HOMOTOPY THEORY AND THE MAPPING CLASS GROUP: MUMFORD'S CONJECTURE

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I. THE CLASSICAL MODULI SPACE

$F = F_g$ closed, oriented smooth surface, genus $g \geq 2$.

$$\mathcal{S}_{\mathbb{C}}(F) = \left\{ \begin{array}{l} \text{complex structures} \\ \text{compatible w. orient.} \end{array} \right. = \left\{ \begin{array}{l} \text{max. hol. atlasses} \\ \text{compatible w. orient} \end{array} \right.$$

$$\big\downarrow d$$

$$\mathcal{S}_{\mathbb{C}}(TF) = \{ J : TF \to TF \mid J^2 = -1,\ \{v, Jv\} \text{ orient},\ v \in TF \}$$

Gauss: d is a bijection!

Give $\mathcal{S}_{\mathbb{C}}(F)$ topology from $\mathcal{S}_{\mathbb{C}}(TF)$.

$$\mathcal{S}_{\mathbb{C}}(\mathbb{R}^2) = Gl_2^+(\mathbb{R}) / Gl_1(\mathbb{C}) \simeq * \text{ (contractible)}$$

$$\mathcal{S}_{\mathbb{C}}(TF) = \Gamma(F, \mathcal{S}_{\mathbb{C}}^{fib}(TF)) \simeq * \Rightarrow \mathcal{S}_{\mathbb{C}}(F) \simeq *$$

$\mathrm{Diff}(F)$ group of orient. diffeo^s of F; acts on $\mathcal{S}_{\mathbb{C}}(F)$

$\mathcal{M}(F) := \mathcal{S}_{\mathbb{C}}(F) / \mathrm{Diff}(F)$. Moduli space

Uniformazation (Koebke):

$H \subseteq \mathbb{C}$ upper half plane; hyperbolic metric $ds^2 = |dz|^2 / y^2$

$I(H) = \{\text{orient. pres. isometries}\} = \{\mathbb{C}\text{-automorphisms}\}$

$\Sigma \in \mathcal{S}_{\mathbb{C}}(F) \Rightarrow \tilde{\Sigma} = H \Rightarrow \Sigma = H/\Gamma$ hyp. space form

(Remark:
$\mathcal{S}_{\mathbb{C}}(F) = \{\text{hyp. metrics on } F\} / I(H)$.)

$\mathrm{Diff}_1(F) \subset \mathrm{Diff}(F)$ component of the identity.

$\mathrm{Diff}(F)/\mathrm{Diff}_1(F) = \pi_0 \mathrm{Diff}(F) = \Gamma(F)$ (MCG)

THEOREM. (i) $\mathrm{Diff}_1(F)$ acts freely on $\mathcal{S}_{\mathbb{C}}(F)$, and

$\mathcal{S}_{\mathbb{C}}(F) \longrightarrow \mathcal{S}_{\mathbb{C}}(F)/\mathrm{Diff}_1(F)$ principal fiber bdl.

(ii) $\mathcal{J}(F) = \mathcal{S}_{\mathbb{C}}(F)/\mathrm{Diff}_1(F) = \mathbb{R}^{6g-6}$

(iii) $\Gamma(F)$ acts on $\mathcal{J}(F)$ w. finite isotropy gps

((i): Earle-Eells, (ii): Teichmüller)

Note: $\mathcal{M}(F) = \mathcal{S}_{\mathbb{C}}(F)/\mathrm{Diff}(F) = \mathcal{J}(F)/\Gamma(F)$

(i) + (ii) $\Rightarrow$ $B\mathrm{Diff}_1(F) \simeq \mathcal{S}_{\mathbb{C}}(F)/\mathrm{Diff}_1(F) \simeq *$

$\Rightarrow$ $B\mathrm{Diff}(F) \xrightarrow{\simeq} B\Gamma(F)$.

Let $E\Gamma(F) \to B\Gamma(F)$ be universal covering space

$$B\Gamma(F) \underset{\simeq}{\xleftarrow{\mathrm{proj}_1}} E\Gamma(F) \times_{\Gamma(F)} \mathcal{J}(F) \underset{H_*(\,;\mathbb{Q})\text{-isom}}{\xrightarrow{\mathrm{proj}_2}} \mathcal{M}(F)$$

Conclusion:

$B\Gamma(F) \simeq B\mathrm{Diff}(F) \longrightarrow \mathcal{M}(F)$

induces isomorphism on $H_*(-;\mathbb{Q})$.

II EMBEDDED SURFACES

$\mathcal{S}^n_{top}(F)$ = set of orient. submanifolds $\Sigma \subset \mathbb{R}^{n+2}$ which are diffeomorphic to F.

Topology on $\mathcal{S}^n_{top}(F)$: Take tubular neighborhood

$$\mathcal{O}_\varepsilon = \{p+v \in \mathbb{R}^{n+2} \mid p \in \Sigma,\ v \perp T_p\Sigma,\ |v|<\varepsilon\}$$

$(\varepsilon > 0$ small$)$; $\mathcal{O}_\varepsilon \underset{i}{\overset{\pi}{\rightleftarrows}} \Sigma$, $\pi(p+v) = p$, $i(p) = p$

$\mathcal{O}_\varepsilon$ = open ε-disk normal bundle

$\{s(\Sigma) \subset \mathbb{R}^{n+2} \mid s \in \Gamma(\Sigma, \mathcal{O}_\varepsilon)\}$ are open ngbh^s of Σ!

Alternative description of $\mathcal{S}^n_{top}(F)$:

$\mathrm{Emb}(F, \mathbb{R}^{n+2})$ space of smooth embeddings

$\mathrm{Diff}(F)$ acts freely (by composition)

$$\mathcal{S}^n_{top}(F) = \mathrm{Emb}(F, \mathbb{R}^{n+2}) / \mathrm{Diff}(F)$$

THEOREM (Kriegl–Michor) The orbit map

$$\mathrm{Emb}(F, \mathbb{R}^{n+2}) \longrightarrow \mathrm{Emb}(F, \mathbb{R}^{n+2}) / \mathrm{Diff}(F)$$

is a princ. bundle of smooth infinite dim. mfd^s.

DEFINITION $\mathcal{S}_{top}(F)$ is the union of

$$\mathcal{S}^n_{top}(F) \hookrightarrow \mathcal{S}^{n+1}_{top}(F) \hookrightarrow \mathcal{S}^{n+2}_{top}(F) \hookrightarrow \dots$$

Whitney: $\pi_i \, Emb(F, \mathbb{R}^{n+2}) = 0$ for $i < n-2$

$\Rightarrow \quad Emb(F, \mathbb{R}^{\infty+2}) \simeq *$

$\Rightarrow \quad Emb(F, \mathbb{R}^{\infty+2})/Diff(F) \simeq BDiff(F)$

Conclusions:

$\mathcal{S}_{top}(F) \simeq BDiff(F) \simeq B\Gamma(F)$

$H_*(\mathcal{S}_{top}(F);\mathbb{Z}) = H_*(B\Gamma(F);\mathbb{Z})$

$H_*(B\Gamma(F);\mathbb{Q}) = H_*(\mathcal{M}(F);\mathbb{Q})$

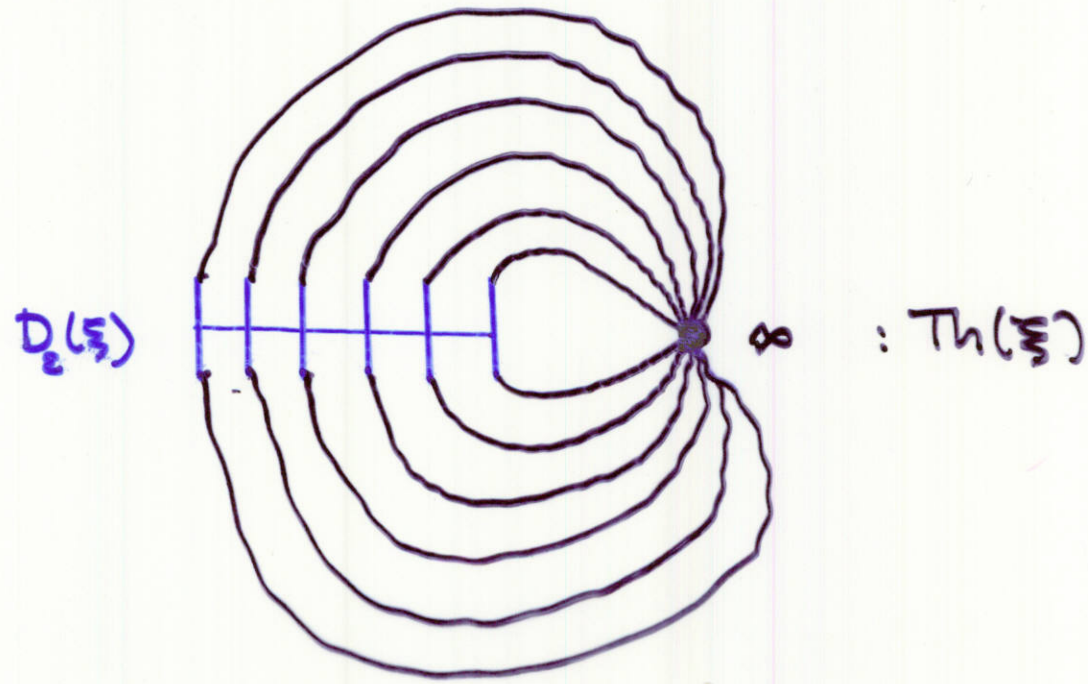
III THE MAP $\alpha_F : \mathcal{S}_{top}(F) \to \Omega^n \mathbb{C}P^\infty_{-1}$

$G(2,n)$ = Grassmannian of 2-planes in $\mathbb{R}^{n+2}$

$U^\perp_{2,n} = \{(V,v) \mid V \in G(2,n),\ v \perp V\}$

(n-dimensional vector bundle over $G(2,n)$)

The one-point compactification of $U^\perp_{2,n}$ is called the Thom space (in topology) and is denoted $Th(U^\perp_{2,n})$



- Key geometric property of Thom complex:

  $Th(\xi) \setminus$ nghb of 0-section $\simeq \infty$!

- Key algebraic property:

  $$\tilde{H}^*(Th(\xi);\mathbb{Z}) \cong H^{*-n}(X;\mathbb{Z})$$

  ξ n-dim, oriented vector bundle over X.

Given $\Sigma \in \mathcal{S}^n_{top}(F)$; choose tubular neighborhood $\Sigma \underset{\pi}{\overset{i}{\rightleftarrows}} \mathcal{O}_\varepsilon \overset{j}{\hookrightarrow} \mathbb{R}^{n+2}$. Get diagram

$$\begin{tikzcd} \mathcal{O}_\varepsilon \arrow[r, "\hat{t}"] \arrow[d] & U^\perp_{2,n} \arrow[d] \\ \Sigma \arrow[r, "t"] & G(2,n) \end{tikzcd} \qquad \begin{array}{l} t(p) = T_p\Sigma \subset \mathbb{R}^{n+2} \\ \hat{t}(p+v) = (T_p\Sigma, \tan(\frac{\pi}{2\varepsilon}|v|)v) \end{array}$$

Note: $j : \mathcal{O}_\varepsilon \hookrightarrow \mathbb{R}^{n+2}$ open; $\hat{t}$ is proper:

- $S^{n+2} = \widehat{\mathbb{R}}^{n+2} \xrightarrow{\hat{j}} \widehat{\mathcal{O}}_\varepsilon \xrightarrow{\hat{t}} Th(U^\perp_{2,n}) \quad (\infty \mapsto \infty)$

($\hat{j}$ is the Pontrjagin-Thom collapse map)

We get a well-defined homotopy class

$$\mathcal{S}^n_{top}(F) \longrightarrow Map_*(S^{n+2}, Th(U^\perp_{2,n})) =: \Omega^{n+2} Th(U^\perp_{2,n})$$

Changing from n to $n+1$:

$G(2,n) \subset G(2,n+1)$; $U^\perp_{2,n+1}|_{G(2,n)} = U^\perp_{2,n} \times \mathbb{R}$

The inclusion $U^\perp_{2,n} \times \mathbb{R} \longrightarrow U^\perp_{2,n+1}$ is proper $\Rightarrow$

$$Th(U^\perp_{2,n}) \wedge S^1 \xrightarrow{\varepsilon_n} Th(U^\perp_{2,n+1})$$

Topologists call $\{Th(U^\perp_{2,n}), \varepsilon_n\}_n$ a spectrum and denote it $\mathbb{CP}^\infty_{-1}$!

$S^{n+2} = S^{n+1} \wedge S^1$, so using ε_n we get
$\mathrm{Map}_*(S^{n+2}; \mathrm{Th}(U_{2,n}^\perp)) \longrightarrow \mathrm{Map}_*(S^{n+3}; \mathrm{Th}(U_{2,n+1}^\perp))$
and diagrams

$$\begin{array}{ccccc} \mathcal{S}^n_{top}(F) & \longrightarrow & \mathcal{S}^{n+1}_{top}(F) & \rightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \Omega^{n+2}\mathrm{Th}(U_{2,n}^\perp) & \longrightarrow & \Omega^{n+3}\mathrm{Th}(U_{2,n+1}^\perp) & \rightarrow & \cdots \end{array}$$

DEFINITION $\Omega^\infty \mathbb{C}P^\infty_{-1} = \mathrm{colim}_n \, \Omega^{n+2}\mathrm{Th}(U_{2,n}^\perp)$.
(infinite loop space of spectrum $\mathbb{C}P^\infty_{-1}$)

$\pi_0 \Omega^\infty \mathbb{C}P^\infty_{-1} = \mathbb{Z}$; all components are homotopy equivalent. Let $\Omega^\infty_k \mathbb{C}P^\infty_{-1}$ be kth component. Get

$\alpha_F : \mathcal{S}_{top}(F) \longrightarrow \Omega^\infty_{g-1}\mathbb{C}P^\infty_{-1}$; $g = g(F)$ genus.

Remember $\mathcal{S}_{top}(F) \simeq B\Gamma(F)$

THEOREM A (M.- Weiss) The map

$$\alpha_F : B\Gamma(F) \longrightarrow \Omega^\infty_{g-1}\mathbb{C}P^\infty_{-1}$$

induces isomorphism on $H_q(-;\mathbb{Z})$ for $2q < g-1$.

IV HOMOLOGY OF $\Omega^\infty \mathbb{C}P^\infty_{-1}$

$U_{2,n} = \{(V,v) \mid V \in G(2,n),\ v \in V\}$ 2-plane bundle $/G(2,n)$

$$U^\perp_{2,n} \longrightarrow U^\perp_{2,n} \oplus U_{2,n} = G(2,n) \times \mathbb{R}^{n+2}, \text{ proper}$$
$$(w \mapsto w \oplus 0)$$

$\Rightarrow \quad Th(U^\perp_{2,n}) \to G(2,n)_+ \wedge S^{n+2} \quad (G(2,n)_+ = G(2,n) \sqcup \{*\})$

$\Rightarrow \quad \Omega^{n+2} Th(U^\perp_{2,n}) \longrightarrow \Omega^{n+2}(G(2,n)_+ \wedge S^{n+2})$

Let $n \to \infty$ and use $G(2,\infty) \simeq \mathbb{C}P^\infty$ to get

$$\omega : \Omega^\infty_0 \mathbb{C}P^\infty_{-1} \longrightarrow \Omega^\infty_0(\mathbb{C}P^\infty_+ \wedge S^\infty)$$

Fact: ω is homotopic to a Serre fibration with fiber $\Omega^{\infty+2} S^\infty$

$\tilde{H}^*(\Omega^{\infty+2} S^\infty; \mathbb{Q}) = 0$ (since $\pi_*(\Omega^{\infty+2}S^\infty)$ finite; Serre)

$\Rightarrow$ (1) $H_*(\omega; \mathbb{Q})$ is an isomorphism

Infinite loop spaces are the abelian groups of homotopy theory. $\Omega^\infty S^\infty(X_+) = \Omega^\infty(X_+ \wedge S^\infty)$ is the free infinite loop space generated by X, similar in spirit to the free abelian group $\mathbb{Z}[X]$ generated by a set

The canonical complex line bundle gives

$$L : \mathbb{C}P^\infty \longrightarrow BU$$

BU is an infinite loop space (Bott periodicity) so L extends to a map

$$\hat{L} : \Omega^\infty S^\infty_0(\mathbb{C}P^\infty_+) \longrightarrow BU$$

(2) $\hat{L}$ is split surjective (up to homotopy), and

$H^*(\hat{L}; \mathbb{Q})$ is an isomorphism

(Segal).

(1) + (2) $\Longrightarrow$

$$H^*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}; \mathbb{Q}) = \mathbb{Q}[\hat{\kappa}_1, \hat{\kappa}_2, \dots]$$

$$\hat{\kappa}_i = \omega^* \hat{L}^*(i!\,ch_i)$$

($\hat{\kappa}_i$ corresponds to the Miller-Morita-Mumford classes under α_F of theorem A)

- $H^*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}; \mathbb{F}_p)$ was calculated by Galatius; Topology '04
- $H^*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}; \mathbb{Z})$ contain torsion of all orders

Let $H^*_{free}(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}) \subset H^*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}; \mathbb{Q})$ integral lattice

THEOREM (Galatius-M.-Tillmann) $\hat{\kappa}_{2i}$ is divisible by precisely 2, $\hat{\kappa}_{2i-1}$ by precisely denom$(B_i/2i)$ in the integral lattice.

$H^q(\Omega_0^\infty \mathbb{C}P_{-1}^\infty ; \mathbb{Z})$:

$q=1$: 0

$q=2$: $\mathbb{Z}$

$q=3$: 0

$q=4$: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/12$

$q=5$: 0

$q=6$: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$

$q=7$: $\mathbb{Z}/4$

V HARER STABILITY

Notation: $F^s_{g,b}$ genus g surface, b boundary circles, s marked points (in the interior)

$\Gamma^s_{g,b} = \pi_0(\mathrm{Diff}(F^s_{g,b} ; \partial F^s_{g,b} \sqcup \{x_1, \dots, x_s\}))$

- Components of $\mathrm{Diff}(;)$ are contractible (as before)
- $B\Gamma^s_{g,b} \longrightarrow \mathcal{M}(F^s_{g,b})$ is a homotopy equivalence if $b>0$ or $s>2g+2$

$F_{0,1} =$  $F_{1,2}$: 

Note: $F^s_{g,b-1} = F^s_{g,b} \cup_{S^1} F_{0,1}$; $F^s_{g+1,b} = F^s_{g,b} \cup_{S^1} F_{1,2}$

Get maps $\Gamma^s_{g,b} \to \Gamma^s_{g,b-1}$, $\Gamma^s_{g,b} \to \Gamma^s_{g+1,b}$

STABILITY THEOREM (Harer; Ivanov) For $b>0$ the maps

$$B\Gamma^s_{g,b-1} \longleftarrow B\Gamma^s_{g,b} \longrightarrow B\Gamma^s_{g+1,b}$$

induce isomorphism on $H_k(-;\mathbb{Z})$ for $2k < g-1$.

$$\Gamma^s_{g,b} \longrightarrow \Gamma^s_{g+1,b} \longrightarrow \cdots \qquad \mathrm{colim} = \Gamma^s_{\infty,b}$$

Harer stability $\Rightarrow$ $H_*(B\Gamma^s_{\infty,b})$ independant of $b>0$.

Dependance on s: There is a group extension

$\mathbb{Z}^s \longrightarrow \Gamma_{g,b+s} \longrightarrow \Gamma^s_{g,b}$. It induces Serre fibration

$$B\Gamma_{g,b+s} \longrightarrow B\Gamma^s_{g,b} \xrightarrow{\Pi L_i} (\mathbb{C}P^\infty)^s$$

$g=\infty$: $B\Gamma_{\infty,b+s} \longrightarrow B\Gamma^s_{\infty,b} \longrightarrow B\Gamma_{\infty,b}$ H_*-isomorphism

Conclusion:

$$H^*(B\Gamma^s_{\infty,b}) = H^*(B\Gamma_{\infty,b+s}) \otimes \mathbb{Z}[e_1,\dots,e_s]$$

$(e_i = c_1(L_i))$

Miller - Morita - Mumford classes $\kappa_i \in H^{2i}(B\Gamma_\infty;\mathbb{Z})$

$\mathbb{C}P^\infty \xleftarrow{L} B\Gamma^1_\infty \xrightarrow{\pi} B\Gamma_\infty$ (π forgets marked pt)

$\kappa_i = \pi_!(c_1(L)^{i+1})$ ($\pi_!$ integration along fiber)

Quillen's plus construction: $\Gamma^s_{g,b} = [\Gamma^s_{g,b}, \Gamma^s_{g,b}]$ $\overset{(g>3)}{\Rightarrow}$

$Q: B\Gamma^s_{g,b} \longrightarrow (B\Gamma^s_{g,b})^+$ with two properties

(i) $H_*(B\Gamma^s_{g,b};\mathbb{Z}) \longrightarrow H_*((B\Gamma^s_{g,b})^+;\mathbb{Z})$ isomorphism

(ii) $(B\Gamma^s_{g,b})^+$ is simply connected.

$H_*(B\Gamma^s_{\infty,b};\mathbb{Z})$ independant of b $\overset{\text{(J.H.C. Whitehead)}}{\Longrightarrow}$

$(B\Gamma^s_{\infty,b})^+$ is independant of b up to homotopy!

THEOREM A (M.- Weiss) There is a homotopy equivalence

$$\alpha : B\Gamma^+_{\infty,b} \xrightarrow{\simeq} \Omega^\infty_0 \mathbb{C}P^\infty_{-1}$$

Moreover $\alpha^*(\hat{\varkappa}_i) = \varkappa_i$.

Example: $\mathcal{M}_{0,2} = \mathcal{M}(F_{0,2})$ space of complex annuli with parametrized boundaries.

Standard annuli: $A_r = \{z \in \mathbb{C} \mid r \leq |z| \leq 1\}$, $r \in (0,1)$

Forgetting parametrization of ∂, any annulus is equivalent to some A_r; the holomorphic equivalences of A_r are the rotations. $\Rightarrow$

- $\mathcal{M}_{0,2} = \mathrm{Diff}(S^1) \times_{SO(2)} \mathrm{Diff}(S^1) \times (0,1)$.
- $\Gamma_{0,2} = \mathbb{Z}$ (Dehn twists) $\Rightarrow B\Gamma_{0,2} = S^1$

Since $\mathrm{Diff}(S^1) \simeq SO(2)$ we get

$$\mathcal{M}_{0,2} \simeq S^1 \simeq B\Gamma_{0,2}.$$

VI COBORDISM CATEGORIES

- Segal's category $\mathcal{S}$: $\mathrm{ob}\,\mathcal{S} = \{0,1,2,\dots\}$

A morphism from m to n is a R. surface with m incoming parametrized ∂-circles and n outgoing.

Composition by sewing surfaces together

$$\mathrm{mor}\,\mathcal{S} = \coprod_{g,m,n} \mathcal{M}_{g,n+m} : \mathcal{S} \text{ is a topological category}$$

- Embedded cobordism category $\mathcal{C}$:

$\mathrm{ob}\,\mathcal{C}$ = space of closed, oriented 1-mfd^s $C \subset \{a\} \times \mathbb{R}^{\infty+1}$

$\mathrm{mor}\,\mathcal{C}$ = space of cpt. oriented 2-mfd^s $\Sigma \subset [a_0, a_1] \times \mathbb{R}^{\infty+1}$

that meets the walls $\{a_i\} \times \mathbb{R}^{\infty+1}$ orthogonally

$(\partial \Sigma = \Sigma \cap \{a_1\} \times \mathbb{R}^{\infty+1} - \Sigma \cap \{a_0\} \times \mathbb{R}^{\infty+1}$; $-$ = opposite orient)

composition = union in $\mathbb{R} \times \mathbb{R}^{\infty+1}$.

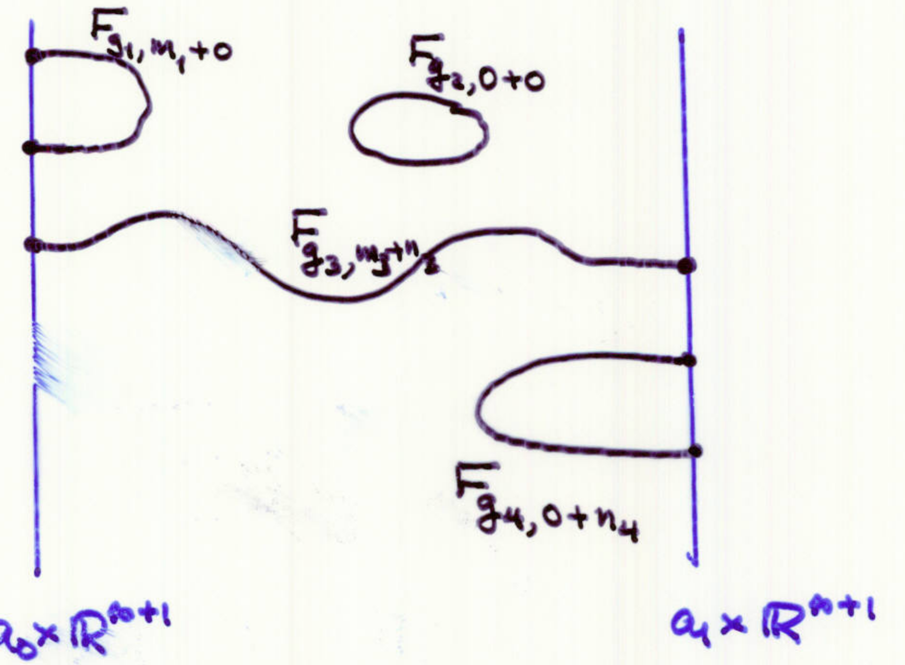
Abstractly, $C \cong \coprod_1^m S^1$; $\Sigma \cong F_{g,m+n}$ (or union of such)

$$\mathrm{ob}\,\mathcal{C} \simeq \coprod B\mathrm{Diff}(\textstyle\coprod_1^n S^1) \simeq \coprod B(\Sigma_n \wr \mathrm{Diff}(S^1))$$

$$\mathrm{mor}\,\mathcal{C} \simeq \coprod B\mathrm{Diff}(F_{g,N}; \partial)$$

(as in section II).

Skematic picture of $\Sigma : C_0 \to C_1$ in $\mathcal{C}$



Classifying space of a (topological) category $\mathcal{X}$:

$N_k\mathcal{X} = \{x_0 \to x_1 \to \dots \to x_k\} \subset (\mathrm{mor}\,\mathcal{X})^k$; k'th nerve

$d_i : N_k\mathcal{X} \longrightarrow N_{k-1}\mathcal{X}$ (delete x_i) ; $i = 0, 1, \dots, k$.

Δ^k standard k-simplex; $d^i : \Delta^{k-1} \to \Delta^k$ kth face

$$B\mathcal{X} := \coprod_{k=0}^{\infty} N_k\mathcal{X} \times \Delta^k \Big/ (d_i \underline{x}, \underline{t}) \equiv (\underline{x}, d^i \underline{t})$$

$(\underline{x} \in N_k\mathcal{X}\,,\ \underline{t} \in \Delta^{k-1})$.

• Fact: $B\mathcal{S} \simeq B\mathcal{C}$.

Examples: 1) $\mathrm{ob}\,\mathcal{X} = *$, $\mathrm{mor}\,\mathcal{X} = G$ (top. monoid)

$B\mathcal{X}$ is the classifying space BG.

e.g. $G = \coprod_{g \geq 0} B\Gamma_{g,2}$. This is a top. monoid, since by gluing $\Gamma_{g,2} \times \Gamma_{h,2} \to \Gamma_{g+h,2} \Rightarrow B\Gamma_{g,2} \times B\Gamma_{h,2} \to B\Gamma_{g+h,2}$

• Quillen: $\Omega B(\coprod B\Gamma_{g,2}) \simeq \mathbb{Z} \times B\Gamma_{\infty,2}^{+}$!

2) X any top. space; $\mathcal{X} = \mathrm{Path}(X)$:

$\mathrm{ob}\,\mathcal{X} = \mathbb{R} \times X$

$\mathrm{Hom}_{\mathcal{X}}((a_0, x_0), (a_1, x_1))$ = space of paths $\gamma : [a_0, a_1] \to X$ with $\gamma(a_0) = x_0$, $\gamma(a_1) = x_1$.

$B\,\mathrm{Path}(X) \simeq X$!

THEOREM B (Galatius, M., Tillmann, Weiss) The classifying space of the embedded cobordism category $\mathcal{C}$ is homotopy equivalent to $\Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1}$.

$(\Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1} = \operatorname{colim} \Omega^{n+1} Th(U^{\perp}_{2,n}))$

The map $B\mathcal{C} \to \Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1}$: Let $\Sigma \subset [a_0,a_1] \times \mathbb{R}^{n+1}$ represent a morphism in $\mathcal{C}$, $\mathcal{O}_\Sigma$ a tubular nbh.

Pontjagin-Thom collapse map gives

$$[a_0,a_1]_+ \wedge S^{n+1} \to \mathcal{O}_\Sigma \to Th(U^{\perp}_{2,n}) \Rightarrow$$

$$N_1\mathcal{C} \to N_1 \operatorname{Path}(\Omega^{n+1} Th(U^{\perp}_{2,n})) \rightsquigarrow$$

$$N_k\mathcal{C} \to N_k \operatorname{Path}(\Omega^{n+1} Th(U^{\perp}_{2,n})),\ k \geq 0 \text{ and}$$

compatible with the face operators $d_i \Rightarrow$

$$B\mathcal{C} \to B\operatorname{Path}(\Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1}) \simeq \Omega^{\infty-1}\mathbb{C}P^{\infty}_{-1}$$

This map turns out to be a homotopy equivalence.

VII THE REDUCED COBORDISM CATEGORY

$\mathcal{X}$ any top. category ; $x, y \in \mathrm{ob}\,\mathcal{X}$ $(= N_0 \mathcal{X})$

$$\mathrm{Hom}_{\mathcal{X}}(x,y) \times \Delta^1 \hookrightarrow N_1 \mathcal{X} \times \Delta^1 \longrightarrow B\mathcal{X} \Rightarrow$$

$$\mathrm{Hom}_{\mathcal{X}}(x,y) \longrightarrow \Omega_{x,y} B\mathcal{X} \simeq \Omega B\mathcal{X} \text{ if } \pi_0 B\mathcal{X} = 0.$$

$\mathcal{C}$ embedded cobordism category. Subcatgory $\mathcal{C}^{red}$:
$\mathrm{ob}\,\mathcal{C}^{red} = \mathrm{ob}\,\mathcal{C}$; $\Sigma \in \mathrm{mor}\,\mathcal{C}^{red}$ if each connectd component of Σ has at least one outgoing ∂.

THEOREM C (Galatius, M., Tillmann, Weiss)
$B\mathcal{C}^{red} \longrightarrow B\mathcal{C}$ is a homotopy equivalence.

THEOREM (Tillmann): $\Omega B\mathcal{C}^{red} \simeq \mathbb{Z} \times B\Gamma_\infty^+$.
(Invent. 1997).

Serre : $f : X \to Y$ any map, $y \in Y$

$$hF(f)_y := \{(x, \lambda(t)) \mid \lambda(t) \text{ path in } Y \text{ from } y \text{ to } f(x)\}$$

- Notice : $f^{-1}(y) \longrightarrow hF(f)_y$; take $\lambda(t)$ constant;
  if $X \simeq *$ then $hF(f)_y \simeq \Omega_y Y$.

- f is homotopic to a Serre fibration with fibers homotopy equivalent to $hF(f)_y$.

In practice one also has:

- If $f^{-1}(y)$ are all homotopy equiv. then $f^{-1}(y) \simeq hF(f)_y$
- If $H_*(f^{-1}(y))$ are all isomorphic then $H_*(f^{-1}(y)) \cong H_*(hF(f)_y)$

(cf. McDuff-Segal, Invent. 1976)

Sketch proof of Tillmann's theorem: Consider

1) functors $E_0, E : \mathcal{C}^{red} \to$ Spaces (contravariant)

$$E_0(C) = \mathrm{Hom}_{\mathcal{C}^{red}}(C, S^1)$$

$$E(C) = \mathrm{colim}\left(E_0(C) \xrightarrow{T_0} E_0(C) \xrightarrow{T_0} E_0(C) \to \dots \right)$$

$(T = F_{1,2} =$  $)$; $E(C) = E_0(C)[\frac{1}{T}]$

$(C = \sqcup_1^3 S^1)$

2) Category $E \wr \mathcal{C}^{red}$. $\quad ob = \{(C, z) \mid z \in E(C)\}$

$$mor(E \wr \mathcal{C}^{red}) = \{ \Sigma : C_0 \to C_1 , z \in E(C_1) \}$$

$$N_k(E \wr \mathcal{C}^{red}) \xrightarrow{\pi_k} N_k \mathcal{C}^{red} \Rightarrow$$

$$\pi : B(E \wr \mathcal{C}^{red}) \longrightarrow B\mathcal{C}^{red}.$$

$C \in N_0 \mathcal{C}^{red}$; $\pi^{-1}(C) = E(C) \simeq \mathbb{Z} \times B\Gamma_{\infty,n}$

$n = \#\, S^1$ in C.

3) Harer stability $\Rightarrow H_*(\pi^{-1}(C))$ independant of C

$\Rightarrow \quad \pi^{-1}(C) \to hF(\pi)_C \quad H_*$-isomorphism.

Now, $\quad B(E \int \mathcal{C}^{red}) = B(E_0 \int \mathcal{C}^{red})[\frac{1}{T}]$ and

$B(E_0 \int \mathcal{C}^{red}) \simeq *$ since $E_0 \int \mathcal{C}^{red}$

has a terminal object $((S^1, id))$.

$\Rightarrow B(E \int \mathcal{C}^{red}) \simeq *$, so

$\Omega B \mathcal{C}^{red} \simeq hF(\pi)_C \xleftarrow{H_*\text{-isom}} \mathbb{Z} \times B\Gamma_{\infty,n}$

$\Rightarrow \mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega B \mathcal{C}^{red} \qquad \square$

Mumford's conjecture:

$$\left.\begin{array}{l} \text{Theorem B}: \ B\mathcal{C} \simeq \Omega^{\infty-1} \mathbb{C}P^\infty_{-1} \\ \text{Theorem C}: \ B\mathcal{C}^{red} \simeq B\mathcal{C} \\ \text{Tillmann's theorem}: \ \mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega B\mathcal{C}^{red} \end{array}\right\} \Rightarrow$$

Theorem A: $\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega^\infty \mathbb{C}P^\infty_{-1}$ and hence

$H_*(B\Gamma_\infty; \mathbb{Q}) \cong H_*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}; \mathbb{Q}) \cong$

$\mathbb{Q}[\kappa_1, \kappa_2, \dots]$.

ADDENDUM: Theorem B and Theorem C are true in all dimensions.

$\mathcal{C}_d$ d-dimensional embedded cobordism category

$\mathrm{ob}\,\mathcal{C}_d = \{$closed, orient., d-1 dimensional submanifolds of $\{a\} \times \mathbb{R}^{\infty+1}\}$

$\mathrm{mor}\,\mathcal{C}_d = \{$compact d-dimensional, orient. cobordisms in $[a_0, a_1] \times \mathbb{R}^{\infty+1}\}$

Then

$$B\mathcal{C}_d \simeq B\mathcal{C}_d^{red} \simeq \operatorname{colim}_n \Omega^{n+d-1} Th(U_{d,n}^{\perp})$$

where

$U_{d,n}^{\perp}$ n-dim. vector bundle over $G(d,n)$

$G(d,n) =$ Grassmannian of orient. d-dim. linear subspaces of $\mathbb{R}^{n+d}$.

But we lack the analogue of Harer stability!

IX SURFACES IN A BACKGROUND SPACE

(joint work with R. Cohen)

Y connected space with base point $* \in Y$.

$F = F_{g,b}$, $b>0$.

$$\mathcal{M}^{top}_{g,b}(Y) = EDiff(F;\partial) \times_{Diff(F;\partial)} Map((F,\partial F),(Y,*))$$

$$\simeq \{(\Sigma,\varphi) \mid \Sigma \text{ R. surface of type } F_{g,b};$$
$$\varphi : (\Sigma,\partial) \to (Y,*) \text{ continuous}\}$$

STABILITY THEOREM: If Y is simply connected then $H_*(\mathcal{M}^{top}_{g,b}(Y))$ is independant of g and $b>0$ for $2* < g-1$.

THEOREM There is a homology equivalence

$$\mathcal{M}^{top}_{\infty,b}(Y) \longrightarrow \Omega^\infty(\mathbb{C}P^\infty_{-1} \wedge Y)$$

Remark: $Y \mapsto \pi_*(\Omega^\infty(\mathbb{C}P^\infty_{-1} \wedge Y))$ is a (generalized) homology theory

Relations to Gromov-Witten theory ?

X AN ANNOUNCMENT

Let F_n be the free group on n generators and $\mathrm{Aut}(F_n)$ the group of automorphisms of F_n.

$\mathrm{Aut}(F_\infty) = \operatorname{colim}_n \mathrm{Aut}(F_n)$

$\Sigma_n \subset \mathrm{Aut}(F_n)$; Σ_n symmetric group on n letters

$B\Sigma_n \longrightarrow B\mathrm{Aut}(F_n)$ and hence

$B\Sigma_\infty \longrightarrow B\mathrm{Aut}(F_\infty)$.

Quillen : $B\Sigma_\infty^+ \simeq \Omega_0^\infty S^\infty$

Hatcher : $B\Sigma_\infty^+ \longrightarrow B\mathrm{Aut}(F_\infty)^+$ is split injective (up to homotopy)

Søren Galatius has just proved that $B\Sigma_\infty^+ \longrightarrow B\mathrm{Aut}(F_\infty)^+$ is a homotopy equivalence

COROLLARY : $\tilde{H}^*(B\mathrm{Aut}(F_\infty); \mathbb{Q}) = 0$.