On the existence of flips

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  - K_S is ample. S is of general type. Note that S is forced to be singular in general.
- Unfortunately we can destroy this picture by blowing up. It is the aim of the MMP to reverse the process of blowing up.

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- If the fibres of f have dimension at least one, then STOP. We have a Mori fibre space.
- If f is birational and the exceptional locus is a divisor, replace X by Y and keep going.

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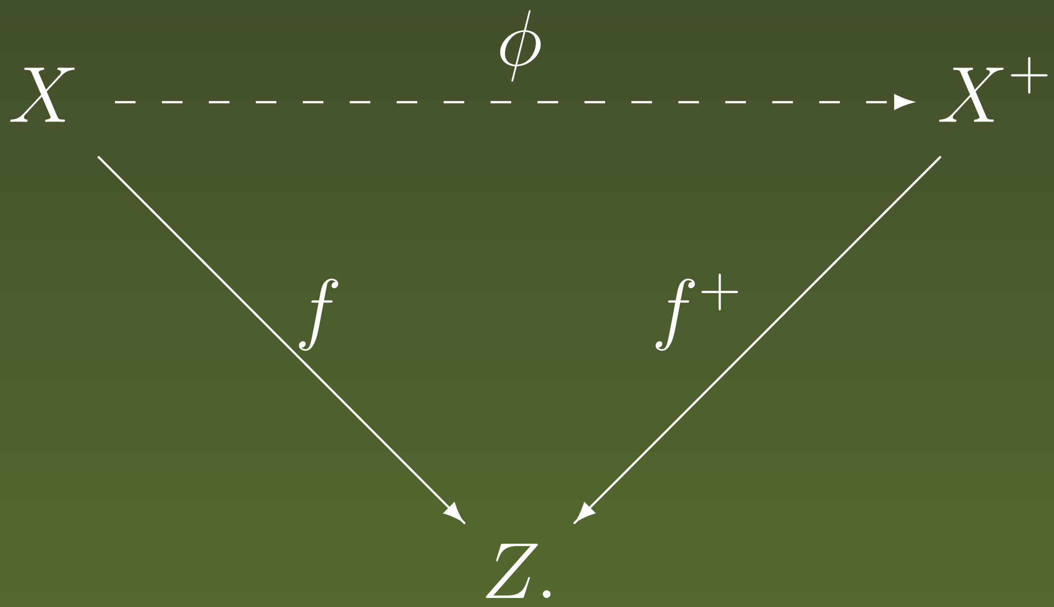
- Instead of contracting C, we try to replace X by another birational model X^+, $X \dashrightarrow X^+$, such that $f^+ \colon X^+ \longrightarrow Y$ is K_{X^+}-ample.



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- Even supposing we can perform a flip, how do we know that this process terminates?
- It is clear that we cannot keep contracting divisors, but why could there not be an infinite sequence of flips?

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- Both of these results have far reaching generalisations, whose form dictates the main definitions of the subject.

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- So suppose we can write $\Delta = S + B$, where S has coefficient one. Then

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- Moreover if $K_X + S + B$ is plt then $K_S + D$ is klt.

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- If we take a cover with appropriate ramification, then we can eliminate any component with coefficient less than one.
- (Kawamata-Viehweg vanishing) Suppose that $K_X + \Delta$ is klt and L is a line bundle such that $L - (K_X + \Delta)$ is big and nef. Then, for $i > 0$,

$$H^i(X, L) = 0.$$

Three main Conjectures

Conjecture. *(Existence) Suppose that* $K_X + \Delta$ *is kawamata log terminal. Let* $f : X \longrightarrow Y$ *be a small extremal contraction.*
Then the flip of f *exists.*

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Conjecture. *(Termination) There is no infinite sequence of kawamata log terminal flips.*

Conjecture. *(Abundance) Suppose that* $K_X + \Delta$ *is kawamata log terminal and nef.*
Then $K_X + \Delta$ *is semiample.*

Some interesting consequences

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Karu has shown that the first two conjectures imply the existence of a geometrically meaningful compactification of the moduli space of varieties of general type.

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- Kawamata proved the termination of threefold flips, and Shokurov/Birkar have proved that acc for the set of log discrepancies/thresholds implies termination.
- I predict that these three conjectures, existence, termination and abundance, will be proved within five years.

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No such implication holds for termination. In practice, however, most proofs of the termination of rational flips, extend to the case of real coefficients. In particular Shokurov has proved that real flips terminate in dimension three. This gives a new proof of the existence of flips in dimension four.

Finite Generation

Start with a small birational contraction $f\colon X \longrightarrow Z$, such that $-(K_X + \Delta)$ is ample. We want $X \dashrightarrow X^+$, where $f^+\colon X^+ \longrightarrow Y$ is $K_{X^+} + \Delta^+$-ample.

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Suppose that the ring $R = \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_X(mk(K_X + \Delta))$ is finitely generated. Then $X^+ = \operatorname{Proj}_Z R$.

Some consequences

- The flip exists iff the ring

$$R = R(X, D) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mD)),$$

where $D = k(K_X + \Delta)$, is a finitely generated A-algebra, where $Z = \operatorname{Spec} A$.

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- In particular, if the flip exists it is unique.
- Shokurov proved that if one assumes termination of flips in dimension $n - 1$, then to prove the existence of flips, it suffices to prove the existence of pl flips.
- For a pl flip, $K_X + \Delta$ is plt, $S = \llcorner \Delta \lrcorner$ is irreducible and $-S$ is ample.

Restricted algebras

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- This is too much to expect.
- However, something like this does happen.

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- Let $mD = N_m + G_m$ be the decomposition of mD into its mobile and fixed parts.

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- The kernel of the restriction map is principal. So R is finitely generated iff R_S is finitely generated.
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- Finite generation is a property of the sequence $M_\bullet$, even up to a birational map.

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- The limit Θ is klt, but the coefficients of Θ are real.
- To prove the existence of $\Theta_\bullet$, we use the methods of multiplier ideal sheaves, due to Siu and Kawamata.

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- Let D be the limit. If M_i is free, then R is finitely generated iff $D = D_m$, some m.

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- Unfortunately, for each m, we might need to go higher and higher. This is clearly an issue of birational geometry.
- Even if there is a single model, on which everything is free, the sequence might vary. This happens even on $\mathbb{P}^1$.

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- Thus there is a model $W \longrightarrow T$ on which the mobile part of $mk(K_T + \Theta_m)$ is free, and the limit D of the characteristic sequence is semiample.
- By a result of Shokurov, this proves that the restricted algebra is finitely generated.

Saturation

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- We say that a divisor D is saturated with respect to E if
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- D is not saturated with respect to E, as above.
- If $g\colon Y \longrightarrow X$ is any birational morphism, then the pullback of any divisor from Y is saturated with respect to any effective and g-exceptional divisor.

An application of vanishing

- Thus for all i and j, and all effective divisors E, exceptional for $g\colon Y \longrightarrow X$,

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- By vanishing, this implies that

$$\operatorname{Mov}\ulcorner \frac{j}{i} M_i + F \urcorner \leq M_j.$$

Diophantine approximation

- If $X = C$ a curve, then D_m is a finite sum $\sum b_{m,k} p_k$, $b_{m,k} \geq 0$, converging to $\sum b_k p_k$, and $F = \sum a_k p_k$.

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- The same argument goes through, almost word for word, for $n \geq 2$, provided one has a model Y, on which everything is free. But this is what we proved.