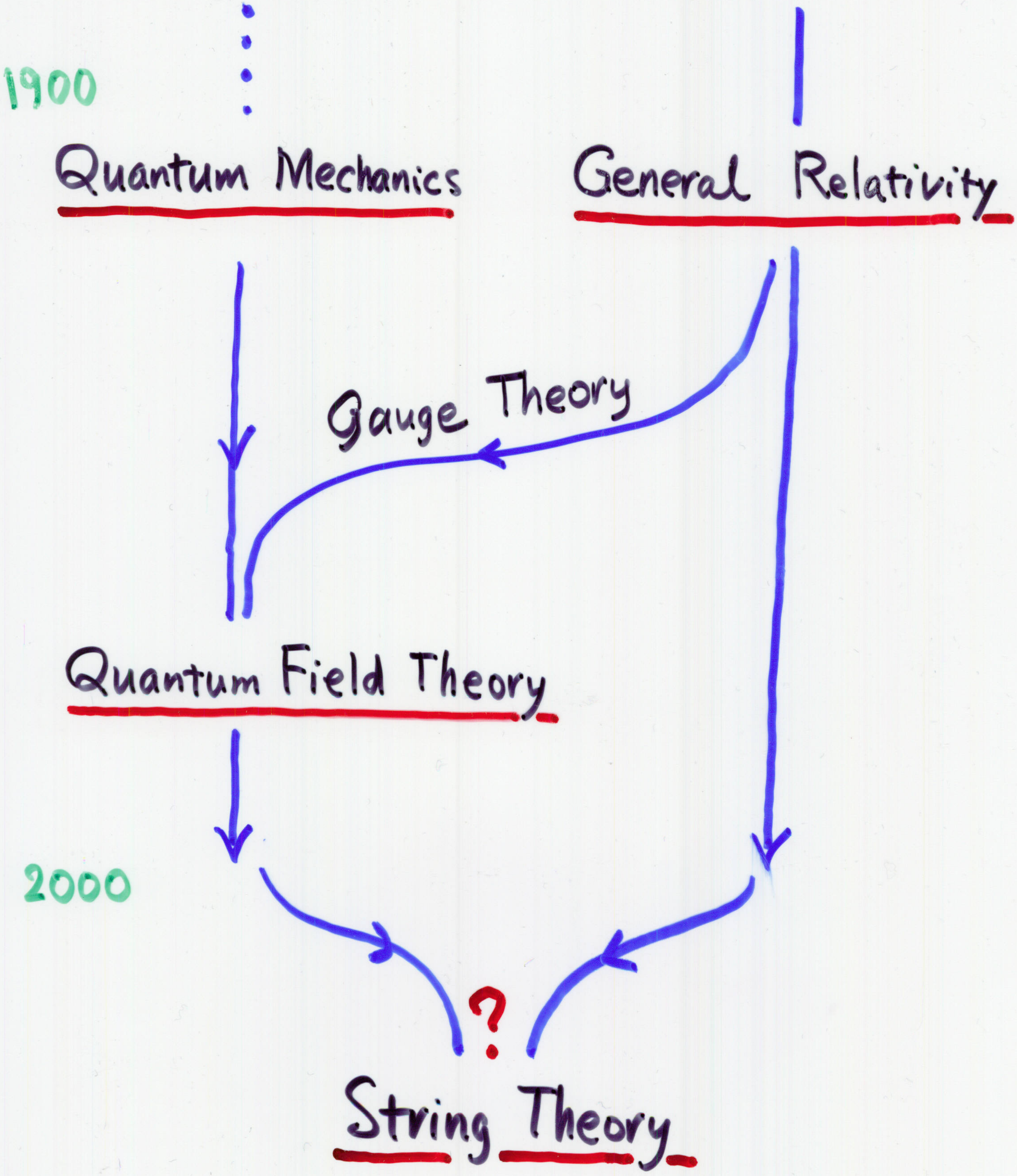
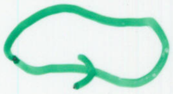

1900
Quantum Mechanics
General Relativity
Gauge Theory
Quantum Field Theory
2000
?
String Theory

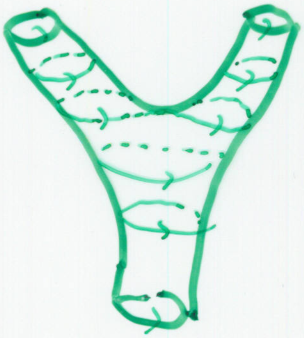
String Theory

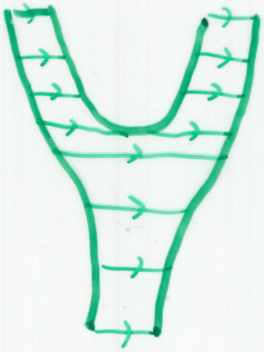
closed



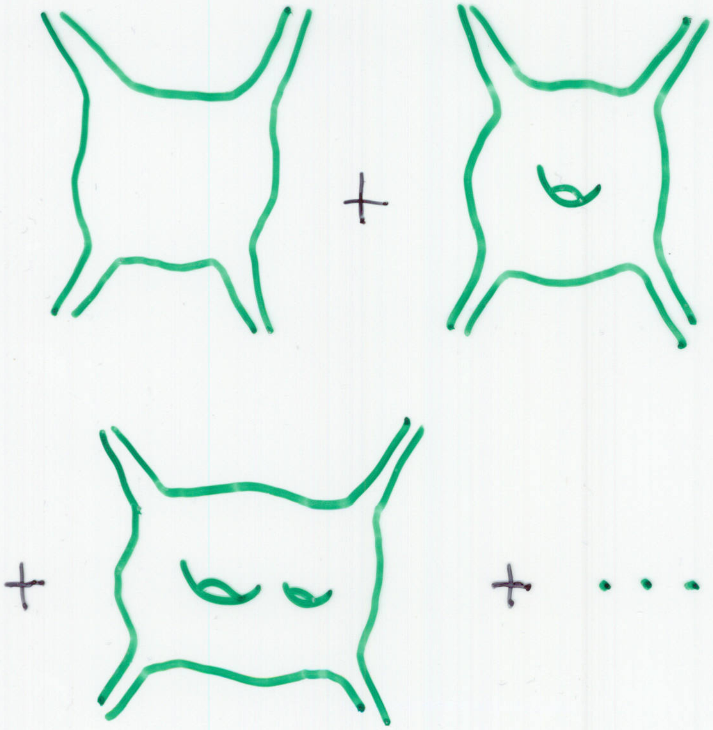
open

vibration modes → species of particles





→ interaction



scattering amplitudes

⋯> Quantum Field Theory on 2d surfaces

world sheets

Non-linear σ-model

Geometry $\leftrightarrow$ 2d QFT

(M, g) Riemannian manifold

variable = map $\phi : \Sigma \longrightarrow M$

Σ: worldsheet

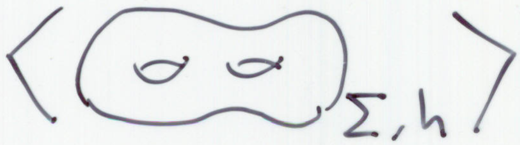
M: target space

action

$$S = \frac{1}{4\pi\alpha'}\int_{\Sigma} g_{IJ}(\phi(\sigma))\, h^{\mu\nu}(\sigma)\, \partial_\mu \phi^I(\sigma)\, \partial_\nu \phi^J(\sigma) \sqrt{h}\, d^2\sigma$$

$h_{\mu\nu}(\sigma)$: worldsheet metric

correlation function



$$= \int_{\mathrm{Map}(\Sigma, M)} \mathcal{D}\phi\; e^{-S(\phi)}$$

or

$$= \int_{\mathrm{Map}(\Sigma, M)} \mathcal{D}\phi\; e^{-S(\phi)}\, \mathcal{O}_1(\phi) \cdots \mathcal{O}_s(\phi)$$

Note : $S = S(g, h ; \Phi)$ is invariant under local rescaling $h_{\mu\nu}(\sigma) \to e^{2\lambda(\sigma)} h_{\mu\nu}(\sigma)$

In quantum theory, this conformal invariance may be **broken** (anomalous) ....

.... the measure $\mathcal{D}_{g,h}\Phi$ may **not** be invariant

Indeed, $h_{\mu\nu} \to e^{2\lambda} h_{\mu\nu}$ effectively chages $g_{IJ} \to g_{IJ}(\lambda)$: i.e.

$$\mathcal{D}_{g, e^{2\lambda}h}\Phi = \mathcal{D}_{g(\lambda), h}\Phi$$

where

$$\frac{d}{d\lambda} g_{IJ}(0) = -R_{IJ} + \frac{\alpha'}{2} R_{IKLM} R_J{}^{KLM} + O(\alpha'^2)$$

$\uparrow$ Ricci tensor $\qquad$ $\uparrow \nearrow$ Riemannian Curvature

Conformal invariance ⇒

$$R_{IJ} \underline{- \frac{\alpha'}{2} R_{IKLM} R_J{}^{KLM} + \cdots} = 0$$

Einstein equation modified by

Stringy quantum correction.

The Correction is large for large curvature

(≡ small Volume)

→ NLσ-model (= relation to geometry)

Stops to make sense at $\underset{g \lesssim \alpha'}{\underline{\text{small } M}}$!

But 2d QFT still makes sense!!

Mirror symmetry is an equivalence of string theories formulated on different "target spaces"

$$\underline{\text{String theory on } M} \cong \underline{\text{String theory on } \tilde{M}}$$

s.t.

$$\left[\begin{array}{c}\text{Symplectic Geometry}\\ \text{of } M\\ + \text{ Stringy quantum}\\ \text{Correction}\end{array}\right] \leftrightarrow \left[\begin{array}{c}\text{Algebraic Geometry}\\ \text{of } \tilde{M}\\ + \text{ Stringy quantum}\\ \text{Correction}\end{array}\right]$$

$$\left[\begin{array}{c}\text{Algebraic } \cdots\\ \cdots\ M\\ + \sim\sim\end{array}\right] \leftrightarrow \left[\begin{array}{c}\text{Symplectic } \cdots\\ \cdots\ \tilde{M}\\ + \sim\sim\end{array}\right]$$

← we need $\boxed{\mathcal{N}=(2,2) \text{ supersymmetry in } 1+1 \text{ dimensions}}$

Supersymmetric σ-model

$(\overset{\text{time}}{t}, \overset{\text{space}}{\sigma})$

$\phi : \Sigma \to M$ ($\Sigma =$ Minkowski space $\mathbb{R}^{1+1}$)

$\psi_\pm$: anticommuting (fermionic) spinors with values in $\phi^* TM$

$$S = \frac{1}{4\pi\alpha'}\int \Big[\, g_{IJ}\left(\partial_t\phi^I\partial_t\phi^J - \partial_\sigma\phi^I\partial_\sigma\phi^J\right)$$

$$+\, i g_{IJ}\,\psi_-^I(\nabla_t+\nabla_\sigma)\psi_-^J + i g_{IJ}\,\psi_+^I(\nabla_t-\nabla_\sigma)\psi_+^J$$

$$+\, \frac{1}{2} R_{IJKL}\,\psi_+^I\psi_+^J\psi_-^K\psi_-^L \,\Big]\, dt\, d\sigma$$

$$\nabla_\mu\psi_\pm^I = \partial_\mu\psi_\pm^I + \partial_\mu\phi^J\,\Gamma^I_{JK}\,\psi_\pm^K \cdots \phi^*(\text{Levi-Civita})$$

(M, g) **Kähler** $\Rightarrow$ S is invariant under

holomorphic coordin

$$\begin{cases} \delta\phi^i = \epsilon_+\psi_-^i - \epsilon_-\psi_+^i\,, & \delta\phi^{\bar i} = \text{c.c.} \\ \delta\psi_\pm^i = \pm i\,\bar\epsilon_\mp(\partial_t\pm\partial_\sigma)\phi^i + \epsilon_\pm\Gamma^i_{jk}\psi_+^j\psi_-^k\,, & \delta\psi_\pm^{\bar i} = \text{c.c.} \end{cases}$$

.... (2,2) supersymmetry

$\epsilon_\pm$: complex parameter (real 2)

$[\delta_1, \delta_2] \propto \epsilon^1_\mp\epsilon^2_\mp(\partial_t \pm \partial_\sigma)$

Another example :

(M, g) Kähler , $W : M \to \mathbb{C}$ holomorphic function with isolated critical points.

e.g. $M = \mathbb{C}^n$, W = polynomial of n-variables

$$S = \frac{1}{2\pi\alpha'}\int\Big[\sum_{i=1}^{n}\Big(|\partial_t\phi^i|^2 - |\partial_\sigma\phi^i|^2 + i\overline{\psi_-^i}(\partial_t+\partial_\sigma)\psi_-^i + i\overline{\psi_+^i}(\partial_t-\partial_\sigma)\psi$$

$$-\underbrace{\sum_{i=1}^{n}|\partial_i W(\phi)|^2}_{\text{potential}} - \sum_{i,j=1}^{n}\Big(\underbrace{\partial_i\partial_j W(\phi)\psi_+^i\psi_-^j}_{\text{'Yukawa' coupling}} + \text{c.c.}\Big)\Big]d\tau d\sigma$$

S is invariant under

$$\begin{cases} \delta\phi^i = \epsilon_+\psi_-^i - \epsilon_-\psi_+^i & \delta\phi^{\bar{i}} = \text{c.c.} \\ \delta\psi_\pm^i = \pm i\overline{\epsilon_\mp}(\partial_t \pm \partial_\sigma)\phi^i \,[\text{illegible}]\, \epsilon_\pm\overline{\partial_i W} & , \delta\psi_\pm^{\bar{i}} = \text{c.c.} \end{cases}$$

… (2,2) Landau-Ginzburg model

with superpotential $W(\phi^1, \ldots, \phi^n)$

$$\underbrace{\text{Symmetry} \xrightarrow{\text{Nöther}}}_{\text{Classical}} \underbrace{\text{Conserved charge} \xrightarrow{\text{Ward}} \text{Symmetry Generator}}_{\text{quantum}}$$

$$\begin{tikzcd}
(2,2)\ \text{SUSY} \arrow[r] & Q_+, \arrow[rr, leftrightarrow, bend left, "\text{c.c.}"] & Q_-, \arrow[rr, leftrightarrow, bend left, "\text{c.c.}"] & \overline{Q}_+, & \overline{Q}_-, & H, P, M \\
 & & \text{supercharges} & & & \text{Poincaré}
\end{tikzcd}$$

$\{A, B\} = AB + BA$

$[A, B] = AB - BA$

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0$$

$$\{ Q_\pm , \overline{Q}_\pm \} = H \pm P$$

$$\{ Q_+, Q_- \} = \{ Q_+, \overline{Q}_- \} = \text{c.c.} = 0$$

$$i[M, Q_\pm] = \mp Q_\pm , \quad i[M, \overline{Q}_\pm] = \mp Q_\pm$$

In addition, we may have R-charges :

depends on theory

$$F_V \text{ (vector) and } F_A \text{ (axial)}$$

$$[F_V, Q_\pm] = -Q_\pm , \quad [F_V, \overline{Q}_\pm] = \overline{Q}_\pm$$

$$[F_A, Q_\pm] = \mp Q_\pm , \quad [F_A, \overline{Q}_\pm] = \pm \overline{Q}_\pm$$

R-symmetry (continued)

NLσ-M $U(1)_V : \psi^i_\pm \to e^{-i\alpha}\psi^i_\pm$

$U(1)_A : \psi^i_\pm \to e^{\mp i\beta}\psi^i_\pm$ $\Big\}$ Classically both symmetries

Quantum: $U(1)_V$ ··· always a symmetry (F_V exists)

$U(1)_A$ ···· anomalous if $c_1(T_M) \neq 0$

$U(1)_A$ is a symmetry iff $c_1(M) = 0$ (Calabi-Yau mfd)

(F_A exists)

LG model look at $W''(\Phi)\psi_+\psi_-$

$U(1)_A : \psi^i_\pm \to e^{\mp i\beta}\psi^i_\pm$ symmetry (F_A exists)

$U(1)_V$ ··· not always a symmetry

If $\exists\ \phi^j \to e^{i\alpha q_j}\phi^j$ s.t. $W(e^{i\alpha q}\phi) = e^{2i\alpha}W(\phi)$,

quasi homogeneous

then $U(1)_V : \phi^j \to e^{i\alpha q_j}\phi^j,\ \psi^j_\pm \to e^{i\alpha(q_j-1)}\psi^j_\pm$

is an $U(1)$ vector R-symmetry (F_V exists)

Supersymmetric ground states

quantize a (2,2) theory on a periodic cylinder

$$H = \frac{1}{2}\{Q_+, \overline{Q}_+\} + \frac{1}{2}\{Q_-, \overline{Q}_-\} \qquad \overline{Q}_\pm = (Q_\pm)^\dagger$$

$$\Rightarrow H \geq 0 \, , \quad \underline{= 0 \text{ iff } Q_+ = \overline{Q}_+ = Q_- = \overline{Q}_- = 0}$$

·· SUSY ground states $\mathcal{H}_{SUSY} \subset \mathcal{H}$

Write $Q_A = \overline{Q}_+ + Q_-$

$Q_B = \overline{Q}_+ + \overline{Q}_-$

$(Q, F) = (Q_A, F_A)$ or (Q_B, F_V) obey

(1) $\{Q, Q^\dagger\} = 2H$

(2) $Q^2 = 0$

(3) $[F, Q] = Q$

When F has integral eigenvalues only ($\Rightarrow \mathbb{Z}$-grading)

(3)⇒ $\cdots \xrightarrow{Q} \mathcal{H}^{q-1} \xrightarrow{Q} \mathcal{H}^{q} \xrightarrow{Q} \mathcal{H}^{q+1} \xrightarrow{Q} \cdots$ (2)⇒ Complex

$$\mathcal{H}^{q}_{SUSY} \overset{(1)}{\cong} H^{q}(Q) := \frac{\mathrm{Ker}\, Q : \mathcal{H}^{q} \to \mathcal{H}^{q+1}}{\mathrm{Im}\, Q : \mathcal{H}^{q-1} \to \mathcal{H}^{q}}$$

Witten index $\mathrm{Tr}(-1)^F := \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} = \sum_q (-1)^q \dim H^q(Q)$

⋯ Euler characteristic of Q-complex

NLσ·M on a Kähler mfd M^n :

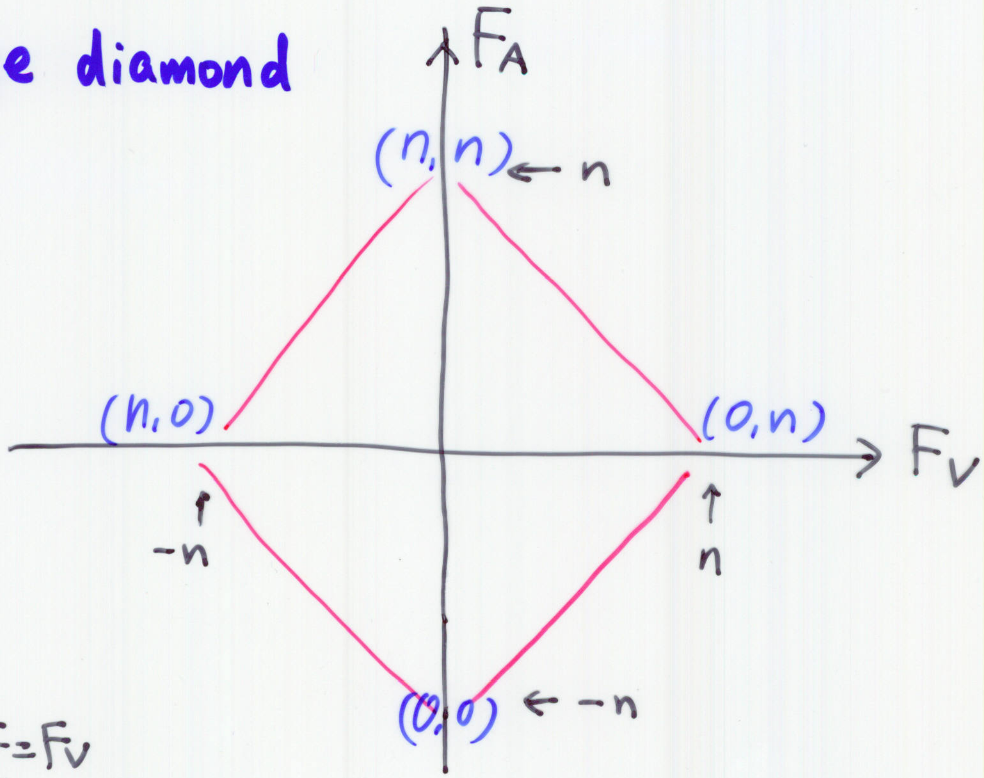
$$\mathcal{H}_{SUSY} \cong \underbrace{\bigoplus_{p,q=1}^{n} H^{p,q}(M)}_{\text{Dolbeault cohomology}}$$

M : Calabi-Yau ⇒ F_V, F_A both exist

$H^{p,q}(M)$ has $F_V = -p+q$

$F_A = p+q-n$

Hodge diamond



we $F=F_V$

$$\mathrm{Tr}(-1)^F = \sum_{p,q} (-1)^{p+q} \dim H^{p,q}(M) = \chi(M) \text{ Euler \# of } M$$

Twisting to Topological Field Theory

When Σ is curved, SUSY is lost: $\delta S = \int \underbrace{\overset{\neq 0}{D_\mu \epsilon}}_{\nexists \text{ covariantly const spinor}} G^\mu \sqrt{h}\, d^2\sigma$

If a supercharge were scalar, it would be preserved

Scalar supercharges are obtained by **Twisting**

re-definition of spin

	F_V	F_A	iM	$iM_A = iM + F_V$	$iM_B = iM + F_A$
$\overline{Q}_+$	1	1	−1	**0**	**0**
$\overline{Q}_-$	1	−1	1	2	**0**
Q_+	−1	−1	−1	−2	−2
Q_-	−1	1	1	**0**	2
				A-twist	B-twist

※ A-twist (B) is possible if F_V (F_A) exists and is integral

※ After twisting we have $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$

where $Q = Q_A = \overline{Q}_+ + Q_-$ for A-twist.

$Q = Q_B = \overline{Q}_+ + \overline{Q}_-$ for B-twist.

Correlation functions of Q_B-closed fields in B-twisted model are independent of worldsheet metric $h_{\mu\nu}$

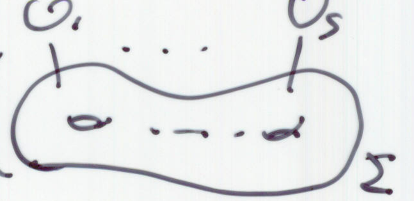
$$h \to h + \delta h \ : \ \delta\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \langle \int \delta h^{\mu\nu} \{Q_B, G_{\mu\nu}\} \mathcal{O}_1 \cdots \mathcal{O}_s \rangle$$
$$= \langle \{Q_B, \int \delta h^{\mu\nu} G_{\mu\nu} \mathcal{O}_1 \cdots \mathcal{O}_s\} \rangle = 0$$

Also $\langle (\mathcal{O}_1 + \{Q_B, \Psi_1\}) \cdots (\mathcal{O}_s + \{Q_B, \Psi_s\}) \rangle = \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle$

<u>Define</u> $\mathcal{R}_B$ = Q_B-cohomology classes of fields

($\cong$ Q_B-cohomology classes of states $\cong \mathcal{H}_{SUSY}$)

Then

$(\mathcal{O}_1, \cdots, \mathcal{O}_s) \in \mathcal{R}_B^s \mapsto \langle$  $\rangle \in \mathbb{C}$

is a ($\mathbb{Z}_2$ graded) symmetric function on $\mathcal{R}_B$ depending only on the topology of Σ.

Same can be said on A-twisted model :

$\mathcal{R}_A$ = Q_A-cohomology classes of fields

($\cong$ — states $\cong \mathcal{H}_{SUSY}$)

$R = R_B$ or R_A obey

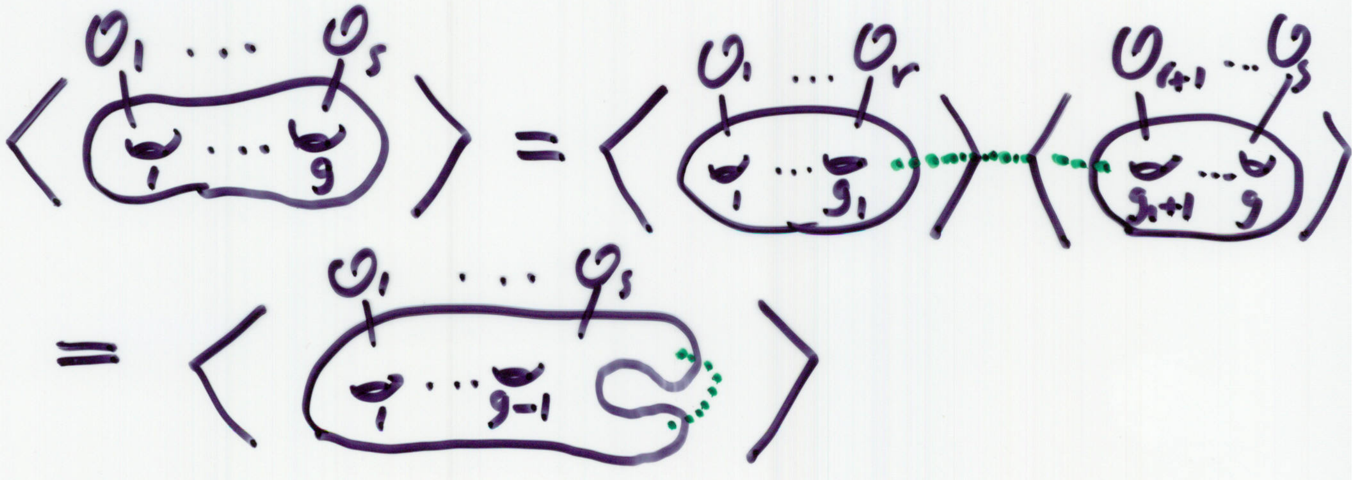
① "1" $\in R$ $\quad \langle 1\, O_1 \cdots O_s \rangle_g = \langle O_1 \cdots O_s \rangle_g$

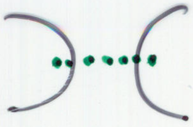
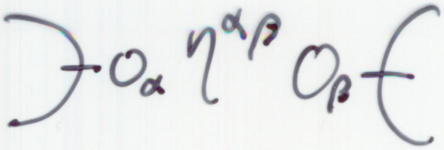
② $\langle O_1 O_2 \rangle_0 =$  is a

non-degenerate bilinear form on R.

thus $\{O_\alpha\} \subset R$ linear basis

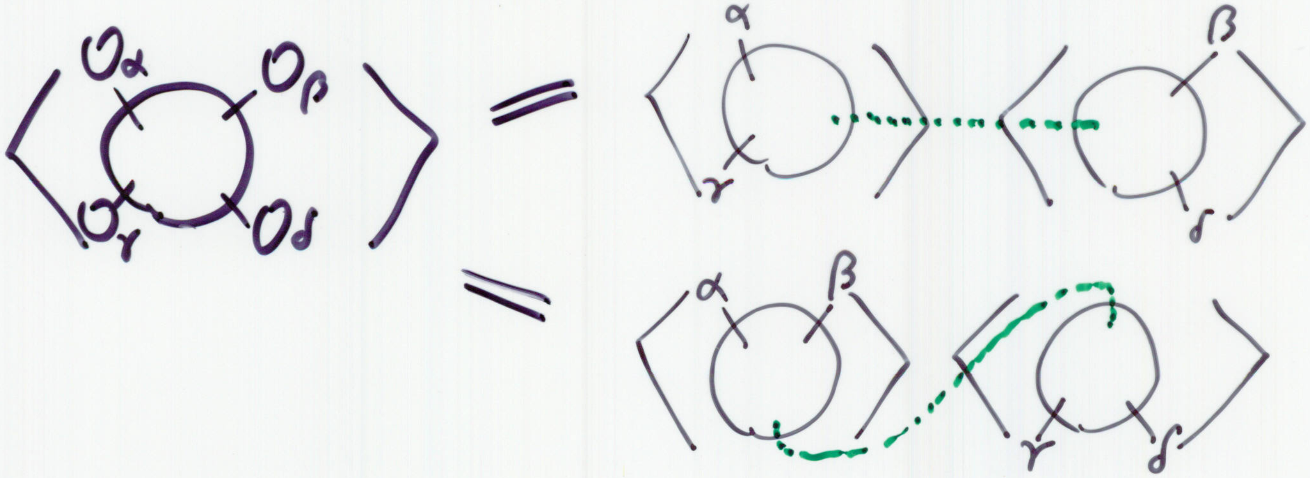
$\eta_{\alpha\beta} = \langle O_\alpha O_\beta \rangle_0$ is invertible $\left(\eta^{\alpha\beta} = \text{inverse} \right)$

③ 

where  means $\sum_{\alpha,\beta}$ 

Consequence :

define $C^{\alpha}_{\beta\gamma} = \eta^{\alpha\delta}\langle O_\delta O_\beta O_\gamma \rangle_0$



i.e. $C^{\lambda}_{\alpha\gamma} C^{\delta}_{\beta\lambda} = C^{\lambda}_{\beta\alpha} C^{\delta}_{\lambda\gamma}$ ——— (☆)

define a <u>product</u> $\mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ by

$$O_\alpha \cdot O_\beta = \sum_\gamma O_\gamma C^{\gamma}_{\alpha\beta}$$

Then (☆) $\Longleftrightarrow$ $O_\beta \cdot (O_\alpha \cdot O_\gamma) = (O_\beta \cdot O_\alpha) \cdot O_\gamma$

$\mathcal{R}$ forms an associative algebra

$\mathcal{R}_B$: chiral ring $\qquad$ $\mathcal{R}_A$: twisted chiral ring

Examples

topological A-model (A-twisted NLσ-model)

M : compact Kähler mfd

$$\left.\begin{array}{l}\omega : \text{Kähler form} \\ [B] \in H^2(M,\mathbb{R})\end{array}\right\} \ t = [\omega] - i[B]$$

$$R = H^{\bullet}_{DR}(M)$$

$$\langle O_1 O_2 O_3 \rangle_0 = \int_M O_1 \wedge O_2 \wedge O_3 + \sum_{d \in H_2(M,\mathbb{Z})} \underbrace{n_d(O_1, O_2, O_3)}_{\text{GW inv.}} e^{-\int_d t}$$

$$O_i \overset{\text{P.D.}}{\longleftrightarrow} C_i \left(\begin{array}{c}\text{integral} \\ \text{cycle}\end{array}\right) : n_d = \text{"\# holomorphic spheres passing through } C_1, C_2, C_3\text{"}$$

$$\langle O_1 O_2 \rangle_0 = \int_M O_1 \wedge O_2 \qquad (\text{non-degenerate})$$

R = quantum cohomology ring of M

topological B-model (B-twisted NL σ-model)

M^n : compact Calabi-Yau mfd (dim=n)

$\Omega \in H^{n,0}(M)$ holomorphic volume form

$$R = \bigoplus_{p,q} H^{0,p}(M, \Lambda^q T_M)$$

with **obvious product**

$\mu_i \in H^{0,p_i}(M, \Lambda^{q_i} T_M)$

$$\langle \mu_1 \mu_2 \mu_3 \rangle_0 = \begin{cases} \int_M \Omega \wedge (\Omega \cdot \mu_1 \wedge \mu_2 \wedge \mu_3) & \text{if } p_1+p_2+p_3 \\ & = q_1+q_2+q_3 = n \\ 0 & \text{Otherwise.} \end{cases}$$

topological LG model (B-twisted LG model)

W = polynomial of $X_1, \dots, X_n$ $(\# \mathrm{Crit}\, W < \infty)$

fix $\Omega = dX_1 \wedge \cdots \wedge dX_n$

$R = \mathbb{C}[X_1, \dots, X_n] / (\partial_1 W, \dots, \partial_n W)$ Jacobi ring

$$\langle f_1 f_2 f_3 \rangle_0 = \mathrm{res}_{W,\Omega}(f_1 f_2 f_3)$$

$$= \sum_{p \in \mathrm{Crit}(W)} \mathrm{Res}_p \left(f_1 f_2 f_3 \frac{dX_1 \wedge \cdots \wedge dX_n}{\partial_1 W \cdots \partial_n W} \right)$$

- If $\mathrm{Crit}(W)$ all non-degenerate $\left(\text{i.e. } \det \partial_i \partial_j W \neq 0 \right)$

$$\mathrm{Res}_p (\cdots) = \frac{f_1(p) f_2(p) f_3(p)}{\det \partial_i \partial_j W(p)}$$

non-degeneracy of $\langle f_1 f_2 \rangle_0$ easy to see

- In general, non-deg. of $\langle f_1 f_2 \rangle_0$: local duality theorem.

Mirror Symmetry

Two (2,2) SUSY QFT's are mirror to each other when they are equivalent as QFT's

$$\mathrm{QFT}_1 \cong \mathrm{QFT}_2$$

under which

$$(Q_+, \bar{Q}_+, \underline{Q_-, \bar{Q}_-}) \longleftrightarrow (Q_+, \bar{Q}_+, \underline{\bar{Q}_-, Q_-})$$

$$(F_V, F_A) \longleftrightarrow (F_A, F_V)$$

$$\text{A-twist} \longleftrightarrow \text{B-twist}$$

↓ consequence

$$R_A \longleftrightarrow R_B$$

GW invariants ⟷ Period integrals

quantum cohomology — Jacobi ring

residue paring

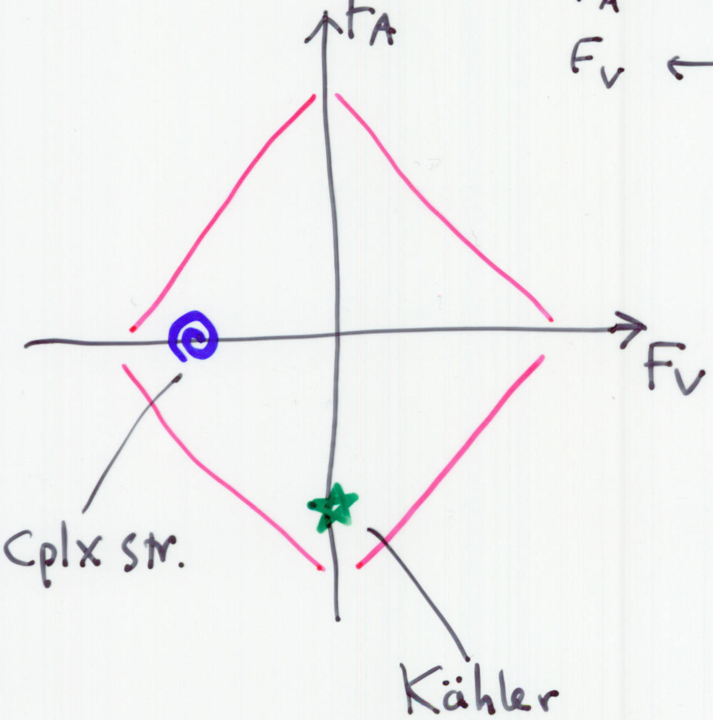
⋮ ⋮

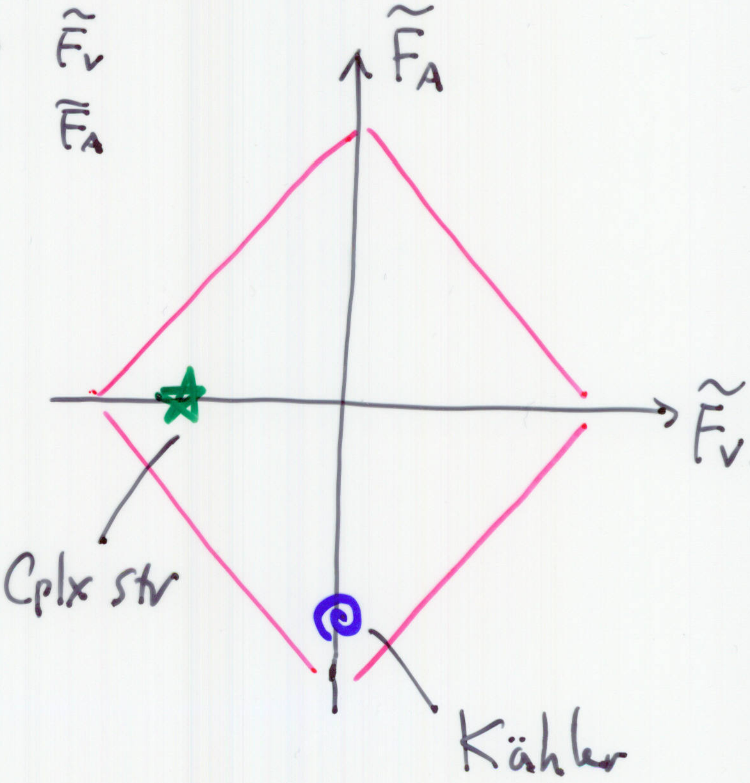
Vice versa

M (Calabi-Yau) $\xleftrightarrow{\text{mirror}}$ $\tilde{M}$ (Calabi-Yau)

$F_A \longleftrightarrow \tilde{F}_V$

$F_V \longleftrightarrow \tilde{F}_A$





Moduli space of (2,2) theories

$$\mathcal{M} = \underset{\substack{\uparrow \\ \text{Chiral} \\ \text{deformations}}}{\mathcal{M}_c} \times \underset{\substack{\uparrow \\ \text{twisted chiral} \\ \text{deformations}}}{\mathcal{M}_t}$$

Decoupling theorem

A-model correlators .... holomorphic functions on $\mathcal{M}_t$ constant along $\mathcal{M}_c$

B-model correlators .... holomorphic functions on $\mathcal{M}_c$ constant along $\mathcal{M}_t$

NLσ-model on M (CY mfd)

$\mathcal{M}_c$ = moduli space of complex structure of M.

(M : CY 3-fold $\Rightarrow$ "Special geometry"
··· determined by period integrals
of Ω on $H_3(M.\mathbb{Z})$)

$\mathcal{M}_t = \{$ complexified Kähler class $[\omega - iB] \} \subset H^2(M; \mathbb{C}/2\pi i\mathbb{Z})$

\+ stringy quantum correction

\+ analytic continuation ???

Use of Mirror Symmetry $M \leftrightarrow \tilde{M}$

$$\mathcal{M}_t(M) = \mathcal{M}_c(\tilde{M}) = \text{classical}$$

Example : quintic

$$M = \{ \underset{\text{degree 5}}{G(X_1, \ldots, X_5)} = 0 \} \subset \mathbb{CP}^4$$

$$\xleftrightarrow{\text{mirror}} \tilde{M} = \text{a resolution of the orbifold of}$$

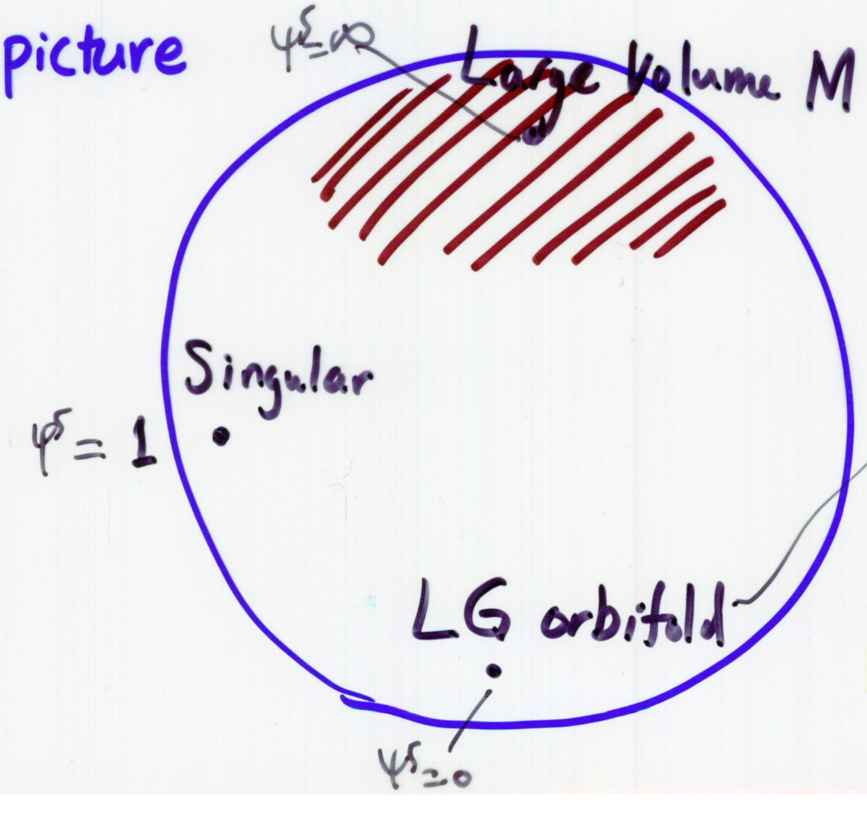
$$Z_1^5 + \cdots + Z_5^5 - 5\psi Z_1 \cdots Z_5 = 0 \quad \text{in } \mathbb{CP}^4$$

$$\text{by } (\mathbb{Z}_5)^3 : Z_i \to \omega_i Z_i \qquad \omega_i^5 = \omega_1 \cdots \omega_5 = 1$$

$$\mathcal{M}_t(M) = \mathcal{M}_c(\tilde{M}) = \{ \psi^5 \}$$

3 special points $\psi^5 = \begin{cases} 0 & \tilde{M} = \mathbb{Z}_5 \text{ symmetri} \\ 1 & \tilde{M} : \text{Conifold singular} \\ & \quad (\text{ODP}) \\ \infty & \tilde{M} : \text{union of} \\ & \quad \text{five } \mathbb{CP}^3\text{'s} \end{cases}$

picture



LG model with

$$W = G(X_1, \ldots, X_5)$$

modulo $\mathbb{Z}_5 : X_i \to e^{2\pi i/5} X_i$