Weak approximation for rationally connected varieties

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/ $\mathbb{C}$ although most statements extend to positive characteristic

§1 Geometric weak approximation

Background:

$$F = \begin{cases} \text{number field} \\ \mathbb{C}(B) & B \text{ smooth proj. curve} \end{cases}$$

$$v = \text{place of } F = \begin{cases} p \in \operatorname{Spec} \mathcal{O}_F, \infty_1 \ldots \infty_{r_1+r_2} \\ b \in B \end{cases}$$

$$F_v \quad \text{completion} = \begin{cases} \mathbb{Q}_p, \mathbb{R}, \mathbb{C} \\ \mathbb{C}((t_b)) \end{cases}$$

X/F algebraic variety

$$\underset{\substack{\text{rational}\\\text{points}}}{X(F)} \subset \underset{\substack{\text{points}\\\text{over}\\\text{completion}}}{X(F_v)}$$

②

Defn X satisfies

weak approximation (WA)

if for any places $v_1 \ldots v_r$

and open

$$\emptyset \neq U_i \subset X(F_{v_i})$$
$$i = 1 \ldots r$$

there exists $x \in X(F)$

with $x \in U_i$

Ex $X = \mathbb{P}^1 \quad F = \mathbb{Q}$

Chinese Remainder Theorem

Geometric translation

- $F = \mathbb{C}(B)$
- X proper $/F$

Choose a model $\mathcal{X} \xrightarrow{\pi} B$

flat, proper, $\underbrace{\mathcal{X}_{\mathbb{C}(B)}}_{\text{generic fiber}} = X$

Rational points $\iff$ Sections

$$x \in X(\mathbb{C}(B)) \iff \begin{tikzcd} \mathcal{X} \arrow[r, "\pi"'] & B \arrow[l, bend right, "s"'] \end{tikzcd}$$

(valuative criterion)

WA holds iff for any $b_1, \ldots, b_r \in B$, $N \geq 0$, formal sections

$$\hat{s}_i : \hat{B}_i = \mathrm{Spec}(\hat{\mathcal{O}}_{B\,b_i}) \longrightarrow \mathcal{X} \times_B \hat{B}_i$$

There exists $s : B \longrightarrow \mathcal{X}$ with

$$s \equiv \hat{s}_i \pmod{t_{b_i}^{N+1}}$$

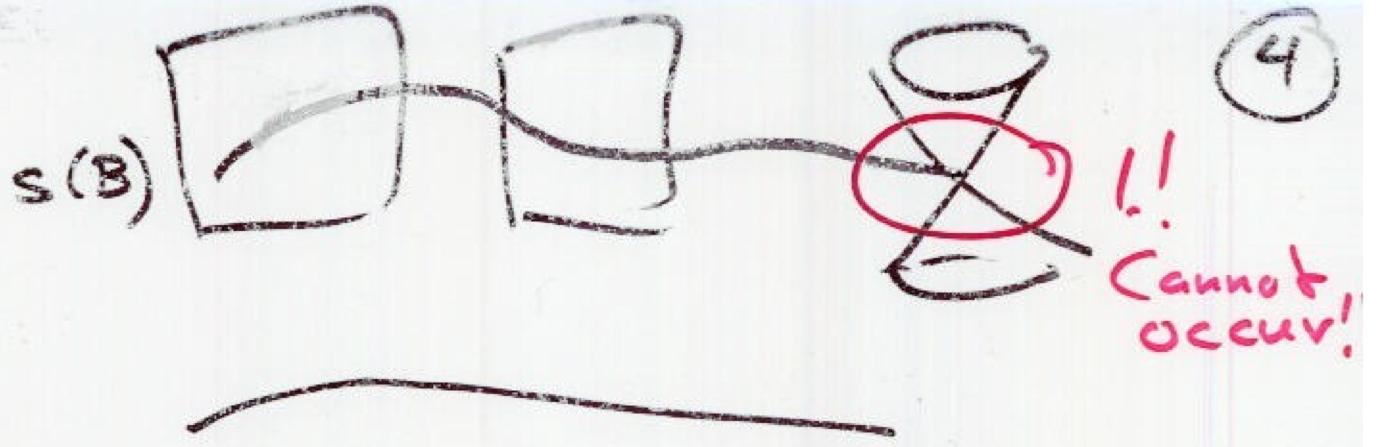
(uniformizer)

Assume

- $\mathcal{X}$ is regular model (resolve singularities)

Then $s(B) \subset \mathcal{X}^{sm}$ smooth locus for π

$$\mathcal{X} \longrightarrow B$$



Hensel's Lemma $\Rightarrow$

WA holds iff for any collection of N jets of sections

$$\begin{tikzcd} & \mathcal{X} \times_B \mathrm{Spec}(-) \arrow[d] \\ & \mathrm{Spec}\left(\mathcal{O}_{B,b_i}/t_{b_i}^{N+1}\right) \arrow[u, bend left, "j_i"] \end{tikzcd}$$

There exists $s : B \longrightarrow \mathcal{X}$ with $s \equiv j_i \pmod{t_{b_i}^{N+1}}$

General results

$/F = \mathbb{C}(B)$

Birationality

$X_1 \overset{\sim}{\dashrightarrow} X_2$ $/F$ $\Rightarrow$ WA holds for X_1 iff it holds for X_2

- $\mathbb{P}^n$, $Gr(k,m)$, $Q_2 \subset \mathbb{P}^n$ $n \geq 2$ smooth quadric

(WA holds)

Fibration property

Colliot Thélène / Gille

$X \longrightarrow Y$ fibration

WA holds for Y and for fibers $\Rightarrow$ WA holds for X

✓ · Conic bundles $X \longrightarrow \mathbb{P}^{n-1}$

✓ · Cubic hypersurfaces containing line $/F$ $\ell \subset X_3 \subset \mathbb{P}^n$ smooth $n \geq 3$

e.g. when $n > 5$

Conjecture??

X/F rationally connected

$F = \mathbb{C}(B)$

WA holds for X

E.g. $X_d \subset \mathbb{P}^n$

$d \leq n$ smooth

WA holds for X_d.

Problem

$X_3 \subset \mathbb{P}^3$
smooth cubic
surface $/F$

Does weak approximation
hold?

§3 Weak approximation at places of good reduction

$X / F = \mathbb{C}(B)$ smooth proper

Defn: $b \in B$ is of good reduction if there exists a smooth proper model

$$\hat{\mathcal{X}} \longrightarrow \hat{B}_b = \mathrm{Spec}(\hat{\mathcal{O}}_{B,b})$$

(Easy) **Theorem** There exists a proper flat algebraic space

Good

$$\pi : \mathcal{X}' \longrightarrow B$$

with $\mathcal{X}'_{\mathbb{C}(B)} = X$ and π smooth at places of good reduction

$\# \{ \text{places of bad reduction} \} < \infty$

Why algebraic spaces?

$$\mathfrak{X} = \{ s_0 f_0 + s_1 f_1 = 0 \} \subset \mathbb{P}^3 \times \mathbb{P}^1$$

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "\pi"'] \\ \mathbb{P}^1 \end{tikzcd}$$

generic pencil of cubic surfaces

$\{ p_1 \, \dots \, p_{32} \} \subset \mathbb{P}^1$

places of bad reduction

$B \longrightarrow \mathbb{P}^1$ branched double cover

Brieskorn gives resolution

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\rho"] \arrow[dr] & \mathfrak{X} \times_{\mathbb{P}^1} B \arrow[d] \\ & B \end{tikzcd}$$

minimal resolution

$\mathcal{Y}$ is not a scheme - no divisors meeting exceptional $\mathbb{P}^1$'s

Main Theorem

X proper rationally connected / $F = \mathbb{C}(B)$

Then X satisfies weak approximation at places of good reduction

proof $\mathcal{X} \longrightarrow B$ good regular model

$b_1 \ldots b_r \in B$
$j_1 \ldots j_r$ N-jets

Graber - Harris - Starr $\Longrightarrow$
$\exists$ section $s' : B \longrightarrow \mathcal{X}$

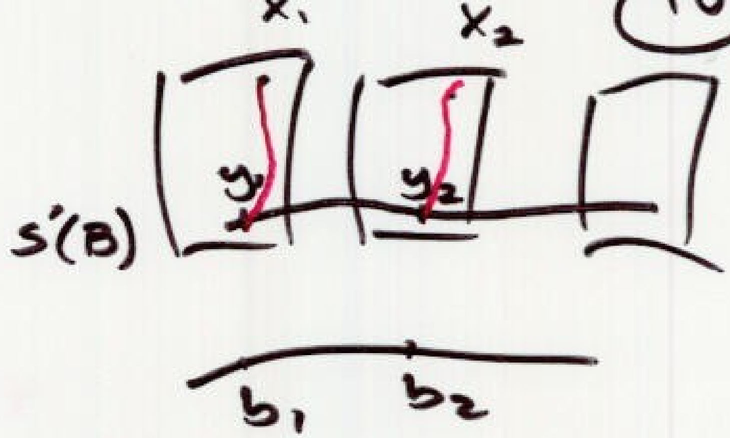
Induction on N

$(N=0)$ (Kollár Miyaoka Mori)

Goal Given $x_i \in \mathcal{X}_{b_i}$ smooth fibers
Produce $s : B \longrightarrow \mathcal{X}$ $s(b_i) = x_i$

$y_i = s'(b_i)$

$i = 1 \dots r$



Choose very free curves

$f_i : \mathbb{P}^1 \longrightarrow \mathcal{X}_{b_i}$

$f(0) = x_i \quad f(\infty) = y_i$

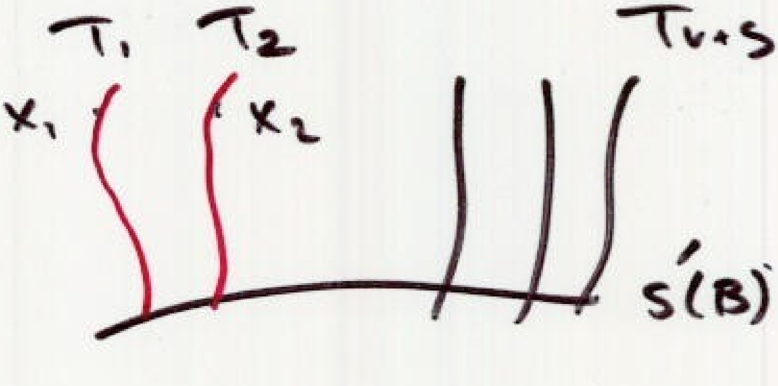
$T_i = f_i(\mathbb{P}^1)$

__Claim__ There exist additional very free curves

$f_k : \mathbb{P}^1 \longrightarrow \mathcal{X}_{b_k} \qquad k = r+1, \dots, r+s$

$f_k(\infty) = s(b_k)$

so that the "comb"

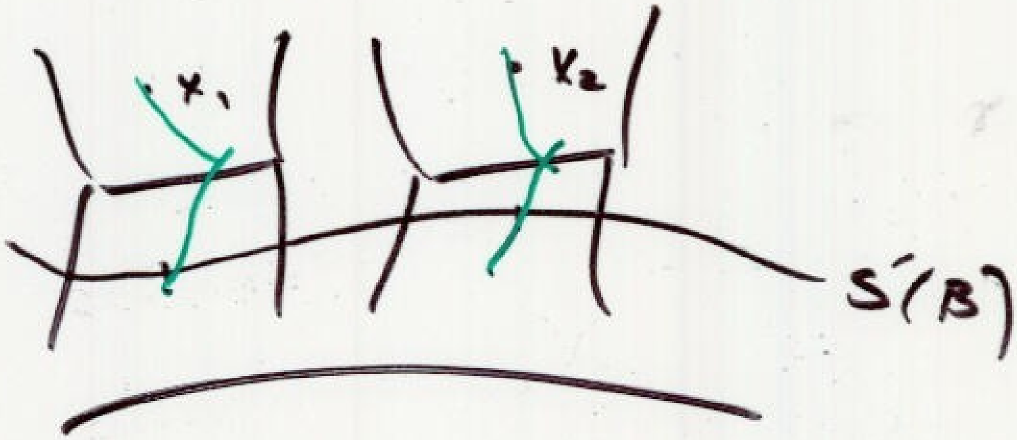


deforms to a smooth curve in $\mathcal{X}$ containing $x_1 \dots x_r$

$\leadsto$ desired section

Problem

Generalise to case where $x_i \in \mathfrak{X}_{b_i}$ are smooth points of singular fibers



$\Longrightarrow$ full weak approximation

(N=1)

$\boxed{\text{Goal}}$ Given $x_i \in \mathcal{X}_{b_i}$

tangent directions $v_i \in T_{x_i}\mathcal{X} - T_{x_i}\mathcal{X}_{b_i}$

Find a section

$s : B \longrightarrow \mathcal{X}$

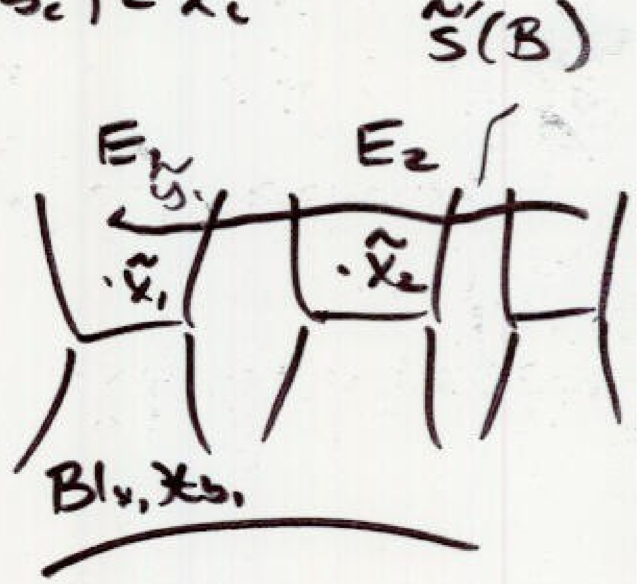
$s(b_i) = x_i$

$s'(b_i) = v_i$

Have $s' : B \longrightarrow \mathcal{X}$

with $s'(b_i) = x_i$

$$\begin{array}{c} \tilde{\mathcal{X}} = Bl_{\{x_1, \dots, x_n\}} \mathcal{X} \\ \downarrow \\ B \end{array}$$



$[v_i] = \tilde{x}_i \in E_i \cong \mathbb{P}^n$

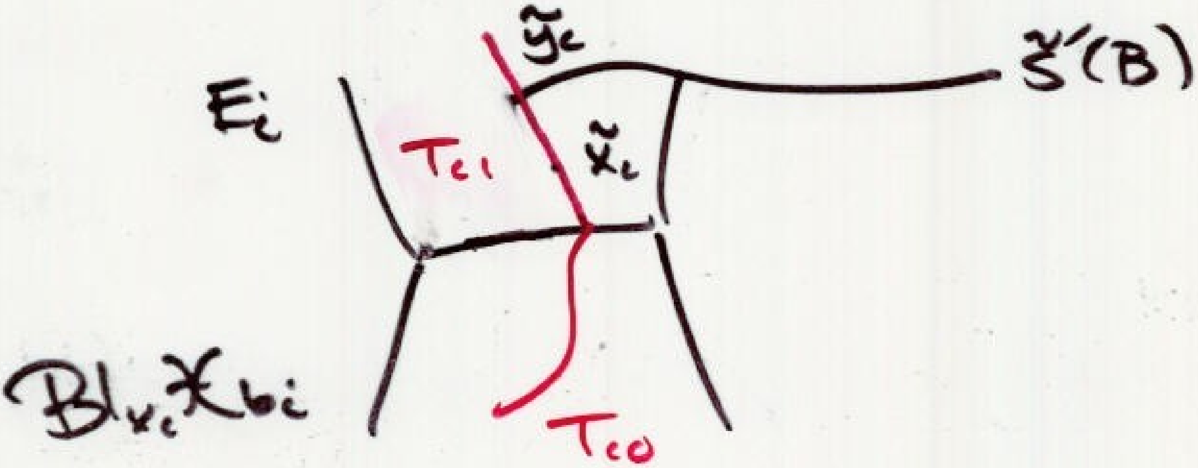
$\tilde{y}_i = \tilde{s}'(b_i)$

$\tilde{s}' : B \longrightarrow \tilde{\mathcal{X}}$

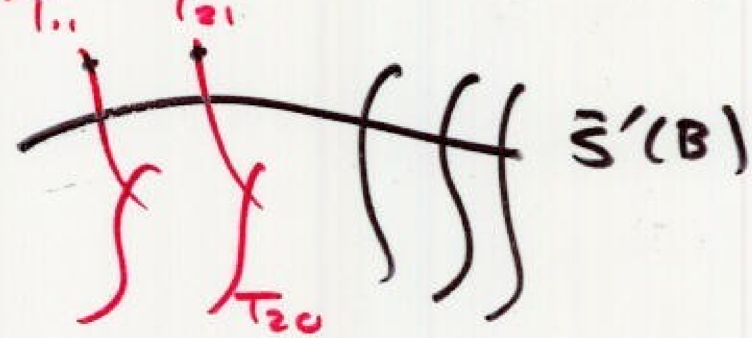
proper transform

$T_{c,1}$ = line joining
$\tilde{x}_c$ and $\tilde{y}_c$

$T_{c,0}$ = very free curve in
$Bl_{x_c}\mathcal{X}_{b_i}$ with
$T_{c,0} \cap E_c = T_{c,1} \cap Bl_{x_c}\mathcal{X}_{b_i}$



<u>Claim</u> There exists "broken
comb"



deforming to a section $\hat{s}: B \to \hat{\mathcal{X}}$
containing $\tilde{x}_1 \ldots \tilde{x}_r$

§4 Places of bad reduction? (14)

Theorem (with same proof)

$\mathcal{X} \longrightarrow B$ regular proper model

Assume: For each $b \in B$

$\mathcal{X}_b^{sm}$ is "strongly rationally connected" singular non-proper

(For each $x \in \mathcal{X}_b^{sm}$ there exists $f: \mathbb{P}^1 \longrightarrow \mathcal{X}_b^{sm}$ $f(0) = x$ $f(\infty) = \text{generic}$)

Then weak approximation holds

N.B. Excludes reducible fibers completely!!

Corollary

1. $\mathcal{X} \longrightarrow B$ regular proper with fibers cubic surfaces with at worst rational double points

Then weak approximation holds

2. (Subcase of 1)
Weak approximation holds for cubic surfaces with square-free discriminant

"generic"