree of curve}$$

$$g = \text{ domain genus of a map}$$

$$\chi = \text{ Euler char of an ideal sheaf } \approx 1 - g$$

The geometry of these moduli spaces is very different.

One feature $\overline{\mathcal{M}}_g(X;\beta)$ and $\mathrm{Hilb}(X;\beta,\chi)$ share is the existence of a virtual fundamental class (*Li & Tian, Behrend & Fantechi, Thomas*) of dimension

$$\dim [\quad]^{\mathrm{vir}} = -\beta \cdot K_X \, .$$

GW and DT invariants of X are defined by evaluating natural cohomology classes on $[\quad]^{\mathrm{vir}}$.

Today, we will focus on the Calabi-Yau case $K_X = 0$. In this case, $\dim[\quad]^{\mathrm{vir}} = 0$ and there is one invariant, “the number of curves”, in every degree and genus.

The case of general X will, probably, be discussed in Rahul’s lecture …

GW/DT partition function

multi-index

Form the following generating function

$$Z_{GW}(t,u) = \sum_{\beta,g} u^{2g-2}\, t^{\beta} \int_{\left[\overline{\mathcal{M}}_g(X;\beta)^{\bullet}\right]^{\mathrm{vir}}} 1$$

disconnected

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Define Z_{DT} and Z'_{DT} by the same formulas with

$$u^{2g-2} \Rightarrow q^{\chi}$$

Conjecture ([MNOP])

$$\boxed{Z'_{GW}(t,u) = Z'_{DT}(t,-e^{iu})}$$

has a generalization to Descendent, relative, etc. invariants of an arbitrary smooth projective 3-fold X.

Known cases:

- X = canonical bundle of a smooth toric surface ([ORV] + [MNOP] + Liu-Liu-Zhou)
- X = any rank 2 bundle over a smooth curve (Bryan-Pandharipande + [OP])

noncompact

In this talk, we will explore the case X = toric CY, for which:

- GW=DT is "almost proven",
- we can compute Z_{DT} by localization as a sum over torus-fixed points in $\mathrm{Hilb}(X; \beta, \chi)$

isolated

How to visualize the T-fixed points $\mathrm{Hilb}(X;\beta,\chi)^T$?

Warm-up: $\mathrm{Hilb}(\mathbb{C}^2; d, n)$

By definition, it is formed by ideals $I \subset \mathbb{C}[x, y]$ such that

$$\operatorname{codim} I_{\leq k} = dk + n\,. \quad k \gg 0$$

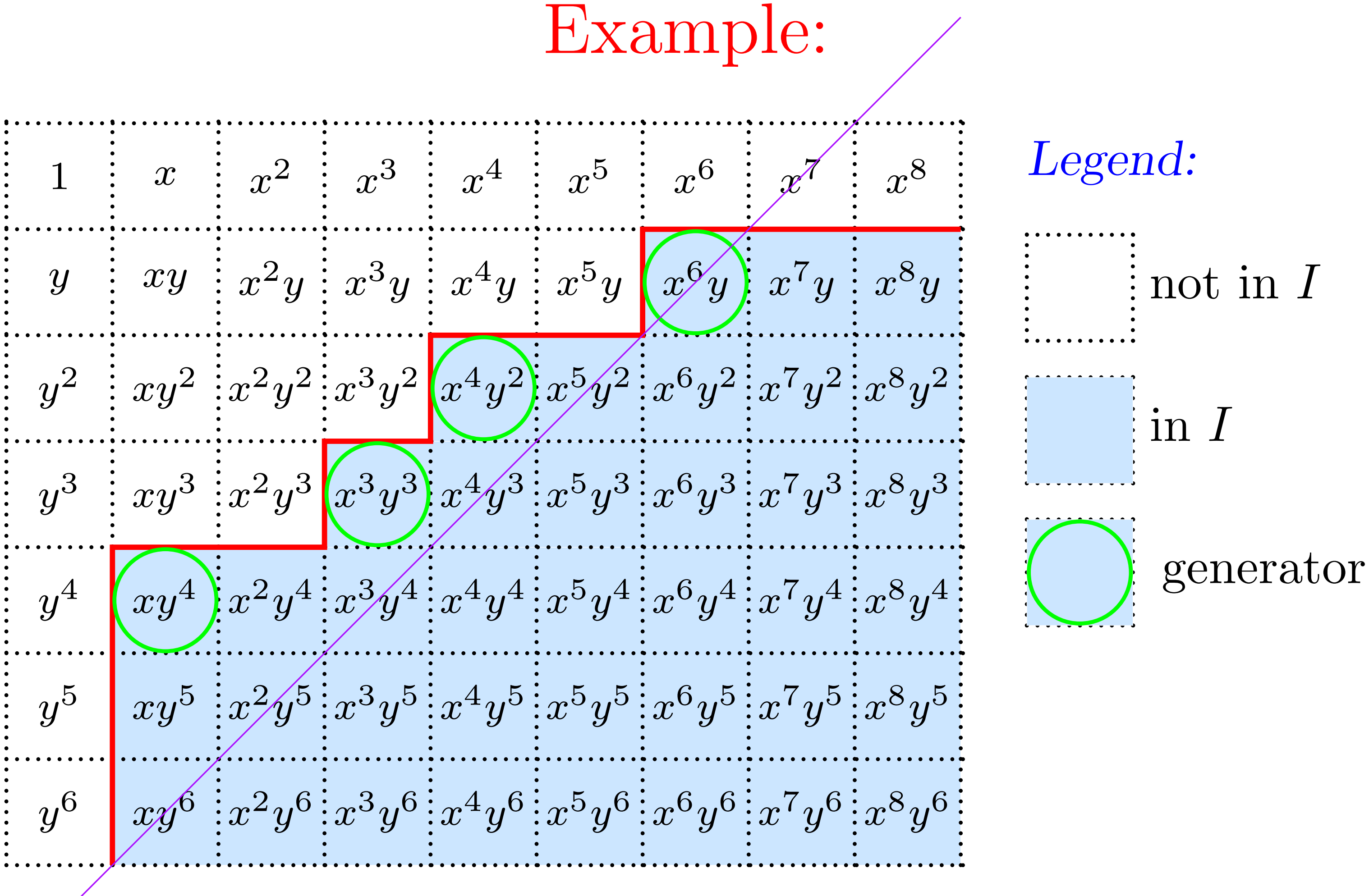
degree

The torus $(\mathbb{C}^*)^2$ acts on $\mathrm{Hilb}(\mathbb{C}^2; d, n)$ by rescaling x and y.

Monomials $x^i y^j$ are eigenvectors of the torus action with distinct eigenvalues.

Torus-fixed ideals I are spanned by monomials.

Example:

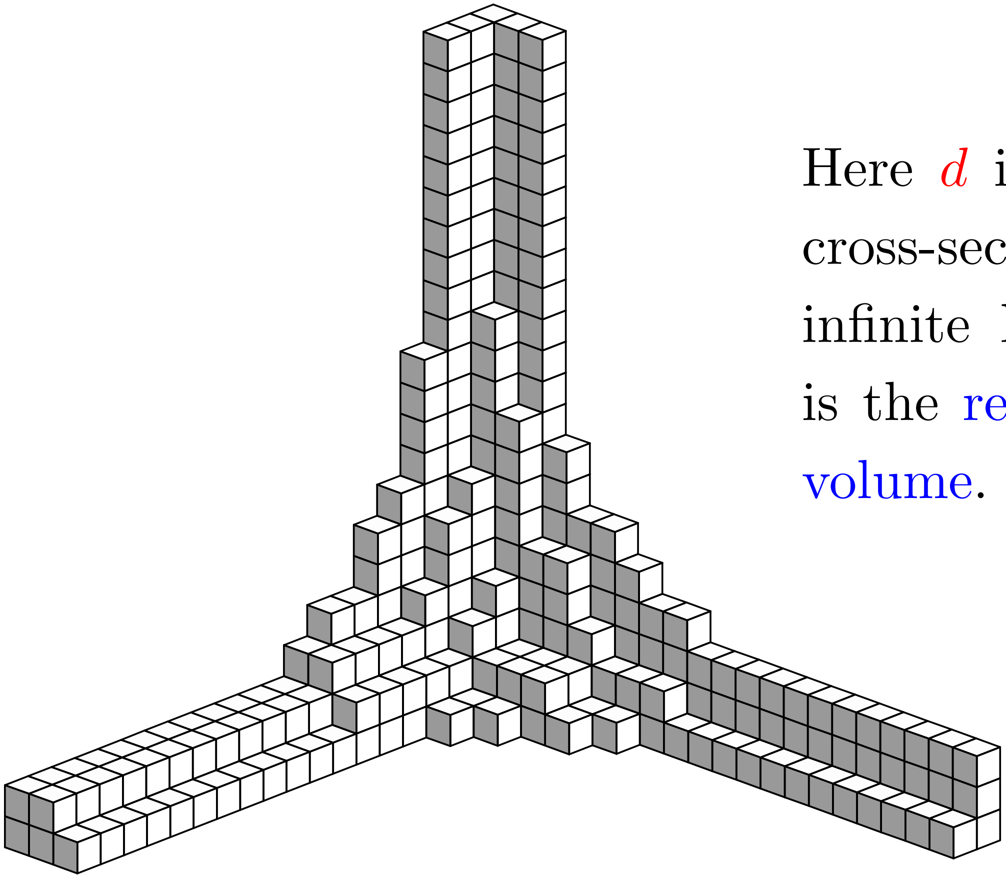


1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	x^8y
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2	x^7y^2	x^8y^2
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	x^5y^3	x^6y^3	x^7y^3	x^8y^3
y^4	xy^4	x^2y^4	x^3y^4	x^4y^4	x^5y^4	x^6y^4	x^7y^4	x^8y^4
y^5	xy^5	x^2y^5	x^3y^5	x^4y^5	x^5y^5	x^6y^5	x^7y^5	x^8y^5
y^6	xy^6	x^2y^6	x^3y^6	x^4y^6	x^5y^6	x^6y^6	x^7y^6	x^8y^6

d = total width of infinite rows and columns (= 2 here)

χ = renormalized area (= 9 here) .

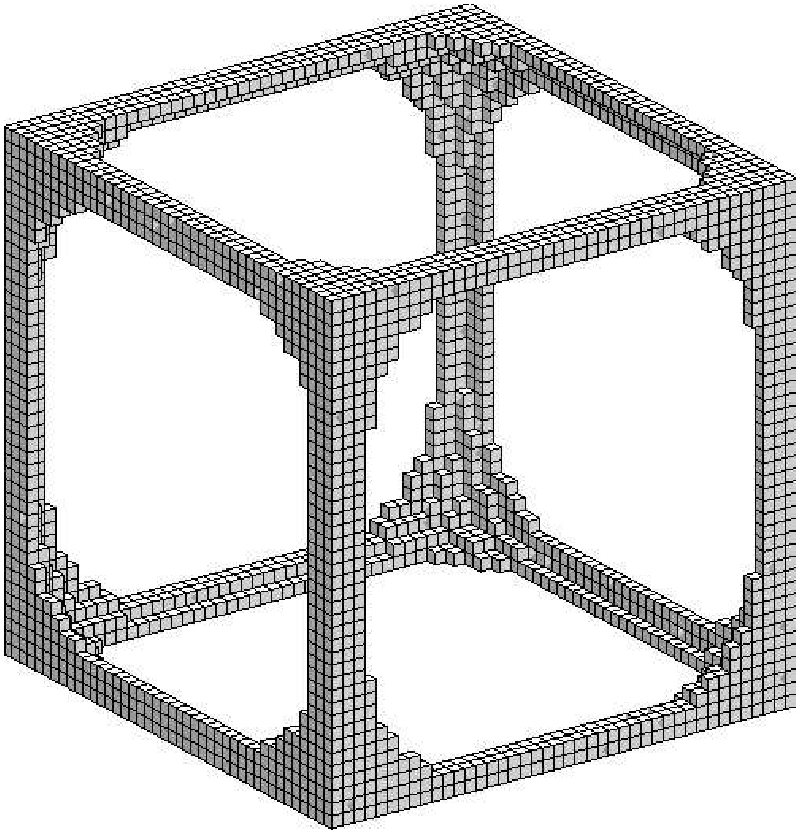
For $\mathrm{Hilb}(\mathbb{C}^3; d, \chi)$, torus-fixed points correspond to 3D partitions, with possibly infinite legs along the coordinate axes:



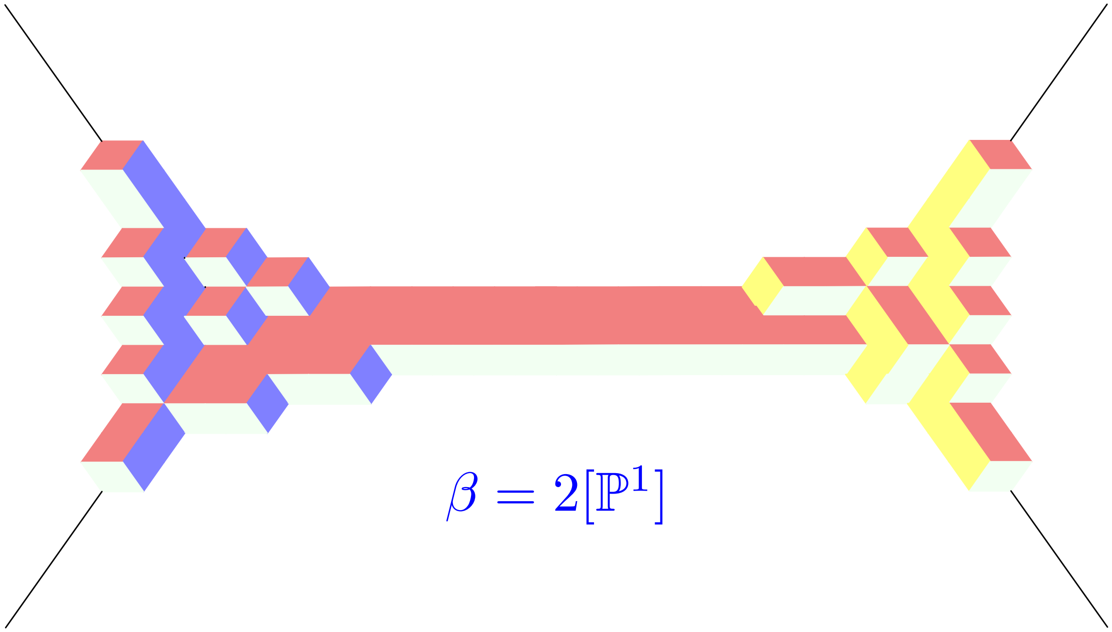
Here d is the total cross-section of the infinite legs and χ is the renormalized volume.

Now for a picture of a torus-fixed point in $\mathrm{Hilb}(X;\beta,\chi)$ assemble 3D partitions according to toric combinatorics of X.

Here is an example for $X = \left(\mathbb{P}^1\right)^3 \longleftarrow$ not CY



Here is a CY example: $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$



$\beta = 2[\mathbb{P}^1]$

What about the localization weight ?

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The CY condition implies it always equals ±1, in fact,

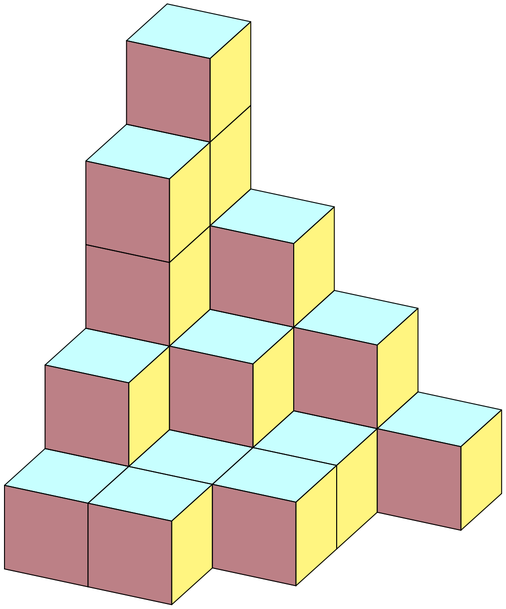
$$Z_{DT} = \sum_{\text{fixed points}} (-q)^{\chi} \, (\pm t)^{\beta} \, .$$

For example, for $X = \mathbb{C}^3$, $\beta = 0$, we get

$$Z_{DT} = \sum_{\text{3D partitions } \pi} (-q)^{\mathrm{vol}(\pi)}$$

$$= \prod_{n>0} \frac{1}{(1-(-q)^n)^n}$$

McMahon



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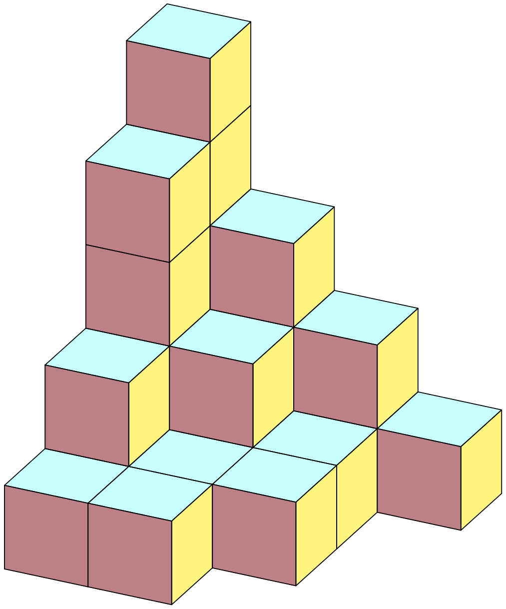
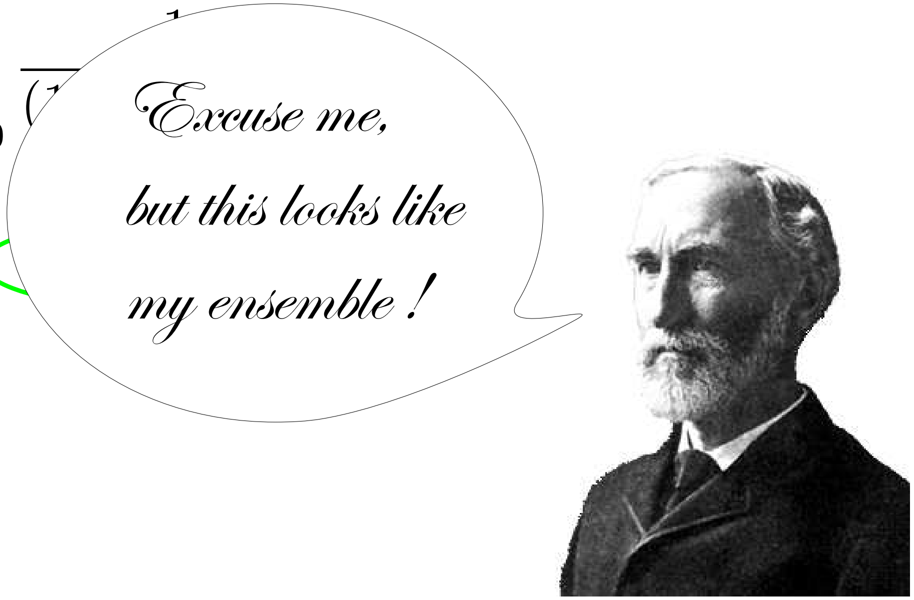
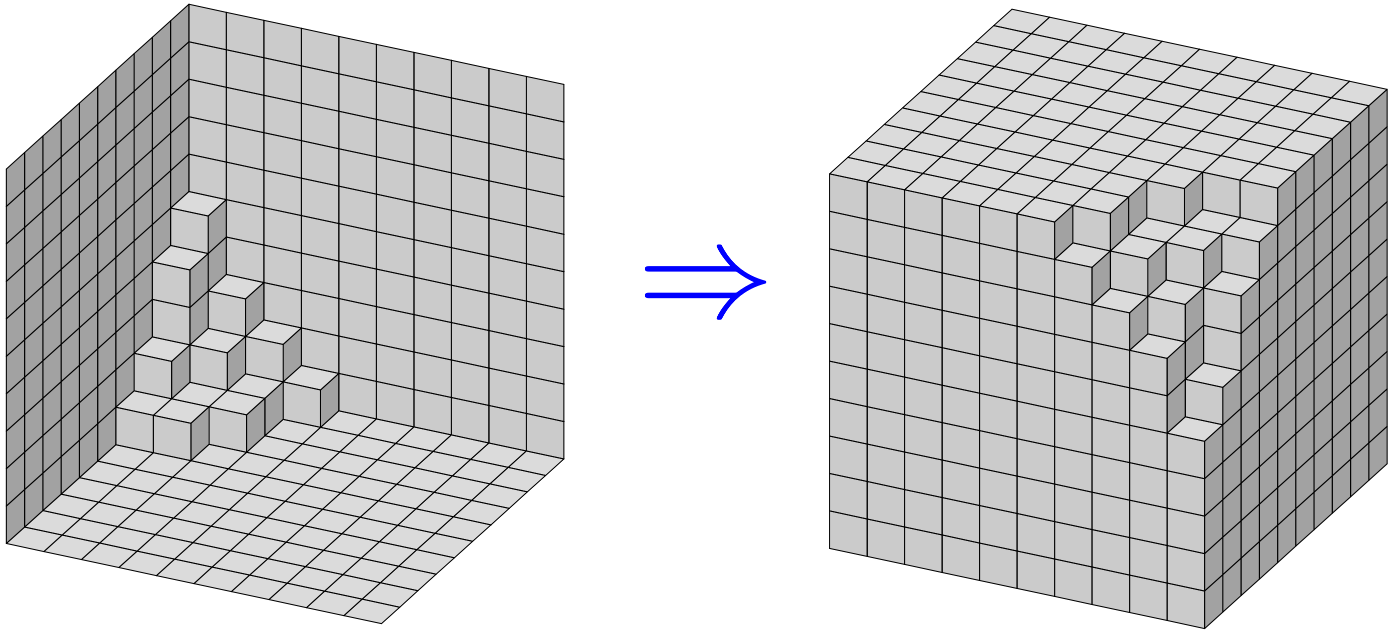




And, indeed, it is time for a

Stat Mech 101 interruption

Bottle duality takes 3D partitions to dissolving crystal corners

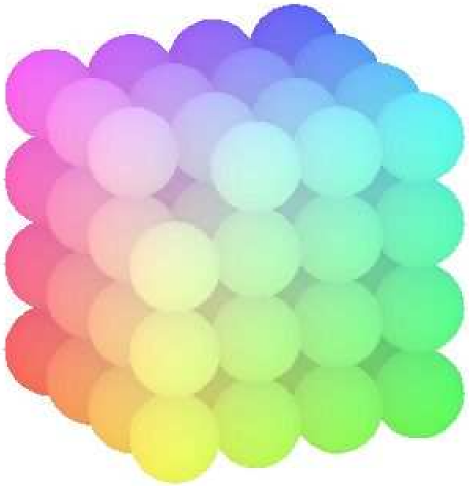


Gibbs ensemble

In equilibrium, the probability of a given configuration decays exponentially with its energy.

For a model of crystal, we can take

$$\text{Energy} = -\underset{\boxed{\text{chemical potential}}}{\mu_c}\ \text{Volume} + \underset{\boxed{\text{another const} > 0}}{\mu_s}\ \text{Surface}$$



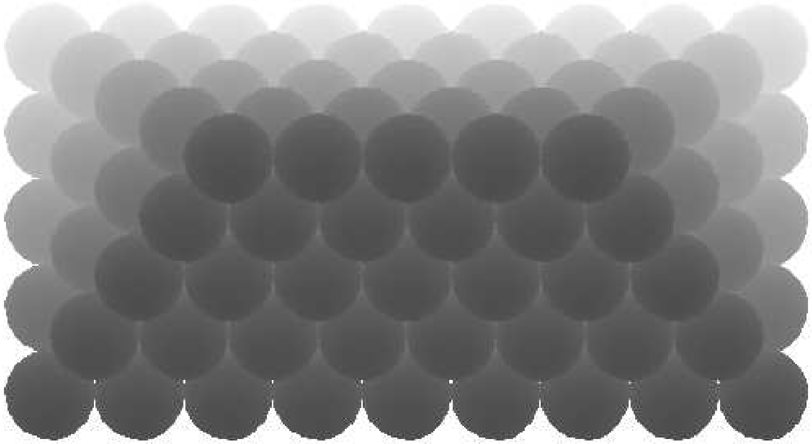
Chemical potential is the energy bill for removing an atom.

When $\mu_s \gg 0$ (low temperature), only configuration that minimize surface ($=$ 3D partitions) survive.

The shape of our dissolving crystal is given by the moment polytope of X. For example, for local $\mathbb{P}^1$

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$$

the atoms are arranged like this:



Seagram distillery
Waterloo, ON

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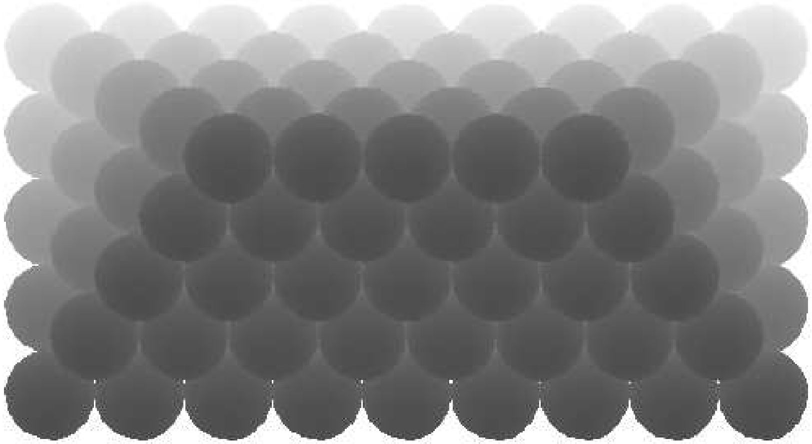
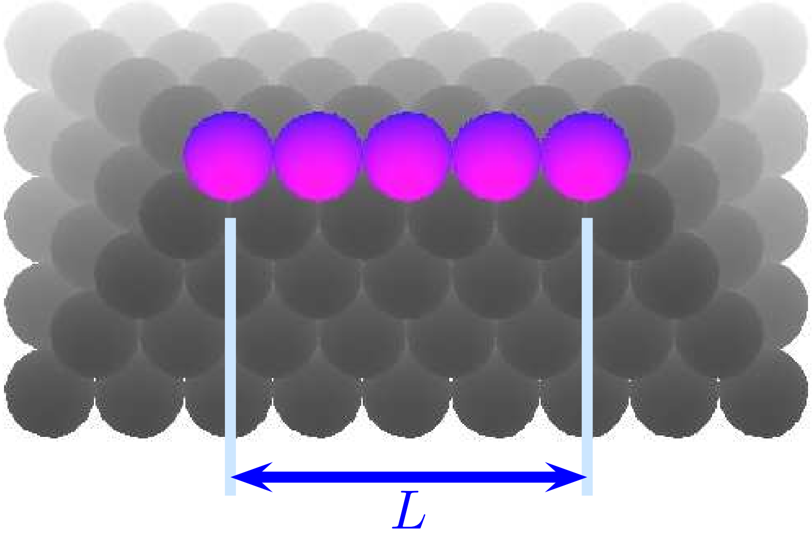


Removing a whole row of atoms changes: degree by 1, surface by -2, and volume by $\approx L$, where L is the edge length.



For $\mu_s, L \gg 0$ (large cold crystal), approximate balance occurs when

$$\mu_c L \approx 2\,\mu_s\,.$$

Conclusion:

For X = local $\mathbb{P}^1$, the DT/GW partition partition function Z is the large size, low temperature expansion of the Seagram crystal partition function with

$$u = i\mu_c \qquad \text{genus} \leftrightarrow \text{chem potential}$$

$$t = -\exp\left(-L\mu_c + 2\,\mu_s\right) \qquad \text{degree} \leftrightarrow \text{edge length}$$

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Known random partition/matrix models for GW invariants are special/limit cases of this.

Did we loose sight of geometry ? . . .

Did we loose sight of geometry ? . . . Geometry reappears as the

Limit shape

The connected GW invariants are the coefficients in the expansion of $\ln Z$ in powers of u.

The everyday life limit

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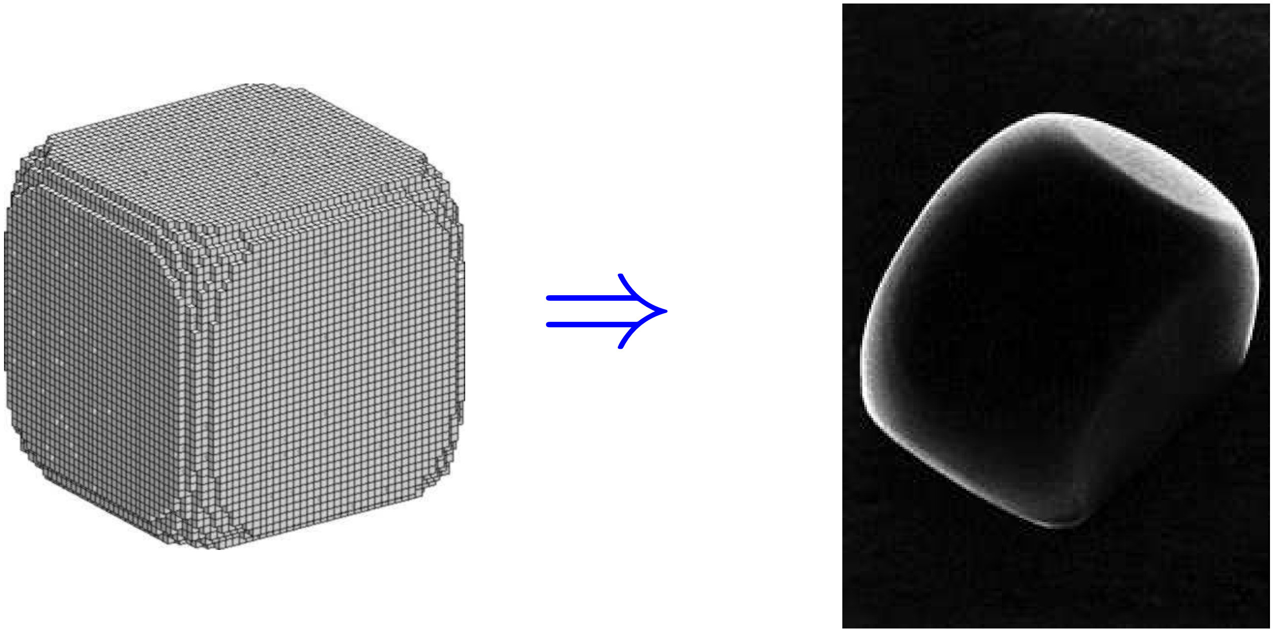
Thermodynamic limit

Law of large numbers

A nonrandom macroscopic limit shape forms *



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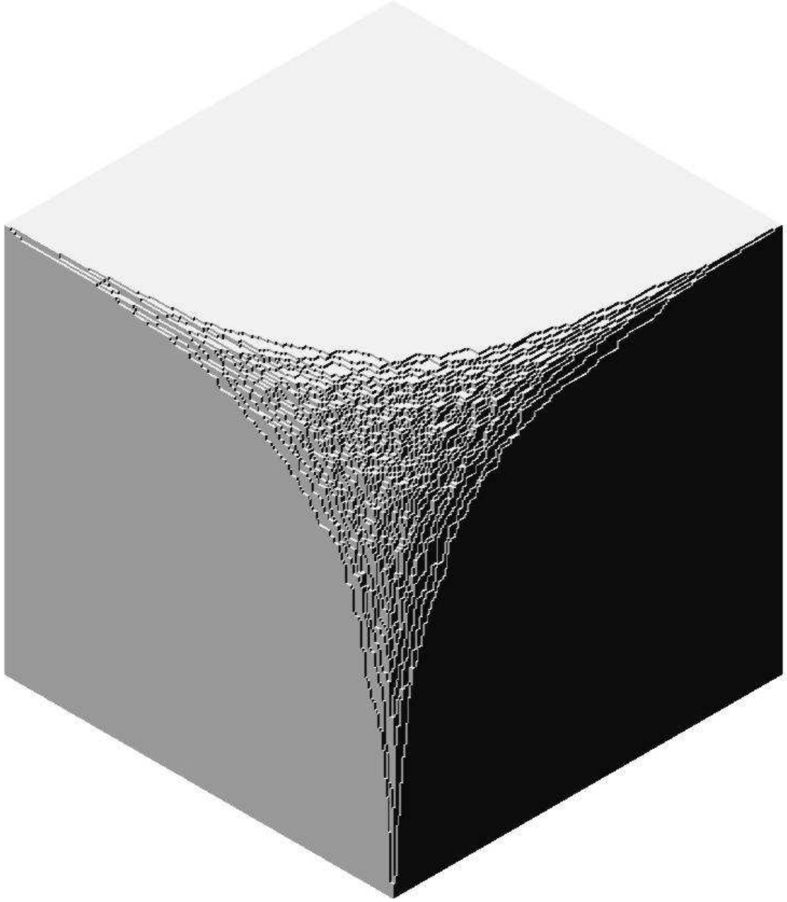
*Rigorously: as the parameter μ_c goes to 0, our random surface (=measure on Lipschitz surfaces), scaled by μ_c in all directions, converges weakly to the δ-measure on a single surface — the *limit shape*

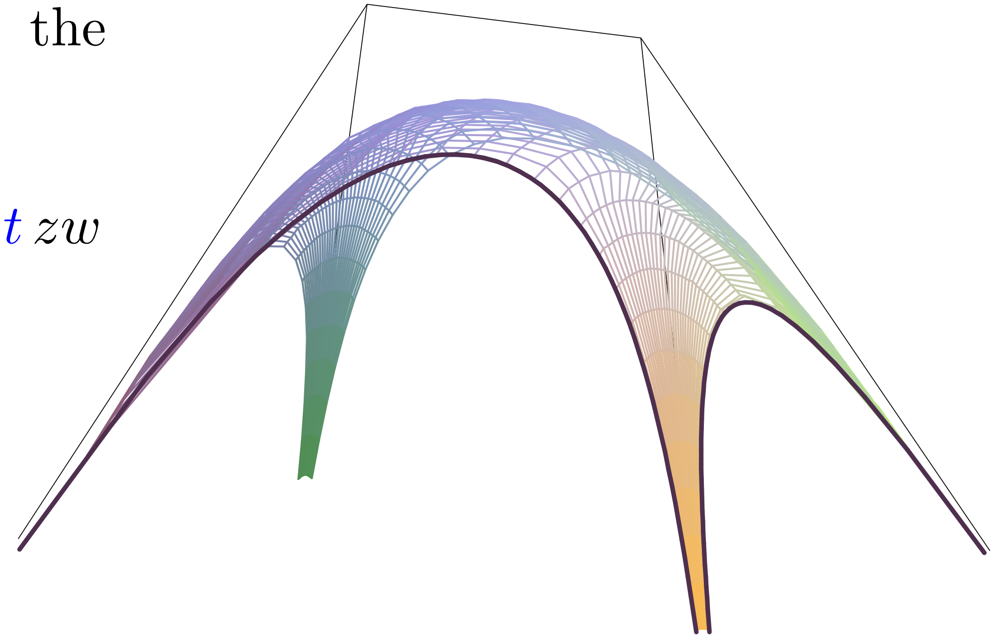
Here is a computer simulation of the limit shape formation near a corner.



For our random surfaces (and all dimer models in general), the limit shapes can be computed exactly.

For local $\mathbb{P}^1$, we get the Ronkin function R of

$$P(z,w) = 1 + z + w + t\,zw$$



$$R(x,y) = \frac{1}{(2\pi i)^2} \iint_{\substack{|z|=e^x \\ |w|=e^y}} \log P(z,w)\, \frac{dz}{z}\, \frac{dw}{w}$$

It may look surprising that the limit shape is an algebraic curve in disguise, but, in fact, this happens for any “polygonal” boundary conditions [KO]

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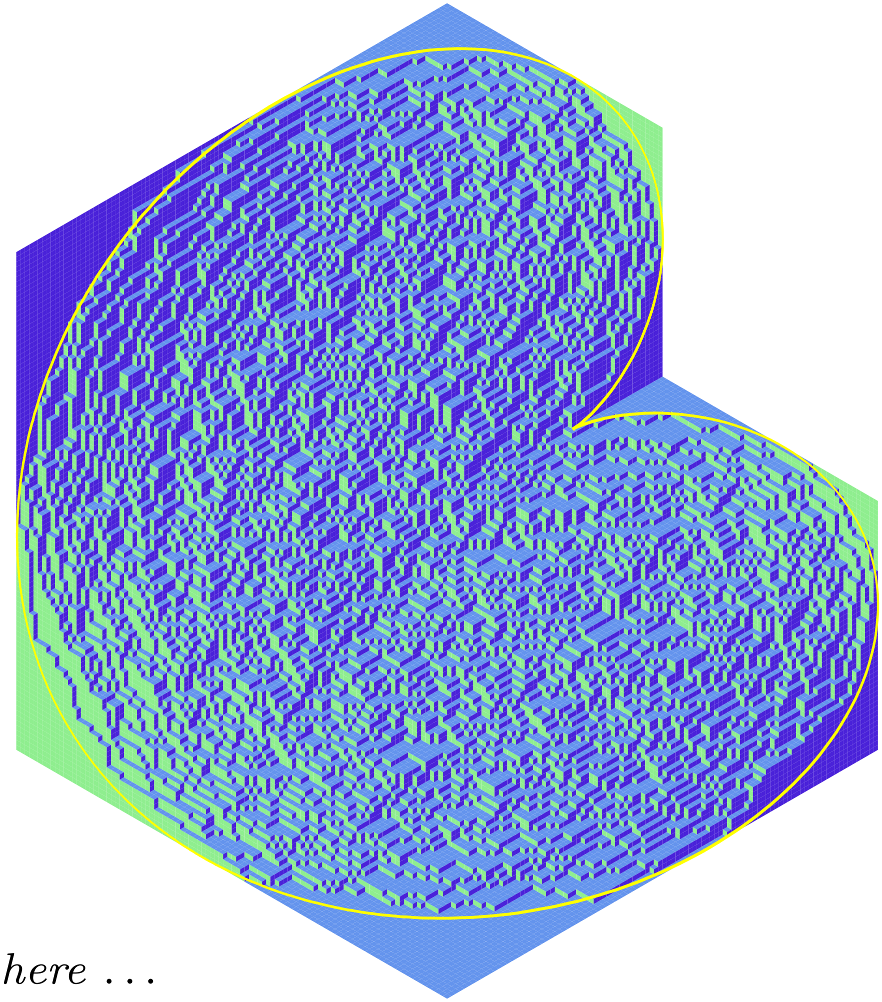
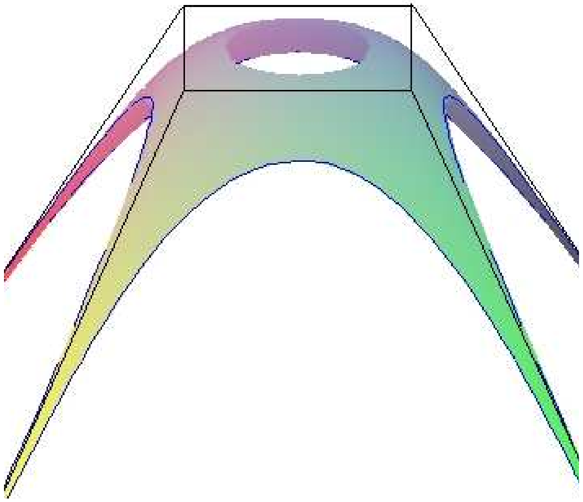
Here we see a cardioid.



Theorems about real curves are hidden here . . .

For a general CY toric X, the limit shape is the Ronkin function of a plane curve P.

This plane curve P it the Hori-Vafa mirror of X



local $\mathbb{P}^1 \times \mathbb{P}^1$

Why mirror ?

By basic probability theory

$$\begin{aligned}\text{genus } 0 \text{ GW invariants of } X &= \lim_{u\to 0} u^2 \ln Z \\ &= \text{surface tension of the limit shape}\end{aligned}$$

$$\Downarrow$$

periods of P

In the setting of instanton counting, the limit shape is the Seiberg-Witten curve. This is how the probabilistic proof of Nekrasov conjecture works.

For pure gauge theory, different, nonprobabilistic proofs given by Nakajima-Yoshioka and Braverman.

All orders in the $u \to 0$ asymptotics of $\ln Z$ (= GW invariants of all genera) should be computable in terms of the limit shape.

This has been worked out for random matrices (Eynard), which is a limit case of our random surfaces.

How to do without a torus action ?

How to do without a torus action ?

Not a rhetorical question …