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Lectures on Mirror Symmetry

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Abstract

1 Introduction

The discovery of general relativity and quantum mechanics are the biggest events in the history of twentieth century physics. General relativity is the theory of gravity that is relevant at long distances (celestial and universal scales) while quantum mechanics is the framework to describe physics at tiny scales (of atoms, nuclei, etc). Since then, physicists' dream has been to unify the two. Quantum mechanics successfully absorbed the special relativity, and developed into quantum field theory. Gauge principle that came out of general relativity is included in the framework of quantum field theory and resulted in quantum gauge theory, the prominent example of which is the standard model of particle physics. However, gravity itself is kept aside from this progress. A naive attempt, including gravity in the framework of quantum mechanics, suffers from difficulties originating from the blowing up strength at short distances. It also sacrifices the beauty of general relativity. It looks like some new framework is required. String theory is a candidate for a framework that truely unifies quantum mechanics and general relativity.

The two events in physics had enormous impact in mathematics. General relativity, which is described by Riemannian geometry, promoted progress in differential geometry. Quantum mechanics literally gave birth to new fields in mathematics such as functional analysis and operator algebra. One may expect that the right framework of unified theory is described by mathematical language that unifies also the fields in mathematics that are developed through general relativity and quantum mechanics. In the study of string theory, we indeed encounter unexpected relation between different fields in mathematics. Mirror symmetry, which is the theme of this lecture series, shows an example of such a relation. The fields that are involved are: complex algebraic/analytic geometry and symplectic geometry. It states that the complex geometry of one mathematical object is equivalent to the symplectic geometry of another object, when quantum correction induced by strings is included in both sides. But it will soon become clear that more surprise involving yet other fields is expected.

2 String Theory — An Introduction

The idea of string theory is to consider a string instead of a point particle as the elementary object. A string is a circle (closed string) or a segment (open string), and it can be oriented or unoriented. At long distances, a tiny string looks point-like, and various vibration modes are regarded as species of particles of various masses. Interaction among them corresponds to the process of splitting and joining of strings. In a spacetime manifold, a string sweeps out a two-dimensional surface, which is called the *worldsheet* of a string. The worldsheet for propagation of a closed or open string is just a cylinder or a strip. When the process of splitting/joining as well as creation/annihilation is included, it can be a surface of arbitrary topology. A scattering amplitude is given by the sum over all possible worldsheets with a fixed asymptotic condition specified by the incoming and outgoing particles. Obviously, we need to specify the weight of this summation, or the measure of the so called path-integral.

2.1 Non-linear sigma models

There is a standard way, based on *non-linear sigma models*, in which the worldsheet is regarded as a domain mapped to the target spacetime. The data to fix the theory is the target manifold M with a Riemannian metric g. The summation variable is the worldsheet Σ, its metric h and the map to the target $X : \Sigma \to M$. The non-linear sigma model action, included in the path-integral weight $\mathrm{e}^{-S_{\mathrm{NL}\sigma\mathrm{M}}}$, is given by

$$S_{\mathrm{NL}\sigma\mathrm{M}} = \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{IJ}(X(\sigma)) h^{\mu\nu}(\sigma) \partial_\mu X^I(\sigma) \partial_\nu X^J(\sigma) \sqrt{h} \mathrm{d}^2\sigma, \tag{2.1}$$

when Σ is oriented. $\sqrt{h}\mathrm{d}^2\sigma$ is the volume element. α' is some constant which has the dimension of lenght squared in the target space. This action is invariant under the diffeomorphism of the worldsheet $f : \Sigma \to \Sigma$; $(h, X) \to (f^*h, X \circ f)$ as well as the Weyl rescaling of the worldsheet metric $h(\sigma) \to \mathrm{e}^{\phi(\sigma)} h(\sigma)$. Weyl rescaling sometimes fails to be a symmetry — a certain condituion is required, as I shall explain below. Thus, the summation is over the topology of Σ, and for each topology, we have a path-integral over the space of (h, X) modulo the group of diffeomorphisms and Weyl rescalings. The latter space naturally has a Riemannian metric and the integration measure is formaly given by its volume form. The weight $\mathrm{e}^{-S_{\mathrm{NL}\sigma\mathrm{M}}}$ can be modified by a phase e^{iS_B} where

$$S_B = \frac{1}{2\pi\alpha'} \int_{\Sigma} X^* B \tag{2.2}$$

for a two-form B on M called the *B-field*. Another action that can be added is

$$S_D = \frac{1}{4\pi} \int_{\Sigma} R_h(\sigma) \Phi(X(\sigma)) \sqrt{h} \mathrm{d}^2\sigma \tag{2.3}$$

for a function Φ on M called the *dilaton* field, where R_h is the scalar curvature of the worldsheet. For a constant Φ, this gives a weight $\mathrm{e}^{-\chi\Phi}$ on the worldsheet of Euler number χ. This amounts to assigning the factor of e^{Φ} to each piece of pair of pants, when the worldsheet is decomposed into union of pairs of pants. Since a pair of pants is splitting/joining of closed strings, $g_s = \mathrm{e}^{\Phi}$ can be regarded as the string coupling constant.

Weyl rescaling symmetry, or *conformal invariance* is generically broken in the renormalization procedure. It is maintained in the quantum theory when the data (g, B, Φ) obey certain conditions, which can be written as differential equations in power series in α':

$$R_{IJ} - \frac{1}{4}H_I{}^{KL}H_{JKL} + 2\nabla_I\nabla_J\Phi + O(\alpha') = 0, \tag{2.4}$$

$$\nabla_K H^K_{IJ} - 2(\nabla_K\Phi)H^K_{IJ} + O(\alpha') = 0, \tag{2.5}$$

$$\frac{D-26}{3\alpha'} + 4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12}H^2 + O(\alpha') = 0, \tag{2.6}$$

where D is the dimension of the space. R_{IJ} and R are the Ricci tensor and the scalar curvature. $H = \mathrm{d}B$ is the "curvature" three-form of the B-field called the H-field. These equations can be regarded as the classical equation of motion for the spacetime fields (g, B, Φ). Since α' has dimension of length squared, the terms $O(\alpha')$ consists of higher curvature and higher derivatives of (g, B, Φ). For example, the first equation includes a term $-\frac{\alpha'}{2}R_{IKLT}R_J{}^{KLT}$. These terms are relatively small when the curvature is small and everything is slowly varying. Note that the flat space $R_{IJKL} = H_{IJK} = 0$ with $\Phi =$ constant is a solution if and only if the dimension is 26. As the curvature is increased, for example by rescaling $g \to \lambda g$ and taking λ smaller and smaller, the α' expansion stops to make sense. However, this does not necessarily mean that the theory ceases to exist beyond some small volume (or large curvature). It is just that the non-linear sigma model can no longer be used as a good description of the worldsheet theory. Abstractly, any conformal field theory with central charge 26 defines a classical solution of string theory.

Non-linear sigma model itself makes perfect sense as quantum field theory even without the conditions (2.4)-(2.6). It is simply that (g, B, Φ) change as the energy scale μ is changed (for example, cut-off dependence) — the left hand sides of (2.4)-(2.6) are the $\log\mu$ derivatives of $(g_{IJ}, B_{IJ}, 16\pi^2\Phi/\alpha')$. For example, consider the sigma model on the sphere S^N with $B = 0$, $\Phi =$ constant. Since S^N is Ricci positive, the equation $\mu \mathrm{d}g_{IJ}/\mathrm{d}\mu = R_{IJ} + O(\alpha')$ shows that S^N is larger and the curvature is smaller as we increase the energy scale, and it eventually flattens in the high energy limit. This is called the *asymptotic freedom*. In the low energy limit, on the other hand, S^N becomes smaller and the non-linear sigma model stops to be a good description — the theory must be described by something else. For spaces like hyperbolic space H^N with negative curvature, the situation is opposite. It flattens out in the low energy limit, but at high energies it is highy curved and the theory must be described by something else than the non-linear sigma model.

Coming back to backgrounds obeying (2.4)-(2.6), there is actually a problem at string loop diagrams, i.e. for the worldsheet of genus one and higher. The integration of the worldsheet geometry diverges at the boundary corresponding to a long neck, due to the propagation of tachyon — a particle of negative mass squared. One can actually construct a different string theory without such a problem — superstring theory. The first step is to introduce fermionic fields and supersymmetry on the worldsheet.

2.2 Supersymmetric sigma models

The sigma model action in flat Minkowski space is

$$S_b = \frac{1}{4\pi\alpha'}\int\Big[g_{IJ}\Big(\partial_t X^I\partial_t X^I - \partial_\sigma X^I\partial_\sigma X^J\Big) + B_{IJ}\Big(\partial_t X^I\partial_\sigma X^J - \partial_\sigma X^I\partial_t X^J\Big)\Big]\,\mathrm{d}t\mathrm{d}\sigma, \tag{2.7}$$

where t and σ are the time and space coordinates. To this system, we include anti-commuting (fermionic) fields $\psi_\pm$ which are spinors with values in the pull back by X of the tangent bundle of M. The action to be added to S_b is

$$\begin{aligned} S_f &= \frac{1}{4\pi\alpha'}\int\Big[ig_{IJ}\psi_-^I\Big(\nabla_t^{(-)} + \nabla_\sigma^{(-)}\Big)\psi_-^J + ig_{IJ}\psi_+^I\Big(\nabla_t^{(+)} - \nabla_\sigma^{(+)}\Big)\psi_+^J \\ &\qquad + \frac{1}{2}R^{(-)}_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L\Big]\mathrm{d}t\mathrm{d}\sigma, \end{aligned} \tag{2.8}$$

where $\nabla^{(\pm)}$ is the Levi-Civita connection twisted by the H-field, so that

$$\nabla_\mu^{(\pm)}\psi^I = \partial_\mu\psi^I + \partial_\mu X^J\left(\Gamma^I_{JK} \pm \frac{1}{2}g^{IM}H_{JMK}\right)\psi^K,$$

and $R^{(-)}_{IJKL} = g_{KM}(R^{(-)}_{IJ})^M_L$ is the curvature of $\nabla^{(-)}$. If the B-field is flat, $H = 0$, the two connections agree to the Levi-Civita connection. In addition to the standard Poincaré invariance (translations and Lorentz transformation of (t,σ)), this system has a fermionic symmetry generated by

$$\begin{aligned} \delta X^I &= i\epsilon_+^1\psi_-^I - i\epsilon_-^1\psi_+^I, \\ \delta\psi_\pm^I &= \pm\epsilon_\mp^1(\partial_t \pm \partial_\sigma)X^I + \epsilon_\pm^1 f^I, \\ (f^I &:= i\Gamma^I_{KL}\psi_+^K\psi_-^L + \tfrac{i}{2}g^{IM}H_{KLM}\psi_+^K\psi_-^L) \end{aligned} \tag{2.9}$$

where $\epsilon_\pm^1$ are real fermionic variation parameters. The "square" of such a transformation are proportional to the time and space translations (up to equations of motion)

$$[\delta,\delta']\mathcal{O} = 2i\epsilon_-^1\epsilon_-^{1\prime}(\partial_t + \partial_\sigma)\mathcal{O} + 2i\epsilon_+^1\epsilon_+^{1\prime}(\partial_t - \partial_\sigma)\mathcal{O}.$$

Such a fermionic symmetry is called the *supersymmetry*. Since there is one real right moving and one real left moving parameters, this is technically called the $(1,1)$ supersymmetry. The action also has mod 2 fermion number symmetries

$$(-1)^{F_+} : (\psi_+^I, \psi_-^I) \to (-\psi_+^I, \psi_-^I), \tag{2.10}$$

$$(-1)^{F_-} : (\psi_+^I, \psi_-^I) \to (\psi_+^I, -\psi_-^I), \tag{2.11}$$

The sum over worldsheet geometries must incorporate this supersymmetry, and proceeds as follows (see [1] and references therein for details): First, we couple $S_b + S_f$ to two dimensional supergravity whose variables are the metric h (or the zweibein) and a spinor-valued fermionic one-form $\chi_{\pm\mu}$. The coupled system classically has diffeomorphism (and local Lorentz), local supersymmetry, Weyl rescaling, and local superconformal symmetry. We then integrate over the space of (h, χ, X, ϕ) modulo these gauge symmetry actions. The exact conformal and superconformal invariance in quantum theory requires a condition on the data (g, B, Φ) similar to (2.4)-(2.6). One notable difference is that the critical dimension $D = 26$ is replaced by $D = 10$.

GSO projection

The theory includes spinors and there are choices in the spin structure. For example, on a cylindrical part of the worldsheet the spinors can be periodic or anti-periodic along the circle — these are called Ramond and Neveu-Schwarz sectors respectively. It is then natural to include sum over spin structures in the worldsheet path-integral. There are various ways to do so. One may not correlate the spin structures of left handed fermions ψ_+ and right handed fermions ψ_- (chiral) or one may correlate them (non-chiral), and there are choices in the phase of summation. This result in the projection of the spectrum, called the GSO projections, where we only admit states with plus or minus one eigenvalues under $(-1)^{F_+}$, $(-1)^{F_-}$ (chiral) or their product $(-1)^F$ (non-chiral). There are two ways to perform chiral GSO projection, and the resulting theories are called the Type IIA or Type IIB superstring theory. There are also two ways to perform non-chiral GSO projections, corresponding to Type 0A and 0B string. In Type II superstring formulated on flat ten-dimensional Minkowski space, tachyon is removed from the spectrum by the projection, and the divergence from the long neck boundary is absent. It moreover has space-time supersymmetry – the particle spectrum is invariant under the exchange of bosons (from NS-NS and R-R sectors) and fermions (from R-NS and NS-R sectors). Type IIA string has $(1,1)$ supersymmetry in ten-dimensions, while Type IIB has $(2,0)$ supersymmetry.

2.3 D-branes

The worldsheet of open string has boundaries, and one must specify a boundary condition. For a non-linear sigma model, there is a standard boundary condition for each submanifold W of M. Let us consider the "left-half plane" $\sigma \le 0$, $-\infty < t < \infty$, with the timelike boundary $\sigma = 0$. The boundary condition at $\sigma = 0$ is that

$$X(t,0) \in W \text{ and } \partial_\sigma X(t,0) \text{ is normal to } W \text{ at } X(t,0), \tag{2.12}$$

when $B = 0$ and $\Phi =$ consatnt. This is the boundary condition for the D-brane wrapped on the submanifold W. For supersymmetric sigma model the following condition for fermions

$$\psi_+ + \psi_- \text{ is tangent to } W \text{ and } \psi_+ - \psi_- \text{ is normal to } W \tag{2.13}$$

is compatible with the diagonal ($\mathcal{N} = 1$) supersymmetry generated by $\epsilon^1_- = -\epsilon^1_+$, while the one with the relative sign of ψ_+ and ψ_- reversed is compatible with $\epsilon^1_- = \epsilon^1_+$. One can introduce a $U(N)$ gauge field A on the D-brane. The open string end point is charged so that the worldsheet path-integral weight has a matrix factor

$$\mathrm{P}\exp\left(-i\int_{\sigma=0}\left[\partial_t X^M A_M + iF_{MN}\psi^M\psi^N\right]\mathrm{d}t\right) \tag{2.14}$$

called the Chan-Paton factor. "P exp" stands for the path-ordered exponential. M,N are the indices of the coordinates of W. Here we have in mind the boundary condition (2.13) and $\psi^M = \psi^M_+ + \psi^M_-$.

Just as in the closed string case, conformal invariance requires certain conditions on the D-brane data (W, A). For the rank one case, it is given by [2]

$$\nabla^N F_{NM} + O(\alpha') = 0, \tag{2.15}$$

$$g^{MN} K^I_{MN} + O(\alpha') = 0, \tag{2.16}$$

where g_{MN} is the metric induced on W from the metric g on M, and K^I_{MN} is the second fundamental form of $W \subset M$. To the leading order in the α' expansion, a solution corresponds to a minimal surface W with a gauge field obeying the Maxwell equation.

In Type II string theory on flat Minkowski space $\mathbf{R}^{9+1}$, D-brane at the flat subspace $\mathbf{R}^{p+1}$ with $A = 0$ is a solution (called the Dp-brane) and it preserves a half of the spacetime supersymmetry if p is even for Type IIA and odd for Type IIB. For sucn values of p, Dp-brane is charged or is a sourse of the $(p+1)$-form gauged potential coming from massless bosons in the R-R sector [3]. For the other values of p, the open string spectrum of Dp brane contains a tachyon, and is unstable.

Tachyon

2.4 Orientifolds

2.5 Heterotic strings

2.6 The five string theories in ten dimensions

2.7 Duality

M theory

Mirror Symmetry

3 (2,2) theories and the moduli space

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A simple example

Let us consider the supersymmetric sigma model with the Euclidean 2-plane $\mathbf{R}^2$ or the complex plane $\mathbf{C}$ as the target space. The variables $(X^1, X^2, \psi^1_\pm, \psi^2_\pm)$ can be combined into complex variables - a complex scalar field ϕ and a Dirac fermion $\psi_\pm$, $\overline{\psi}_\pm = \psi^\dagger_\pm$. The action is expressed as

$$S = \frac{1}{2\pi}\int\Big\{|\partial_t\phi|^2 - |\partial_\sigma\phi|^2 + i\overline{\psi}_-(\partial_t+\partial_\sigma)\psi_- + i\overline{\psi}_+(\partial_t-\partial_\sigma)\psi_+\Big\}\mathrm{d}t\mathrm{d}\sigma. \tag{3.1}$$

Here we set $\alpha' = 1$. The system has a supersymmetry generated by

$$\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+, \quad \delta\overline{\phi} = -\overline{\epsilon}_+\overline{\psi}_- + \overline{\epsilon}_-\overline{\psi}_+,$$
$$\delta\psi_\pm = \pm i\overline{\epsilon}_\mp(\partial_0 \pm \partial_1)\phi, \quad \delta\overline{\psi}_\pm = \mp i\epsilon_\mp(\partial_0 \pm \partial_1)\overline{\phi},$$

where $\epsilon_\pm$ are now *complex* fermionic parameters and $\overline{\epsilon}_\pm = \epsilon^\dagger_\pm$ are the complex conjugates. and thus δ is a supersymmetry. Since there is one complex (or two real) right-handed and left-handed parameters, it is (2,2) supersymmetry. If we set $\epsilon_\pm = i\epsilon^1_\pm$ with real $\epsilon^1_\pm$, we get back the $(1,1)$ supersymmetry. The model has also two $U(1)$ symmetries:

$$U(1)_V : \psi_\pm \to \mathrm{e}^{-i\alpha}\psi_\pm, \quad \overline{\psi}_\pm \to \mathrm{e}^{i\alpha}\overline{\psi}_\pm,$$
$$U(1)_A : \psi_\pm \to \mathrm{e}^{\mp i\beta}\psi_\pm, \quad \overline{\psi}_\pm \to \mathrm{e}^{\pm i\beta}\overline{\psi}_\pm,$$

When the supersymmetry is conjugated with these $U(1)$ symmetries, the parameters $\epsilon_\pm$ receive phase rotations, $U(1)_V : \epsilon_\pm \to \mathrm{e}^{-i\alpha}\epsilon_\pm$, $U(1)_A : \epsilon_\pm \to \mathrm{e}^{\mp i\beta}\epsilon_\pm$. Such $U(1)$ symmetries are called (vector and axial) *R-symmetries.*

3.1 $(2,2)$ supersymmetry algebra

For each continuous symmetry of the classical action, there is a conserved charge found by the Nöther procedure. The charge for the time and space translation symmetries are the Hamiltonian and momentum. In the above example they are

$$H = \frac{1}{2\pi}\int\Big\{|\partial_t\phi|^2 + |\partial_\sigma\phi|^2 - i\overline{\psi}_-\partial_\sigma\psi_- + i\overline{\psi}_+\partial_\sigma\psi_+\Big\}\mathrm{d}\sigma,$$
$$P = \frac{1}{2\pi}\int\Big\{\partial_t\overline{\phi}\partial_\sigma\phi + \partial_\sigma\overline{\phi}\partial_t\phi + i\overline{\psi}_-\partial_\sigma\psi_- + i\overline{\psi}_+\partial_\sigma\psi_+\Big\}\mathrm{d}\sigma.$$

The conserved charges for the supersymmetry are called the *supercharges*. Since there are four real parameters, there are four supercharges $Q_\pm$ and $\overline{Q}_\pm$. In the example, they are

$$Q_\pm = \frac{1}{2\pi}\int(\partial_t \pm \partial_\sigma)\overline{\phi}\cdot\psi_\pm \mathrm{d}\sigma,$$
$$\overline{Q}_\pm = \frac{1}{2\pi}\int\overline{\psi}_\pm(\partial_t \pm \partial_\sigma)\phi\,\mathrm{d}\sigma = Q_\pm^\dagger$$

The ones for the $U(1)$ R-symmetries, (vector and axial) R-charges, are

$$F_V = \frac{1}{2\pi}\int(\overline{\psi}_-\psi_- + \overline{\psi}_+\psi_+)\mathrm{d}\sigma$$
$$F_A = \frac{1}{2\pi}\int(-\overline{\psi}_-\psi_- + \overline{\psi}_+\psi_+)\mathrm{d}\sigma.$$

When the system is quantized, conserved charges gives rise to operators acting on the Hilbert space of states and generate the symmetry transformations. In a $(2,2)$ supersymmetric system, Hamiltonian H, momentum P, charge for Lorentz transformation M, the supercharges $Q_\pm$, $\overline{Q}_\pm$ obey the following (anti-)commutation relations:

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \tag{3.2}$$
$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{3.3}$$
$$\{\overline{Q}_+, \overline{Q}_-\} = Z, \quad \{Q_+, Q_-\} = Z^*, \tag{3.4}$$
$$\{Q_-, \overline{Q}_+\} = \widetilde{Z}, \quad \{Q_+, \overline{Q}_-\} = \widetilde{Z}^*, \tag{3.5}$$
$$[iM, Q_\pm] = \mp Q_\pm, \quad [iM, \overline{Q}_\pm] = \mp\overline{Q}_\pm, \tag{3.6}$$

Z and $\widetilde{Z}$ are central charges which is non-trivial only when the boundary condition at the left infinity ($\sigma \to -\infty$) and the right infinity ($\sigma \to +\infty$) are different. They vanish when the system is quantized on the cylinder where σ is a periodic space coordinate. When there are R-symmetries F_V and/or F_A, they commute with H, P, M and obey

$$[iF_V, Q_\pm] = -iQ_\pm, \quad [iF_V, \overline{Q}_\pm] = i\overline{Q}_\pm, \tag{3.7}$$
$$[iF_A, Q_\pm] = \mp iQ_\pm, \quad [iF_A, \overline{Q}_\pm] = \pm i\overline{Q}_\pm. \tag{3.8}$$

Although the R-charge may not always exist, we require the conservation of mod 2 fermion number

$$(-1)^F,$$

under which the supercharges are odd. When an R-charge is conserved and integral, it can be used to define $(-1)^F$ in the form of $e^{\pi i F_V}$ or $e^{\pi i F_A}$.

In the model (3.1), the fields ϕ, $\overline{\phi}$, $\psi_\pm$ and $\overline{\psi}_\pm$ obey the canonical (anti-)commutation relations $[\phi(\sigma), \partial_0\overline{\phi}(\sigma')] = 2\pi i\delta(\sigma - \sigma')$, $\{\psi_\pm(\sigma), \overline{\psi}_\pm(\sigma')\} = 2\pi\delta(\sigma - \sigma')$ etc. It is a simple exercise to show that the conserved charges in the model (3.1) obey all of the above $(2, 2)$ supersymmetry algebra, in which the central charges are zero, $Z = \widetilde{Z} = 0$.

Mirror Symmetry

Mirror symmetry is an equivalence of two $(2, 2)$ supersymmetric quantum field theories such that the isomorphism between the space of states maps the generators H, P, M, Q_+, $\overline{Q}_+$ in one theory to those in the other, but exchanges the rest of the generators as follows

$$\begin{aligned} Q_- &\longleftrightarrow \overline{Q}_-, \\ F_V &\longleftrightarrow F_A, \\ Z &\longleftrightarrow \widetilde{Z}. \end{aligned} \tag{3.9}$$

3.2 Non-linear sigma models

We have introduced the supersymmetric sigma model for a manifold M with a metric g and a B-field B. The variables are the map $X : \mathbf{R}^2 \to M$ and the fermions $\psi_\pm$ with values in X^*TM, and the action is $S_b + S_f$ where S_b and S_f are given in (2.7) and (2.8). In general it has $(1, 1)$ supersymmetry, but it also has $(2, 2)$ supersymmetry under the following condition [4]: M has two complex structures J_+ and J_- which are both orthogonal with respect to the metric g (i.e. $g(J_\pm v, J_\pm w) = g(v, w)$) and are parallel with respect to $\nabla^{(+)}$ and $\nabla^{(-)}$ respectively; $\nabla^{(+)}J_+ = 0$ and $\nabla^{(-)}J_- = 0$. Such a manifold is called the twisted generalized Kähler manifold.

Kähler manifolds are the special cases, in which $J_+ = -J_- = J$ and $H = 0$ so that $\nabla^{(\pm)}$ is the Levi-Civita connection ∇. In what follows we shall focus most of our attention to Kähler case. The non-linear sigma model action is expressed in terms of the holomorphic variables ϕ^i, $\psi^i_\pm$ and their complex conjugates as

$$\begin{aligned} S_{\mathrm{NL}\sigma\mathrm{M}} &= \frac{1}{2\pi}\int\Big[g_{i\bar{j}}\Big(\partial_t\phi^i\partial_t\overline{\phi}^{\bar{j}} - \partial_\sigma\phi^i\partial_\sigma\overline{\phi}^{\bar{j}}\Big) \\ &\quad + ig_{i\bar{j}}\overline{\psi}^{\bar{j}}_-(\nabla_t+\nabla_\sigma)\psi^i_- + ig_{i\bar{j}}\overline{\psi}^{\bar{j}}_+(\nabla_t-\nabla_\sigma)\psi^i_+ + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^k_-\overline{\psi}^{\bar{j}}_-\overline{\psi}^{\bar{l}}_+\Big] \end{aligned} \quad (3.10)$$

The $(2,2)$ supersymmetry transformation is

$$\begin{aligned} &\delta\phi^i = \epsilon_+\psi^i_- - \epsilon_-\psi^i_+, && \delta\overline{\phi}^{\bar{\imath}} = -\overline{\epsilon}_+\overline{\psi}^{\bar{\imath}}_- + \overline{\epsilon}_-\overline{\psi}^{\bar{\imath}}_+, \\ &\delta\psi^i_+ = i\overline{\epsilon}_-(\partial_t+\partial_\sigma)\phi^i + \epsilon_+F^i, && \delta\overline{\psi}^{\bar{\imath}}_+ = -i\epsilon_-(\partial_t+\partial_\sigma)\overline{\phi}^{\bar{\imath}} + \overline{\epsilon}_+\overline{F}^{\bar{\imath}}, \\ &\delta\psi^i_- = -i\overline{\epsilon}_+(\partial_t-\partial_\sigma)\phi^i + \epsilon_-F^i, && \delta\overline{\psi}^{\bar{\imath}}_- = i\epsilon_+(\partial_t-\partial_\sigma)\overline{\phi}^{\bar{\imath}} + \overline{\epsilon}_-\overline{F}^{\bar{\imath}}, \end{aligned} \quad (3.11)$$

where $F^i = \Gamma^i_{jk}\psi^j_+\psi^k_-$, and the supercharges are

$$Q_\pm = \frac{1}{2\pi}\int g_{i\bar{j}}(\partial_t\pm\partial_\sigma)\overline{\phi}^{\bar{j}}\,\psi^i_\pm\,\mathrm{d}\sigma \quad (3.12)$$

$$\overline{Q}_\pm = \frac{1}{2\pi}\int g_{\bar{\imath}j}\overline{\psi}^{\bar{\imath}}_\pm(\partial_t\pm\partial_\sigma)\phi^j\,\mathrm{d}\sigma \quad (3.13)$$

The classical action has both vector and axial R-symmetry;

$$\begin{aligned} U(1)_V &: \psi^i_\pm \to \mathrm{e}^{-i\alpha}\psi^i_\pm, \quad \overline{\psi}^{\bar{\imath}}_\pm \to \mathrm{e}^{i\alpha}\overline{\psi}^{\bar{\imath}}_\pm, \\ U(1)_A &: \psi^i_\pm \to \mathrm{e}^{\mp i\beta}\psi^i_\pm, \quad \overline{\psi}^{\bar{\imath}}_\pm \to \mathrm{e}^{\pm i\beta}\overline{\psi}^{\bar{\imath}}_\pm. \end{aligned}$$

In the quantum theory, $U(1)_V$ is always a symmetry but $U(1)_A$ is not. In a bosonic background $X:\Sigma\to M$, $U(1)_A$ transforms the path-integral measure by multiplication of the phase

$$\exp\left(-2i\beta\int_\Sigma X^*c_1(M)\right)$$

where $c_1(M)$ is the first Chern class of the holomorphic tangent bundle T_M. Thus, $U(1)_A$ is a symmetry only when $c_1(M)=0$, namely for Calabi-Yau manifolds. If not, it is broken to its discrete subgroup. For example, for $M=\mathbb{CP}^{N-1}$, $c_1(M)$ is N times a generator of the integral second cohomology group. Thus only the $\mathbf{Z}_{2N}$ subgroup of $U(1)_A$ is a symmetry. In general, the $\mathbf{Z}_2$ subgroup is always a symmetry.

3.3 Landau-Ginzburg models

We now introduce a different class of models called the *Landau-Ginzburg models* (LG models). The data to define a LG model is a Kähler manifold (M,g) with a holomorphic function W, called the *superpotential*, whose critical points are all isolated. We present the model in the particular case where $M = \mathbf{C}^n$ and W is a polynomial, but the generalization is obvious. The variables are n complex scalars $\phi^1, ..., \phi^n$ and n Dirac fermions $\psi^1_\pm$,...,$\psi^n_\pm$. The action is

$$\begin{aligned} S &= \frac{1}{2\pi}\int\Big[\sum_{i=1}^n\Big(|\partial_t\phi^i|^2 - |\partial_\sigma\phi^i|^2 + i\overline{\psi}^{\bar\imath}_-(\partial_t+\partial_\sigma)\psi^i_- + i\overline{\psi}^{\bar\imath}_+(\partial_t-\partial_\sigma)\psi^i_+\Big) \\ &\quad -\sum_{i=1}^n\Big|\partial_i W(\phi)\Big|^2 - \sum_{i,j=1}^n\Big(\partial_i\partial_j W(\phi)\psi^i_+\psi^j_- + \partial_{\bar\imath}\partial_{\bar\jmath}\overline{W(\phi)}\overline{\psi}^{\bar\imath}_-\overline{\psi}^{\bar\jmath}_+\Big)\Big]\mathrm{d}t\mathrm{d}\sigma. \end{aligned} \tag{3.14}$$

The new part is the we now have a potential $U = |W'(\phi)|^2$ as well as the coupling $W''(\phi)\psi_+\psi_-$. The supersymmetry transfprmation is given by (3.11) where now $F^i = -\partial_{\bar\imath}\overline{W(\phi)}$, and the supercharges are

$$Q_\pm = \frac{1}{2\pi}\int\sum_{i=1}^n\Big((\partial_t\pm\partial_\sigma)\overline{\phi}^{\bar\imath}\,\psi^i_\pm \mp i\overline{\psi}^{\bar\imath}_\mp\partial_{\bar\imath}\overline{W}\Big)\,\mathrm{d}\sigma \tag{3.15}$$

$$\overline{Q}_\pm = \frac{1}{2\pi}\int\sum_{i=1}^n\Big(\overline{\psi}^{\bar\imath}_\pm(\partial_t\pm\partial_\sigma)\phi^i \pm i\psi_\mp\partial_i W\Big)\,\mathrm{d}\sigma \tag{3.16}$$

The system always have axial $U(1)$ R-symmetry

$$U(1)_A : \psi^i_\pm \to \mathrm{e}^{\mp i\beta}\psi^i_\pm, \quad \overline{\psi}^{\bar\imath}_\pm \to \mathrm{e}^{\pm i\beta}\overline{\psi}^{\bar\imath}_\pm,$$

but vector $U(1)$ R-symmetry is not automatic becuase of the term $-W''\psi_+\psi_-$ in the action. When there is a transformation $\phi^i \mapsto (\mathrm{e}^{i\alpha q})^i{}_j\phi^j$ with hermitian matrix q such that

$$W(\mathrm{e}^{i\alpha q}\phi) = \mathrm{e}^{2i\alpha}W(\phi) \tag{3.17}$$

the action is invariant under

$$U(1)_V : \phi \to \mathrm{e}^{i\alpha q}\phi, \quad \psi_\pm \to \mathrm{e}^{i\alpha(q-1)}\psi_\pm,$$

and $\overline{\phi} \to \mathrm{e}^{-i\alpha\bar q}\overline{\phi}$, $\overline{\psi}_\pm \to \mathrm{e}^{-i\alpha(\bar q-1)}\overline{\psi}_\pm$. A polynomial having the property (3.17) is is said to be *quasi-homogeneous*. For a more general polynomial, the vector R-symmetry group is a discrete subgroup that includes $\mathbf{Z}_2$.

3.4 Supersymmetric ground states

Let us quantize the system on a cylinder with a periodic boundary condition on all fields. The fermions are in particular periodic, and the states are in the RR sector on which the $(2,2)$ supersymmetry generators act with $Z = \widetilde{Z} = 0$. By $Q_\pm^\dagger = \overline{Q}_\pm$ and (3.3), the spectrum is positive semi-definite, $H \geq 0$. Zero energy states are necessarily the ground states and are annihilated by all the supercharges $Q_\pm, \overline{Q}_\pm$. Such states are called *supersymmetric ground states*. They span a finite dimensional subspace $\mathcal{H}_{\rm SUSY}$ of the infinite dimensional space of states $\mathcal{H}$, under the assumption that each energy level is finite dimensional.

Let us denote $Q_A = \overline{Q}_+ + Q_-$ and $Q_B = \overline{Q}_+ + \overline{Q}_-$. The operators $(Q, F) = (Q_A, F_A)$ or (Q_B, F_V) obey the following commutation relations

$$\{Q, Q^\dagger\} = 2H, \tag{3.18}$$
$$Q^2 = 0, \tag{3.19}$$
$$[F, Q] = Q. \tag{3.20}$$

By the second and the third equation, the Hilbert space of states $\mathcal{H}$ can be regarded as the Q-complex;

$$\cdots \xrightarrow{Q} \mathcal{H}^{q-1} \xrightarrow{Q} \mathcal{H}^q \xrightarrow{Q} \mathcal{H}^{q+1} \xrightarrow{Q} \cdots, \tag{3.21}$$

where $\mathcal{H}^q$ is the subspace of R-charge $F = q$. By the equation (3.18), Q-cohomology classes are in one to one correspondence with the supersymmetric ground states;

$$\mathcal{H}^q_{\rm SUSY} \cong H^q(Q) := \frac{{\rm Ker}\, Q : \mathcal{H}^q \to \mathcal{H}^{q+1}}{{\rm Im}\, Q : \mathcal{H}^{q-1} \to \mathcal{H}^q}. \tag{3.22}$$

Witten index is the Q-index (or the Euler characteristic of the complex (3.21))

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F\, \mathrm{e}^{-\beta H} = \sum_q (-1)^q \dim H^q(Q), \tag{3.23}$$

and is independent of any supersymmetric deformation of the system.

In the above, we have assumed that the R-charge F is conserved and the eigenvalues are integers. Otherwise, one can use $(-1)^F$ to define a $\mathbf{Z}_2$ grading so that we have a $\mathbf{Z}_2$ graded (double periodic) Q-complex. Then, we have $\mathcal{H}^{\rm ev,od}_{\rm SUSY} = H^{\rm ev,od}(Q)$ and $\mathrm{Tr}_{\mathcal{H}}(-1)^F\, \mathrm{e}^{-\beta H} = \dim H^{\rm ev}(Q) - \dim H^{\rm od}(Q)$.

Sigma models and LG models

The space of supersymmetric ground states of NLSM on M is isomorphic to the cohomology group of M which is in turn the same as the space of harmonic forms on M (this can be understood when we discuss twisting to topological field theory);

$$\mathcal{H}_{\rm SUSY} \cong \bigoplus_{p,q=1}^{n} H^{p,q}(M). \tag{3.24}$$

Here $H^{p,q}(M)$ is the space of harmonic (p,q) forms, or (p,q)-th Dolbeault cohomology group. If M is Calabi-Yau, the vector and axial R-charges of the ground states are

$$\begin{array}{l} q_V = -p+q, \\ q_A = p+q-n \end{array} \quad \text{on } H^{p,q}(M). \tag{3.25}$$

If M is not Calabi-Yau, the axial R-symmetry is anomalous, and only the expression for q_V makes sense. In either case, using $(-1)^{F_V}$ as the mod 2 fermion number $(-1)^F$, Witten index is given by $I = \sum_{p,q}(-1)^{-p+q}\dim H^{p,q}(M) = \sum_{i=1}^{2n}(-1)^i H^i(M) = \chi(M)$, the Euler number of M. If the non-linear sigma models on two Calabi-Yau manifolds M and $\widetilde{M}$ are mirror to each other, the ground states in $H^{p,q}(M)$ are mapped to the ground states in $H^{n-p,q}(\widetilde{M})$ so that the vector and axial R-charges are exchanged. In particular, there is a relation between M and $\widetilde{M}$ in the Hodge numbers $h^{p,q} = \dim H^{p,q}$:

$$h^{p,q}(M) = h^{n-p,q}(\widetilde{M}). \tag{3.26}$$

The supersymmetric ground states of LG model are in one to one correspondence with the critical points of the superpotential W, if all the critical points are non-degenerate. The axial R-charges of the ground states are all zero

$$q_A = 0 \quad \text{on the ground states.} \tag{3.27}$$

The reason is that the ground state wavefunctions in the dimensionally reduced model (supersymmetric quantum mechanics) is given by middle-dimensional forms. Witten index is thus the number of critical points $I = \#\text{Crit}(W)$.

If a NLSM on M is mirror to a LG model with non-degenerate critical points only, then the vector R-charge of the NLSM ground states has to be zero. Namely, $H^{p,q}(M) = 0$ if $p \neq q$. Thus the Hodge diamond of M is diagonal.

3.5 Twisting to topological field theory

Chiral ring

An operator $\mathcal{O}$ is called a *chiral operator* if it (anti-)commute with Q_B and *twisted chiral operator* if it (anti-)commute with Q_A. It represents a $Q = Q_B/Q_A$ cohomology class of operators. One can show from the supersymmetry algebra (with $Z = \widetilde{Z} = 0$) that if $\mathcal{O}$ is a chiral operator, $[Q_B, \mathcal{O}] = 0$, then $[(H \pm P), \mathcal{O}] = [Q_B, [Q_\pm, \mathcal{O}]]$. Thus, the worlsheet translations do not change the Q_B-cohomology classes. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two chiral operators, the product $\mathcal{O}_1\mathcal{O}_2$ is also a chiral operator. Same can be said on twisted chiral operators. Thus, Q-cohomology classes of operators form a ring, called the *chiral ring* for $Q = Q_B$ and *twisted chiral ring* for $Q = Q_A$.

Twisting

By Wick rotation, we obtain the Euclidean theory where the group of rotation is $Spin(2)$ generated by $M_E = iM$. It makes a perfect sense to formulate the Euclidean theory on an arbitrary two dimensional surface Σ with a Riemannian metric h as long as the spin structure is chosen. However, it generically loses supersymmetry. To see this, we note that a supervariation of the action would be given by

$$\delta S = \int_{\Sigma} \nabla_{\mu}\epsilon G^{\mu}\sqrt{h}\mathrm{d}^2x,$$

where ϵ is the fermionic variation parameter, which is a section of the spinor bundle, and G^{μ} is the supercurrent, a vector with values in the spinor bundle. If (Σ, h) is curved, there is no covaraintly constant spinor and thus the action is *not* invariant under any supervariation. The only way to preserve a supersymmetry is to change the system so that some of the supercharges $Q_{\pm}$, $\overline{Q}_{\pm}$ are scalars and the corresponding variation parameters (being scalars) can be covariantly constant, $\nabla_{\mu}\epsilon = \partial_{\mu}\epsilon = 0$. This is the idea of *twisting*. If a vector R-charge F_V is conserved and integral, one can twist the theory by declaring that $M_E + F_V$ to be the new rotation generator. This is called the *A-twist*. The same procedure for axial R-symmetry is called the *B-twist*. After B(A)-twist, $\overline{Q}_+$ and $\overline{Q}_-$ ($\overline{Q}_+$ and Q_-) become scalars and thus there is a supersymmetry even when the worldsheet is curved. Correlation functions with insertions of only (twisted) chiral operators are independent of the choice of worldsheet metric. This is because the variation of the metric $h_{\mu\nu}$ corresponds to insertion of the energy-momentum tensor $T_{\mu\nu}$ but in the twisted theory it is $Q = Q_B$ (Q_A) exact

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\}$$

so that it annihilates the correlators with only Q-closed operators. For this reason the twisted model is sometimes called topological B(A)-model. Sphere 3-point functions determine the structure constants of the (twisted) chiral ring.

When a B-twist is possible, there is a one to one correspondence with chiral ring elements and supersymmetric ground states. Consider a worlsheet of semi-infinite cigar geometry as in Fig. 1, and perform the B-twisted path-integral in the interior of the cigar, with a chiral ring element ϕ_i inserted at the tip. This leads to a wavefunction at the circle boundary. The flat cylinder region is not affected by twisting, and thus the wavefunction

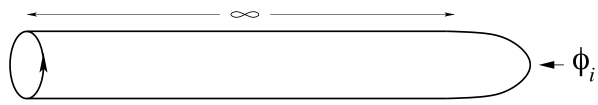

Figure 1: The semi-infinite cigar

can be regarded as a state of the untwisted theory. Because of the twisting in the curved region, the fermions are periodic along the circle — namely the state belongs to RR sector. In the limit of infinite length, all the excited states are projected out and we are left with the zero energy state $|i\rangle$. This is the supersymmetric ground state corresponding to ϕ_i. The same applies to A-twistable theories where the ground states corresponds to twisted chiral ring elements.

Examples

We can consider A-twist of non-linear sigma models (where F_V is conserved) and B-twist of Landau-Ginzburg models on Calabi-Yau target spaces (where F_A is conserved). We recall that $M_E + F_V$ ($M_E + F_A$) is the rotation generator after A-twist (B-twist). The quantum numbers of the fermions change under twisting, as shown in the table.

	M_E	F_V	F_A	$M_E + F_V$	$M_E + F_A$
ψ_-	1	-1	1	0	2
ψ_+	-1	-1	-1	-2	-2
$\overline{\psi}_-$	1	1	-1	2	0
$\overline{\psi}_+$	-1	1	1	0	0

Let us first consider A-twist of NLSM on a Kähler manifold (M, g) with B-field B. The change of the spin motivates to rename the fields as $\chi^i = \psi^i_-$, $\overline{\chi}^{\bar{\imath}} = \overline{\psi}^{\bar{\imath}}_+$, $\rho^{\bar{\imath}}_z = \overline{\psi}^{\bar{\imath}}_-$ and $\rho^i_{\bar{z}} = \psi^i_+$. Then, under the scalar supersymmetry $\delta = \epsilon Q_A$, the fields transform as

$$\begin{aligned} &\delta\phi^i = \epsilon\chi^i, \quad \delta\chi^i = 0, \quad \delta\rho^i_{\bar{z}} = 2i\epsilon\partial_{\bar{z}}\phi^i + \epsilon\Gamma^i_{jk}\rho^j_{\bar{z}}\chi^k, \\ &\delta\overline{\phi}^{\bar{\imath}} = \epsilon\overline{\chi}^{\bar{\imath}}, \quad \delta\overline{\chi}^{\bar{\imath}} = 0, \quad \delta\rho^{\bar{\imath}}_z = -2i\epsilon\partial_z\overline{\phi}^{\bar{\imath}} + \epsilon\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\rho^{\bar{\jmath}}_z\overline{\chi}^{\bar{k}}. \end{aligned} \tag{3.28}$$

For a differential form $\alpha \in \Omega^{p,q}(M)$, $\mathcal{O}_\alpha := \alpha_{i_1\ldots i_p\bar{\jmath}_1\ldots\bar{\jmath}_q}\chi^{i_1}\ldots\chi^{i_p}\overline{\chi}^{\bar{\jmath}_1}\ldots\overline{\chi}^{\bar{\jmath}_q}$ obeys $\delta\mathcal{O}_\alpha = \epsilon\mathcal{O}_{\mathrm{d}\alpha}$ Thus, the Q_A-cohomology classes of fields are in one to one correspondence with the de Rham cohomology classes of M. Configurations invariant under Q_A are those with $\partial_{\bar{z}}\phi^i = 0$, namely holomorphic maps of the worldsheet to M. Thus, A-model correlation functions receive contributions only from holomorphic maps. We note that the classical action for a holomorphic map $\phi : \Sigma \to M$ is expressed as

$$S = \int_\Sigma g_{i\bar{\jmath}}h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\overline{\phi}^{\bar{\jmath}}\sqrt{h}\mathrm{d}^2x - i\int_\Sigma \phi^*B = \int_\Sigma \phi^*(\omega - iB). \tag{3.29}$$

For instance, genus zero three-point functions are expressed as

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\mathcal{O}_{\alpha_3}\rangle_{g=0} = \sum_{\beta\in H_2(M)} n^{\beta}_{\alpha_1\alpha_2\alpha_3}\, \mathrm{e}^{-\int_\beta(\omega-iB)}, \tag{3.30}$$

where the sum is over the homology class of the image of the maps and $n^{\beta}_{\alpha_1\alpha_2\alpha_3}$ is the number of "image β" holomorphic maps such that the three points $0,1,\infty$ of $\Sigma=\mathbb{CP}^1$ are mapped to fixed Poincaré dual cycles of $[\alpha_1],[\alpha_2],[\alpha_3]$ respectively.

Let us next consider B-twist of LG model with superpotential W on a CY space M. This time, an appropriate renaming of the fermions is $\theta_j = g_{i\bar{\jmath}}(\overline{\psi}^{\bar{\jmath}}_- - \overline{\psi}^{\bar{\jmath}}_+)$, $\eta^{\bar{\jmath}} = \overline{\psi}^{\bar{\imath}}_- + \overline{\psi}^{\bar{\imath}}_+$, $\rho^i_z = \psi^i_-$, $\rho^i_{\bar{z}} = \psi^i_+$. Under the scalar supersymmetry $\delta=\epsilon Q_B$ the fields transform as

$$\begin{aligned} &\delta\phi^i = 0, \quad \delta\theta_j = -\epsilon\partial_j W,\\ &\delta\overline{\phi}^{\bar{\imath}} = \epsilon\eta^{\bar{\imath}}, \quad \delta\eta^{\bar{\imath}} = 0,\\ &\delta\rho^i_\mu = 2\epsilon J^{\nu}{}_{\mu}\partial_\nu\phi^i. \end{aligned} \tag{3.31}$$

The Q_B-cohomology group is isomorphic to the space of holomorphic functions on M modulo holomorphic derivatives of W. Q_B-invariant configurations are constant maps to the critical points of W. In particular, genus zero three-point functions are

$$\langle \mathcal{O}_{f_1}\mathcal{O}_{f_2}\mathcal{O}_{f_3}\rangle_{g=0} = \sum_{p\in \mathrm{Crit}(W)} \frac{f_1(p)f_2(p)f_3(p)}{\det\partial^2 W(p)}, \tag{3.32}$$

where the sum is over the critical points of W and $\partial^2 W$ the Hessian matrix of second derivatives of W. Note that the Hessian and the determinant both depend on the choice of the coordinates. The ambiguity is fixed by fixing a nowhere vanishing holomorphic top form Ω and choosing a coordinate system such that Ω is written as $\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n$. One could also consider the case where M is a compact CY manifold and $W=0$. In such a case, the Q_B-cohomology group is identified as $H^{0,*}(M,\wedge^*T_M)$, under the identification $\eta^{\bar{\imath}}\leftrightarrow \mathrm{d}\bar{z}^{\bar{\imath}}$, $\theta_i\leftrightarrow\frac{\partial}{\partial z^i}$. Geneus zero three-point functions for a CY 3-fold M with holomorphic 3-form Ω are given by

$$\langle \mathcal{O}_{\mu_1}\mathcal{O}_{\mu_2}\mathcal{O}_{\mu_3}\rangle_{g=0} = \int_M \mu^i_1\wedge\mu^j_2\wedge\mu^k_3\Omega_{ijk}\wedge\Omega, \tag{3.33}$$

for $\mu_1,\mu_2,\mu_3\in H^{0,1}(M,T_M)$.

Mirror Symmetry

Under mirror symmetry, (Q_B,F_A) of one theory is mapped to (Q_A,F_V) of the mirror theory. Thus, the chiral ring of one theory is the same as the twisted chiral ring of the mirror. Also, B-twist of one theory is mapped to A-twist of the mirror. The problem of counting the number of holomorphic curves in a Kähler manifold, which is sometimes difficult, is mapped to the classical problem of finding the critical points of the superpotential and computing the sum (3.32), or performing the classical integration like (3.33).

3.6 Moduli space of $(2,2)$ theories

Descent relations

Let us consider a bosonic scalar chiral operator $\mathcal{O}$. We define fermionic one-form operators $\mathcal{O}^{(1)}$ and bosonic two-form operator $\mathcal{O}^{(2)}$ by

$$\mathcal{O}^{(1)} = \frac{i}{2}\mathrm{d}x^{+}[Q_{+},\mathcal{O}] + \frac{i}{2}\mathrm{d}x^{-}[Q_{-},\mathcal{O}], \tag{3.34}$$

$$\mathcal{O}^{(2)} = \frac{1}{2}\{Q_{+},[Q_{-},\mathcal{O}]\}\mathrm{d}t\mathrm{d}\sigma \tag{3.35}$$

where $x^{\pm} = t \pm \sigma$. Then, using the supersymmetry algebra and the fact that $\mathcal{O}$ is chiral, it is easy to show the following *decent relations*

$$\begin{aligned} 0 &= [Q_B,\mathcal{O}], \\ \mathrm{d}\mathcal{O} &= \{Q_B,\mathcal{O}^{(1)}\}, \\ \mathrm{d}\mathcal{O}^{(1)} &= [Q_B,\mathcal{O}^{(2)}]. \end{aligned} \tag{3.36}$$

If we choose a representative $\mathcal{O}$ in the same Q_B-cohomology class that commutes with both $\overline{Q}_{+}$ and $\overline{Q}_{-}$, refined decent relations hold. Define $\mathcal{O}^{(0,1)}$ and $\mathcal{O}^{(1,0)}$ to be the $\mathrm{d}x^{+}$ and $\mathrm{d}x^{-}$ parts of $\mathcal{O}^{(1)}$. They fit into the following "ascent diagram"

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & \nearrow & & & \\ & & \mathcal{O}^{(0,1)} & & & & 0 \\ & \nearrow & & \searrow & & \nearrow & \\ \mathcal{O} & & & & \mathcal{O}^{(2)} & & \\ & \searrow & & \nearrow & & \searrow & \\ & & \mathcal{O}^{(1,0)} & & & & 0 \\ & & & \searrow & & & \\ & & & & 0 & & \end{array} \tag{3.37}$$

where the upper-right and lower-right arrows correspond to the action of Q_{+} and Q_{-} respectively. They obey the following descent relations

$$0 = [\overline{Q}_{\pm},\mathcal{O}]$$

$$\overline{\partial}\mathcal{O} = \{\overline{Q}_+, \mathcal{O}^{(0,1)}\}, \quad \partial\mathcal{O} = \{\overline{Q}_-, \mathcal{O}^{(1,0)}\}, \tag{3.38}$$
$$\overline{\partial}\mathcal{O}^{(1,0)} = [\overline{Q}_+, \mathcal{O}^{(2)}], \quad \partial\mathcal{O}^{(0,1)} = [\overline{Q}_-, \mathcal{O}^{(2)}],$$

where $\partial = \mathrm{d}x^-\partial_-$ and $\overline{\partial} = \mathrm{d}x^+\partial_+$.

The same applies to a bosonic twisted-chiral scalar operator $\mathcal{O}$ (that commutes with both $\overline{Q}_+$ and Q_-). One can construct fermionic one-forms $\widetilde{\mathcal{O}}^{(0,1)}$, $\widetilde{\mathcal{O}}^{(1,0)}$ and a bosonic two-form $\widetilde{\mathcal{O}}^{(2)}$ that obeys the descent relations for Q_A or for $\overline{Q}_+$ and Q_-. All we need to do is to replace Q_- by $\overline{Q}_-$ and $\overline{Q}_-$ by Q_- in the construction for the chiral case.

Chiral and twisted-chiral deformations

Chiral and twisted-chiral ring elements can be used to deform the theory while preserving the $(2,2)$ supersymmetry. Let $\mathcal{O}_i$ be chiral operators that are annihilated by both $\overline{Q}_+$ and $\overline{Q}_-$ and $\mathcal{O}_a$ be twisted chiral operators annihilated by both $\overline{Q}_+$ and Q_-. Then, one can deform the theory by adding

$$\Delta S = \int \sum_i t_i \mathcal{O}_i^{(2)} + \int \sum_a \widetilde{t}_a \widetilde{\mathcal{O}}_a^{(2)} + \text{complex conjugate} \tag{3.39}$$

to the original action S_0. By definition and using the descent relation, we see that it preserves the $(2,2)$ supersymmetry, namely, it commutes with all four supercharges $Q_\pm$, $\overline{Q}_\pm$. For example, the integrand $\mathcal{O}_i^{(2)}$ commutes with $Q_\pm$ and the commutators $[\overline{Q}_\pm, \mathcal{O}^{(2)}]$ are total derivatives $\mathrm{d}\mathcal{O}^{(1,0)}/\mathrm{d}\mathcal{O}^{(0,1)}$. We shall call t_i and $\widetilde{t}_a$ chiral and twisted chiral deformation parameters respectively.

Chiral deformation terms $\int t_i \mathcal{O}_i^{(2)} + c.c.$ are Q_A-exact. This can be shown using the "ascent relation" (3.37), $Q_- = Q_A - \overline{Q}_+$, and the descent relations;

$$\begin{aligned}
\mathcal{O}^{(2)} &= \frac{i}{2}\mathrm{d}x^-\{Q_-, \mathcal{O}^{(0,1)}\} \\
&= \frac{i}{2}\mathrm{d}x^-\{Q_A - \overline{Q}_+, \mathcal{O}^{(0,1)}\} \\
&= \frac{i}{2}\mathrm{d}x^-\left[\{Q_A, \mathcal{O}^{(0,1)}\} - \overline{\partial}\mathcal{O}\right] = \{Q_A, \cdots\} + \text{total derivative.}
\end{aligned}$$

For the complex conjugate part, we use the expression $\overline{\mathcal{O}}^{(2)} = -\frac{i}{2}\mathrm{d}x^+\{\overline{Q}_+, \overline{\mathcal{O}}^{(1,0)}\}$ and $\overline{Q}_+ = Q_A - Q_-$, as well as the complex conjugate of $\{\overline{Q}_-, \mathcal{O}^{(1,0)}\} = \partial\mathcal{O}$. This means that the correlation functions of the A-twisted topological field theory does not depend on the chiral deformation parameter. Consersely, the twisted chiral deformation terms $\int \widetilde{t}_a \widetilde{\mathcal{O}}^{(2)} + c.c.$ are Q_B-exact, and does not affect the topological correlation functions of the B-twisted model. In this sense, *chiral and twisted chiral deformations are decoupled from each other.* Also, the $\bar{t}_i$ part of the chiral deformation is Q_B-exact. This follows from

$$\begin{aligned}\overline{\mathcal{O}}^{(2)} &= -\frac{i}{2}\mathrm{d}x^+\{\overline{Q}_+,\overline{\mathcal{O}}^{(1,0)}\}\\ &= -\frac{i}{2}\mathrm{d}x^+\{Q_B-\overline{Q}_-,\overline{\mathcal{O}}^{(1,0)}\}\\ &= \{Q_B,\cdots\}+\frac{i}{2}\mathrm{d}x^+\{\overline{Q}_-,\overline{\mathcal{O}}^{(1,0)}\}=\{Q_B,\cdots\}\end{aligned}$$

where we have used the complex conjugate of the equation $\{Q_-,\mathcal{O}^{(1,0)}\}=0$ (see the arrow at the bottom of the ascent diagram (3.37)). This means that topological correlators of the B-twisted model do not depend on $\bar{t}_i$'s, namely *they are holomorphic functions of the chiral parameters t_i only.* Likewise, *topological correlators of the A-twisted model are holomorphic functions of the twisted chiral parameters $\widetilde{t}_a$ only.*

The moduli space

As the decoupling of the chiral and twisted chiral deformations suggests, the moduli space of $(2,2)$ theories is a product of two spaces

$$\mathcal{M}=\mathcal{M}_c\times\mathcal{M}_t \tag{3.40}$$

The spaces $\mathcal{M}_c$ and $\mathcal{M}_t$ are parametrized respectively by chiral and twisted chiral parameters. Topological correlation functions of B-twisted models are holomorphic functions on $\mathcal{M}_c$ whereas those for A-twisted models are holomorphic functions on $\mathcal{M}_t$.

Let us see what $\mathcal{M}_c$ and $\mathcal{M}_t$ are for the non-linear sigma models on a Kähler manifold M. We have seen that the topological correlators of the A-twisted models depend holomorphically on

$$\exp\left(-\int_\Sigma\phi^*\omega+i\int_\Sigma\phi^*B\right).$$

for a map $\phi:\Sigma\to M$ of the worldsheet to the target, where ω is the Kähler form and B is the closed B-field. This depends only on the class of $\omega-iB$ in the group $H^2(M,\mathbf{C}/2\pi i\mathbf{Z})$ which is a cylinder of dimension r if $h^{1,1}(M)=r$. Thus, we identify $\mathcal{M}_t$ as the space of the Kähler class complexified by the B-field class, (*complexified Kähler class* in short):

$$[\omega-iB]\in H^2(M,\mathbf{C}/2\pi i\mathbf{Z}).$$

On the other hand, the topological correlators in the B-twisted model, when possible (i.e. when M is Calabi-Yau), depends holomorphically on the complex structure of M. Thus, we identify $\mathcal{M}_c$ as the space of complex structures on M.

There is something strange here — since the Kähler form must be positive definite, one cannot take the entire cylinder $H^2(M, \mathbf{C}/2\pi i\mathbf{Z})$ as $\mathcal{M}_t$ but we must chop-off the region where the positivity fails. Is $\mathcal{M}_t$ a space with such a sharp boundary? As noticed earlier, non-linear sigma models stops to be a good description of the theory when the curvature is large (compared to $1/\alpha'$ which we are setting 1) or equivalently when the size is small (compared to $\sqrt{\alpha'} = 1$). Thus, the answer to the question is: as the Kähler class $[\omega]$ decreases, quantum correction becomes large and we can no longer use the non-linear sigma model description. $H^2(M, \mathbf{C}/2\pi i\mathbf{Z})$ is a good approximation to $\mathcal{M}_t$ only in the "large volume region" and it must be glued to something else in the interior! This is a tough but interesting question, and great progress has been made during late 1980's through mid 1990's. As of today, we have mainly two methods to understand the "deep interior" of $\mathcal{M}_t$ — one is linear sigma model and another is mirror symmetry.

Before discussing about it, let us look at the other space, $\mathcal{M}_c$. Here the decoupling of chiral and twisted chiral deformations greatly helps. $\mathcal{M}_c$ does not depend on where you are in $\mathcal{M}_t$. In particular, we can take the large volume limit in $\mathcal{M}_t$ where the metric is scaled up to infinity and the curvature is everywhere vanishingly small. In this limit, the quantum correction is absent and everything is classical. In particular, $\mathcal{M}_c$ is really identified as the moduli space of complex structures of M. For a Calabi-Yau manifold of dimension 3, there is a well-developed machinery to define and study the structure of the moduli space. It has a metric, called the special geometry, which is determined by the period integrals of the holomorphic volume form over symplectic basis elements of $H_3(M, \mathbf{Z})$.

Let us now describe the use of mirror symmetry in studying $\mathcal{M}_t$. Let M be a Calabi-Yau manifold and let $\widetilde{M}$ be the mirror. Since chiral and twisted chiral rings are exchanged under mirror symmetry, we find the relation

$$\mathcal{M}_t(M) = \mathcal{M}_c(\widetilde{M}), \qquad \mathcal{M}_c(M) = \mathcal{M}_t(\widetilde{M}). \tag{3.41}$$

As we have just noted, $\mathcal{M}_c(\widetilde{M})$ is a classical moduli space of complex structures of $\widetilde{M}$ which we can study without worrying about the quantum correction. Thus, we can learn about $\mathcal{M}_t(M)$ by studying $\mathcal{M}_c(\widetilde{M})$.

Let us look at an example. Consider the quintic hypersurface M in $\mathbb{CP}^4$, which is the zero set of a degree five polynomial of homogeneous coordinates.

$$G(x_1, x_2, x_3, x_4, x_5) = 0.$$

It has $h^{1,1}(M) = 1$ coming from the pull-back of the Kähler class of $\mathbb{CP}^4$ and $h^{2,1}(M) = 101$ from the parameters of quintic polynomials $G(x)$. Thus, in the large volume region $\mathcal{M}_t$

looks like the cylinder $H^2(M, \mathbf{C}/2\pi i\mathbf{Z}) \cong \mathbf{R} \times S^1$. The mirror $\widetilde{M}$ is known to be (a resolution of) the quotient of a quintic hypersurface in $\mathbb{CP}^4$ of the type

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0$$

by the $\mathbf{Z}_5^3$ action on the homogeneous coordinates $(z_1, ..., z_5)$ given by $z_i \to \omega_i z_i$, with $\omega_i^5 = \prod_{i=1}^5 \omega_i = 1$. This indeed has $h^{1,1}(\widetilde{M}) = 101$, and $h^{2,1}(\widetilde{M}) = 1$. The complex structure of $\widetilde{M}$ is parametrized by ψ. To be more precise, since $\psi \to e^{2\pi i/5}\psi$ can be abosrbed by the coordinate change, say, $z_1 \to e^{-2\pi i/5} z_1$, the fifth power ψ^5 is the right parameter of $\mathcal{M}_c(\widetilde{M})$. The special geometry of this ψ^5-space $\mathcal{M}_c(\widetilde{M})$ is studied in detail by P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes in [5], where the first prediction on the number of rational curves was also made. We see that there are three special points: $\psi^5 = 0, 1, \infty$. At $\psi^5 = 0$ the manifold $\widetilde{M}$ has $\mathbf{Z}_5$ symmetry given by $z_1 \to e^{-2\pi i/5} z_1$ and $z_i \to z_i$ $(i = 2, 3, 4, 5)$. At $\psi^5 = 1$, $\widetilde{M}$ has a singularity at $z_i = 1$ $(\forall i)$, which looks like the origin of $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0$ in $\mathbf{C}^4$. In physics, it is called the *conifold singularity* while in mathematics it is called the ordinary double point. At this point, the worldsheet theory is expected to break down. At $\psi^5 = \infty$, the term $z_1 z_2 z_3 z_4 z_5$ dominates and the manifold looks like the union of five $\mathbb{CP}^3$'s. It turns out that this last point $\psi^5 = \infty$ corresponds to the large volume limit of the original quintic M. The other two points are in the deep interior of $\mathcal{M}_t(M)$.

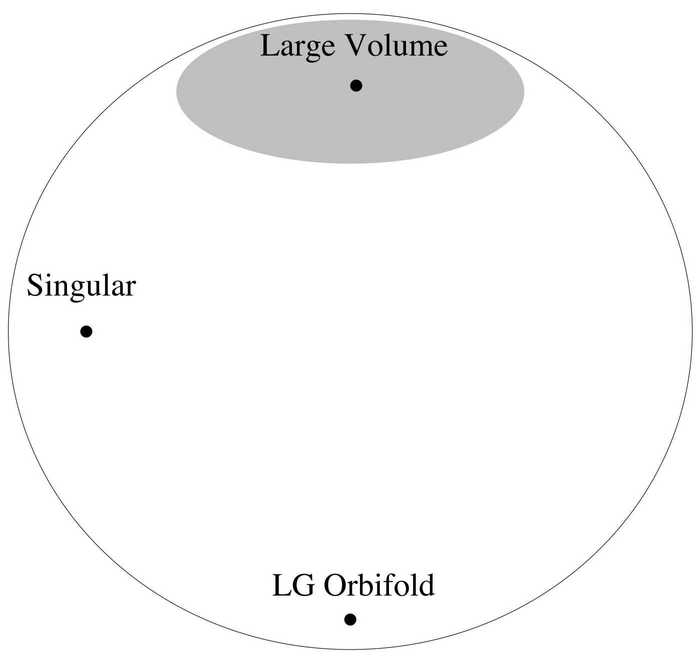

Figure 2: The Kahler moduli space of quintic

This provides a complete understanding of the geometry of $\mathcal{M}_t(M)$, but there remain some physical questions such as the interpretation of the two special points in the original theory. The situation is greatly improved by the use of linear sigma models which are very simple gauge theories with $(2,2)$ supersymmetry [6]. The twisted chiral parameters of a linear sigma model are the theta parameters of the gauge fields and their complex partners. The parameter space include the regions corresponding to non-linear sigma models on large volume Calabi-Yau manifolds but it also contains other regimes that have no such geometrical interpretation. Figure 2 shows the understanding of $\mathcal{M}_t(M)$ obtained by the linear sigma model. The shaded region correspond to the regime where the theory is well-described by non-linear sigma model on the quintic hypersurface M. The point marked "LG Orbifold" is where the theory is the Landau-Ginzburg model of five variables $x_1, ..., x_5$ with superpotential

$$W = G(x_1, x_2, x_3, x_4, x_5)$$

moded out by the orbifold group $\mathbf{Z}_5$ generated by $x_i \to \mathrm{e}^{2\pi i/5} x_i$. This corresponds to the point $\psi^5 = 0$. The point marked "Singular" is where the theory is singular becuase a non-compact flat direction opens out. This corresponds to the point $\psi^5 = 1$.

3.7 Linear sigma models

3.8 Degree d hypersurface in $\mathbb{CP}^{N-1}$

4 T-duality

5 Mirror Symmetry

5.1 Torus

5.2 Toric Fano manifolds

5.3 Degree d Hypersurface in $\mathbb{CP}^{N-1}$

6 SYZ conjecture

7 Topological field theory and D-brane category

8 Homological Mirror Symmetry

9 Recent Progress

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