ACTIONS OF $\mathbf{C}^*$ AND $\mathbf{C}_+$ ON AFFINE ALGEBRAIC VARIETIES

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1. Terminology.

X – a smooth complex affine algebraic variety
$A = \mathbf{C}[X]$ – the algebra of regular functions on X
G – an algebraic group
$\Phi : G \times X \to X$ – an algebraic action (i.e. Φ is an action and a morphims)
A^G -subalgebra of G-invariant regular functions
$X//G = \mathrm{Spec} A^G$ –algebraic quotient.

Remark. $X//G$ is affine for reductive G (Nagata).
$X//G$ is quasi-affine (Winkelmann, 2003) for non-reductive, but not necessarily affine (Nagata, Roberts, Daigle, Freudenburg for $G = \mathbf{C}_+$, $X = \mathbf{C}^n$ with $n \geq 5$)

14-h Hilbert Problem (Nagata): Is $F \cap A$ an affine domain for a subfield F of Frac(A)?
Yes, when transcendence degree of F (over $\mathbf{C}$) ≤ 2 (Zariski, 1953). Otherwise, No (Kuroda, 2004).

Examples. (1) $\bar{x} = (x_1, \ldots, x_n) \in \mathbf{C}^n = X$, $\lambda \in \mathbf{C}^* = G$:
A linear action is given by $\lambda(\bar{x}) = (\lambda^{k_1} x_1, \ldots, \lambda^{k_n} x_n)$ where $k_i \in \mathbf{Z}$.

(2) $\bar{x} \in \mathbf{C}^n = X$, $t \in \mathbf{C}_+ = G$:
A triangular action is given by
$\bar{x} \to (x_1, x_2 + tp_2(x_1), \ldots, x_n + tp_n(x_1, \ldots, x_{n-1}))$
where each p_i is a polynomial.
The fixed point set for this action is $p_2 = \ldots = p_n = 0$ is a cylinder $Y \times \mathbf{C}_{x_n}$.

(2′) A triangular action without no fixed points is free. Say, if each p_i is constant and $p_n \neq 0$ then the action is free (and called a translation).

More generally, a $\mathbf{C}_+$-action is free (resp. a translation) on X if it has no fixed points (resp. $X \simeq Y \times \mathbf{C}$ and the action is a translation on the second factor). In particular, for a translation $X//\mathbf{C}_+ \simeq X/\mathbf{C}_+ \simeq Y$ is affine.

Cancellation Conjecture (Zariski-Ramanujam): For any translation on $\mathbf{C}^n$ we have $\mathbf{C}^n//\mathbf{C}_+ \simeq \mathbf{C}^{n-1}$.

Yes, for $n \leq 3$ (Fujita, 1979).
That is, any translation on $\mathbf{C}^3$ is conjugate in $\mathrm{Aut}\mathbf{C}^3$ to a translation $(x_1, x_2, x_3) \to (x_1, x_2, x_3 + t)$ from Example $(2')$.

Definition. Two G-actions Φ_1 and Φ_2 on X are equivalent if $\Phi_2 = \alpha \circ \Phi_1 \circ \alpha^{-1}$ for some $\alpha \in \mathrm{Aut}X$.

Classification Problem. For G-actions on X with given properties describe all equivalence classes.

(Jung-van der Kulk) Aut$\mathbf{C}^2$ *is the amalgamated product* $\mathcal{A}_2 *_{\mathcal{H}_2} \mathcal{J}_2$ *where* $\mathcal{H}_n = \mathcal{A}_n \cap \mathcal{J}_n$.

Jonquière subgroup and subgroups of affine transformation of Aut$\mathbf{C}^n$:

$$\mathcal{J}_n = \{\varphi = (\varphi_1, \ldots, \varphi_n) | \varphi_i \in \mathbf{C}[x_1, \ldots, x_i] \forall i\}$$

$$\mathcal{A}_n = \{\varphi = (\varphi_1, \ldots, \varphi_n) | \varphi_i \in \mathbf{C}[x_1, \ldots, x_n],\ \deg \varphi_i = 1 \forall i\}.$$

Algebraic subgroup of Aut$\mathbf{C}^2$ is of bounded length in this product (Wright, 1979).
Hence it is isomorphic to a subgroup of one of factors (Serre, 1980).

Corollary. (Gutwirth, Rentchler, 1960's). *Every* $\mathbf{C}^*$*-action on* $\mathbf{C}^2$ *is equivalent to linear one, every* $\mathbf{C}_+$*-action on* $\mathbf{C}^2$ *is equivalent to a triangular one. In particular, every free* $\mathbf{C}_+$*-action on* $\mathbf{C}^2$ *is a translation.*

Nagata's automorphism

$$(x, y, z) \to (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z)$$

is not a composition of Jonquière and affine transformations (Shestakov, Umirbaev, 2004).

Linearization Theorem (M. Koras. P. Russell, 1999). *Every* $\mathbf{C}^*$*-action on* $\mathbf{C}^3$ *is equivalent to a linear one.*

Corollary (Popov, 2001, see also Kraft-Popov). *Every action of a connected reductive group on* $\mathbf{C}^3$ *is linearizable, i.e. it is equivalent to a representation.*

$\exists$ non-linearizable actions of reductive groups different from tori on $\mathbf{C}^4$ (Schwarz, 1989). Actually for any such a group $\exists$ such an action on some $\mathbf{C}^n$, $n \geq 4$ (Knop, 1991).

$\exists$ non-linearizable actions of finite groups on $\mathbf{C}^4$ (Jauslin-Moser, Masuda, Petrie, 1991).

$\exists$ a non-linearizable $\mathbf{R}^*$-action on $\mathbf{R}^4$ (Asanuma, 1999).

$\exists$ a non-linearizable analytic $\mathbf{C}^*$-action on $\mathbf{C}^3$ (Derksen, Kutzschebauch, 1998).

Remark. Asanuma's construction would work for $\mathbf{C}^*$-actions on $\mathbf{C}^4$ if $\exists$ a non-rectifiable embedding of $\mathbf{C}$ into $\mathbf{C}^3$. (Each embedding $\mathbf{C} \hookrightarrow \mathbf{C}^n$ is rectifiable for $n \geq 4$ (Jelonek, 1987) and $n = 2$ (the AMS theorem)).

Elements of proof of Linearization theorem.

Hard case: $\mathbf{C}^*$-action Φ on $\mathbf{C}^3$ has one fixed point o; the induced $\mathbf{C}^*$-action Ψ on $T_o\mathbf{C}^3 \simeq \mathbf{C}^3_{x,y,z}$ is given by $(x, y, z) \to (\lambda^{-a}x, \lambda^b y, \lambda^c z)$ where $a, b, c > 0$.

$T_o\mathbf{C}^3//\Psi \simeq \mathbf{C}^2/\mathbf{Z}_d,\ d = a/\mathrm{GCD}(a, b)\mathrm{GCD}(a, c)$.

$S = \mathbf{C}^3//\Phi$ is contractible with one singular point s_0 of analytic type $\mathbf{C}^2/\mathbf{Z}_d$ and $\bar{\kappa}(S) = -\infty$.

When a, b, and c are pairwise prime then $S \simeq T_o\mathbf{C}^3/\Psi \Rightarrow$ Linearization theorem. One can show $S \simeq T_o\mathbf{C}^3/\Psi$ for $\bar{\kappa}(S \setminus s_0) = -\infty$.

(Koras, Russell, coming) *Let S be a normal contractible surface of $\bar{\kappa}(S) = -\infty$ with quotient singularities only. Then $\bar{\kappa}(S_{\mathrm{reg}}) = -\infty$.*

The case of non-pairwise prime a, b, and c can be reduced to the pairwise prime case provided some special class of contractible (Koras-Russell) threefolds are exotic algebraic structures on $\mathbf{C}^3$, i.e. they are diffeomorphic to $\mathbf{R}^6$ but not isomorphic to $\mathbf{C}^3$.

Remark. Every smooth affine contractible variety of dimension at least 3 is diffeomorphic to a real Euclidean space (Choudary, Dimca, 1994).

(Makar-Limanov, 1996; Kaliman, Makar-Limanov, 1997) *Koras-Russell threefolds are exotic structures.*

The Russell cubic R is given by $x+x^2y+z^2+t^3 = 0$ in $\mathbf{C}^4$.

R is diffeomorphic to $\mathbf{R}^6$, $\bar{\kappa}(R) = -\infty$, R admits dominant morphism from $\mathbf{C}^3$.

Remark. $\exists$ one-to-one correspondence between $\mathbf{C}_+$-actions on X and locally nilpotent derivations (LND) on A.

Makar-Limanov invariant

$$\mathrm{AK}(X) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Ker}\, \partial;$$

in other words it is subalgebra of functions invariant under any $\mathbf{C}_+$-action on X.

$\mathrm{AK}(\mathbf{C}^n) = \mathbf{C}$ while $\mathrm{AK}(R) = \mathbf{C}[x]|_R$, i.e. R is not isomorphic to $\mathbf{C}^3$ (but what about biholomorphic?).

Idea of computation of Makar-Limanov invariant.

Step 1. Introduce associated affine variety $\hat{X}$ and affine domain $\hat{A} = \mathbf{C}[\hat{X}]$ for X and A with a map $A \to \hat{A}$, $a \to \hat{a}$ so that $\forall\ \partial \in \mathrm{LND}(A) \setminus 0\ \exists!$ an associated $\hat{\partial} \in \mathrm{LND}(\hat{A}) \setminus 0$.

C – a germ of a smooth curve at $o \in C$, $C^* = C \backslash o$, $\rho : \mathcal{X} \to C$ – an affine morphism such that $\mathcal{X}$ is normal, $\hat{X} := \rho^*(o)$ is reduced irreducible, $\mathcal{X}^* := \mathcal{X} \setminus \rho^{-1}(o) \simeq X \times C^*$ over C^*.

∂ defines a LND on $\mathcal{X}^*$ unique up to multiplication by a function on C^*. Choose this factor so that it extends to LND δ on $\mathcal{X}$ with $\hat{\partial} = \delta|_{\hat{X}} \neq 0$. ($\hat{a}$ is defined via a similarly.)

Example. $\rho : \mathcal{R} = \{cx + x^2y + z^2 + t^3 = 0\} \to C \simeq \mathbf{C}$. For $c \neq 0$, $\rho^{-1}(c) \simeq R$ while $\rho^{-1}(0) \simeq \hat{R} = \{x^2y + z^2 + t^3 = 0\}$.

Step 2. $\deg_\partial(a) = \min\{n | \partial^{n+1}(a) = 0\}$, i.e. $\operatorname{Ker} \partial = \{a \in A | \deg_\partial(a) = 0\}$.
Use $\deg_{\hat{\partial}}(\hat{a}) \leq \deg_\partial(a)$.

Say, $\deg_\partial(y) \geq \deg_{\hat{\partial}}(\hat{y}) \geq 2$ for R. Using different associated varieties $\Rightarrow$
$\forall\ a \in \mathbf{C}[R]$ with $\deg_\partial(a) \leq 1$ is a restriction of $p \in \mathbf{C}[x, z, t]$.

Fact. $\forall a \in A$, $a = a_2/a_1$ where $a_1 \in \operatorname{Ker}\partial$ and a_2 is from algebra generated by $b \in A$ with $\deg_\partial(b) = 1$ over $\operatorname{Ker}\partial$.

$\Rightarrow$ on R we have $y = p(x,z,t)/q(x,z,t)$ with $q(x,z,t) \in \operatorname{Ker}\partial$. On the other hand $y = -(x + z^2 + t^3)/x \Rightarrow x \in \operatorname{Ker}\partial$.

Limitation of Makar-Limanov invariant.

1. Is $R \times \mathbf{C}$ exotic?

2. Hypersurface $D = \{uv = p(\bar{x}\} \subset \mathbf{C}^{n+2}_{u,v,\bar{x}}$ with $n \geq 2$ and smooth reduced $p^*(0) \subset \mathbf{C}^n$ has $\operatorname{AK}(D) = \mathbf{C}$ (Kaliman, Zaidenberg, 1999). If $p^*(0)$ is contractible then D is diffeomorphic to $\mathbf{R}^{2n+2}$

Example. $uv = x + x^2y + z^2 + t^3$.

Such D has Andersén-Lempert property (1992), i.e. the Lie algebra generated by algebraic integrable vector fields coincides with Lie algebra of all algebraic vector fields (Kaliman, Kutzschebauch, coming).

Free $\mathbf{C}_+$-actions.

(Gutwirth, 1961, Rentschler, 1968) any $\mathbf{C}_+$-action on $\mathbf{C}^2$ is triangular, i.e. $\Phi_t(x_1, x_2) = (x_1, x_2 + tp_2(x_1))$ in a suitable coordinate system

$\Rightarrow$ any free action is a translation.

There are non-triangular $\mathbf{C}_+$-actions on $\mathbf{C}^3$ (Bass, 1984) [1] since the fixed point set may not be a cylinder.

(Winkelmann, 1990) Not all free $\mathbf{C}_+$-actions on $\mathbf{C}^4$ are translations [2] since it may happen that $\mathbf{C}^4//\mathbf{C}_+$ is not isomorphic to $\mathbf{C}^4/\mathbf{C}_+$.

[1] More precisely, for $t \in \mathbf{C}_+$, $(x_1, x_2, x_3) \in \mathbf{C}^3$, and $u = x_1x_3 + x_2^2$ such an action may be given by

$$\Phi_t(x_1, x_2, x_3) = (x_1, x_2 + tx_1u, x_3 - 2tx_2u - t^2x_1u^2)$$

and $\partial(x_1) = 0, \partial(x_2) = x_1u$, and $\partial(x_3) = -2x_2u$ which implies that $\partial(u) = 0$. Hence $x_1x_3 + x_2^2 = 0$ is the fixed point set which is not a cylinder and, therefore, the action cannot be triangular.

[2] $\Phi_t(x_1, x_2, x_3, x_4) = (x_1, x_2 + tx_1, x_3 + tx_2 + t^2x_1/2, x_4 + t(x_2^2 - 2x_1x_3 - 1))$. The reason why this free $\mathbf{C}_+$-action is not a translation is that $\mathbf{C}^4/\mathbf{C}_+$ is not Hausdorff while in the case of translations $\mathbf{C}^4/\mathbf{C}_+ = \mathbf{C}^4//\mathbf{C}_+$ is affine.

Theorem. (Kaliman, 2004) *Let* Φ *be a* $\mathbf{C}_+$*-action on factorial three-dimensional* X *with* $H_2(X) = H_3(X) = 0$*. Suppose that the action is free and* $S = X//\Phi$ *is smooth.*

Then Φ *is a translation, i.e.* X *is isomorphic to* $S \times \mathbf{C}$ *and the action is generated by a translation on the second factor.*

Since $\mathbf{C}^3//\mathbf{C}_+ \simeq \mathbf{C}^2$ for any nontrivial $\mathbf{C}_+$-action (Miyanishi, 1980) we have

Corollary. *A free* $\mathbf{C}_+$*-action on* $\mathbf{C}^3$ *is a translation in a suitable coordinate system.*

Equivalently, every nowhere vanishing (as a vector field) locally nilpotent derivation on $\mathbf{C}^{[3]}$ *is a partial derivative in a suitable coordinate system.*

Theorem (Kaliman, Saveliev, 2004) *Let* Φ *be a* $\mathbf{C}_+$*-action on three-dimensional contractible* X*. Then the quotient* $X//\Phi$ *is a smooth contractible surface.*

Since smooth contractible surfaces are rational (Gurjar, Shastri) we have

Corollary 2. *If a contractible threefold X admits a nontrivial $\mathbf{C}_+$-action, then X is rational.*

Remark. This is a partial answer to the Van de Ven conjecture in dimension 3 which states that smooth contractible affine algebraic varieties are rational.

(For smooth contractible affine threefolds with a nontrivial $\mathbf{C}^*$-action rationality is proven by Gurjar, Shastri, and Pradeep).

Element of proof of smoothness of the quotient for contractible three-dimensional X.

Quotient morphism $\pi : X \to X//\Phi$ is surjective

$\Rightarrow X//\Phi$ is contractible and has at worst quotient singularities whose links are homology spheres

$\Rightarrow$ by theorems of Prill, and Brieskorn (about local fundamental groups of quotient singularities) $\Rightarrow$ $X//\Phi$ has at worst E_8–singularities (i.e. singularities of type $x^2 + y^3 + z^5 = 0$)[3].

The link of an E_8–singularity is a Poincaré homology 3-sphere $\mathcal{P}$.

Link $X//\Phi$ at infinity is also a homology 3-sphere.

$\Rightarrow$ if $X//\Phi$ does have a singularity then there is a simply connected homology cobordism between $\mathcal{P}$ and another homology 3-sphere.

But this is impossible (Taubes, 1987; see also Fintushel and Stern, 1990).

$\Rightarrow$ $X//\Phi$ is smooth.

[3] We use the fact that E_8 is the only quotient singularity with a perfect local fundamental group.

$\mathbf{C}^*$-actions on affine surfaces.

S - a normal affine surface with an effective $\mathbf{C}^*$-action Φ
$B = \mathbf{C}[S]$ -algebra of regular functions so that
$B = \oplus_{i \in \mathbf{Z}} B_i = B_{\geq 0} \oplus_{B_0} B_{\leq 0}$

F - the set of fixed points of Φ
$C = (S \setminus F)/\Phi$ - a curve.

Dolgachev-Pikhman-Demazure (DPD) presentation (Flenner, Zaidenberg, 2003; Kollar)

Elliptic case: C is smooth projective and $\exists$ a $\mathbf{Q}$-divisor D on C so that $B = \oplus_{i \geq 0} H^0(C, \mathcal{O}(\lfloor iD \rfloor) u^i$ where $\lfloor E \rfloor$ is the integral part of a $\mathbf{Q}$-divisor E.
Parabolic case: C is smooth affine and $\exists$ a $\mathbf{Q}$-divisor D on C so that $B = \oplus_{i \geq 0} H^0(C, \mathcal{O}(\lfloor iD \rfloor) u^i$.

Hyperbolic case: C is affine smooth and $\exists$ $\mathbf{Q}$-divisors D_+ and D_- on C so that $D_+ + D_- \leq 0$,
$B_{\geq 0} = \oplus_{i \geq 0} H^0(C, \mathcal{O}(\lfloor iD_+ \rfloor) u^i$ and
$B_{\leq 0} = \oplus_{i \leq 0} H^0(C, \mathcal{O}(-\lfloor iD_- \rfloor) u^i$.

Smooth surfaces with more than two equivalence classes of effective $\mathbf{C}^*$-actions are $\mathbf{C}^2$ and Danilov-Gizatullin surfaces (suggested by P. Russell).

Let $\mathbf{F}_n \to \mathbf{P}^1$ be a Hirzebruch surface over $\mathbf{P}^1$ and L be its section with $L^2 = k + 1$. If L is ample then $\mathbf{F}_n \setminus L$ is a Danilov-Gizatullin surface.

Theorem. *There are k equivalence classes of effective $\mathbf{C}^*$-actions on this surface.*

Theorem. (Flenner, Kaliman, Zaidenberg, coming) *Let Φ be an effective $\mathbf{C}^*$-action on a smooth affine surface S different from $\mathbf{C}^2$ or a Danilov-Gizatullin surface. Then any other effective $\mathbf{C}^*$-action is equivalent either to Φ or to Φ^{-1}. In particular, for such S its DPD-presentation is "essentially" unique.*