Theorem (Tsen). If K is the function field of a curve over $\mathbb{C}$, and $X \subset \mathbb{P}^n_K$ any hypersurface of degree $d \leq n$ over K, then X has a K-rational point.

We can say that the correct extension to the category of all varieties of the condition "$d \leq n$" for hypersurfaces is "rationally connected."

Theorem (Lang). If K is the function field of an r-dimensional variety over $\mathbb{C}$, and $X \subset \mathbb{P}^n_K$ any hypersurface of degree d satisfying $d^r \leq n$, then X has a K-rational point.

Question. What is the correct extension to the category of all varieties of the condition "$d^r \leq n$" for hypersurfaces?

For example, is there a geometrically defined class of varieties such that any morphism $\pi : X \to S$ to a surface S whose general fiber belongs to this class necessarily has a rational section?

Such a class must necessarily be contained in the class of rationally connected varieties. So if we fiber S over a curve B, we can find sections of $X \to S$ over each fiber of $S \to B$.

The question is, can we choose these sections consistently?

This raises in turn another question: when is the space of curves on a given variety itself rationally connected?

What about hypersurfaces? Here we have a very interesting coincidence:

Theorem (Starr, -) Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. If $d^2 + d + 1 < n$, then for each degree e the space of rational curves of degree e on X is itself a rationally connected variety.

(We're not sure if the inequality in the theorem is sharp. But the correct inequality is almost certainly of the form $d^2 + O(d) \leq n$.)

More generally, Barry Mazur proposes a formal analogy, between homotopy theory on the one hand and algebraic geometry on the other:

connected $\longleftrightarrow$ rationally connected

components $\longleftrightarrow$ mrc quotient

loop space $\longleftrightarrow$ space of curves

$\pi_1 \longleftrightarrow$ mrc quotient of the space of curves

So, for example, the analog of "simply connected" would be the condition that "X is rationally connected and the space of rational curves on X is rationally connected."

There are many, many problems with this—especially the dependence of the geometry of the space of rational curves on the class.

By way of good news, we have the

Example: Let X be a cubic threefold. Evidence suggests that the mrc quotient of the space of rational curves of degree d on X stabilizes (after $d = 2$) to the intermediate jacobian of X, with the mrc fibration the Abel-Jacobi map. (Roth, Starr, -)

By way of bad news, we have the

Example: Let X be a cubic fourfold. de Jong and Starr have shown that the dimension of the mrc quotient of the space of rational curves of degree d on X goes to ∞ with d.

It seems clear that the question of finding rational sections of families over higher-dimensional bases is trickier than the one-dimensional case—for one thing, non-trivial Brauer-Severi varieties exist. The best work to date on this problem has been by de Jong and Starr.

Cubic fourfolds

The question of rationality of cubic fourfolds is an intriguing one—for one thing, as we said there is some indication that cubic fourfolds may provide an example where the condition of rationality is neither open nor closed.

For the following, $X \subset \mathbb{P}^5$ will be a smooth cubic fourfold.

Classically, it was known that some smooth cubic fourfolds are rational. For example, if X contains two skew 2-planes Γ and Λ, we get a birational map

$$\Gamma \times \Lambda \ \leftrightsquigarrow \ X$$

defined by sending a pair $(p, q) \in \Gamma \times \Lambda$ to the third point of intersection of the line $\overline{pq}$ with X.

More generally, if X contains a quartic scroll S, we get a similar map from the symmetric square of S to X:

$$S_2 = S \times S/\Sigma_2 \ \leftrightsquigarrow \ X$$

sending a chord $\overline{pq}$ to S to its residual intersection with X.

Note: this map is birational by virtue of the fact that the chords to S fill up $\mathbb{P}^5$ exactly once, i.e., a general point of $\mathbb{P}^5$ lies on a unique chord to S. As far as I know, the quartic scroll and the quintic del Pezzo are the only surfaces in $\mathbb{P}^5$ with this property.

Question Are there any others? More generally, are there any k-folds $X \subset \mathbb{P}^{2k+1}$ (other than rational normal scrolls) with the analogous property?

To understand more about X, we have to look at its Hodge structure. The Hodge diamond of X is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 1 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} \tag{1}$$

So the primitive Hodge structure of X in dimension 4—that is, the orthogonal complement of the square ω^2 of the hyperplane class—is a weight 2 Hodge structure of dimensions (1, 20, 1).

Theorem (Voisin). The Torelli map for cubic fourfolds is an open immersion.

In particular, for a very general X, the primitive Hodge structure $HS(X)$ is irreducible; and the fundamental class of any surface $S \subset X$ is a multiple of the hyperplane class squared.

Now, two facts: first, the

Theorem (Abramovich, Karu, Matsuki, Wlodarczyk). Any birational map can be factored into blow-ups and blow-downs.

More elementary is the fact that when we blow up a fourfold along a surface S, we introduce a copy of the weight 2 Hodge structure of S as a direct summand of the weight 4 Hodge structure of the fourfold.

What all this means is that if X is a very general cubic fourfold—so that $HS(X)$ is irreducible—and X is rational, the Hodge structure of X must appear as a summand of the Hodge structure of an algebraic surface S somewhere.

On the other hand...

Suppose we consider now not a very general cubic fourfold, but one that contains a surface S with class α independent from ω^2.

If we look at the orthogonal complement $\langle \omega^2, \alpha \rangle^\perp \subset H^4(X)$, this has the same dimensions $(1, 19, 1)$ and the same signature $(2, 19)$ as the primitive Hodge structure of a polarized K3 surface!

Such a cubic fourfold (with choice of sublattice $\langle \omega^2, \alpha \rangle \subset H^4(X)$) is called a *special cubic fourfold*; the the orthogonal complement $\langle \omega^2, \alpha \rangle^\perp \subset H^4(X)$ is called the special Hodge structure.

Theorem (Hassett) For each $d \equiv 0, 2 \pmod{6}$, $d \neq 6$, the special cubic fourfolds of discriminant d form an irreducible divisor $\mathcal{C}_d$ in the moduli space $\mathcal{M}$ of cubic fourfolds.

The next question would be, when is the special Hodge structure of such a cubic fourfold actually the primitve Hodge structure of a polarized K3 surface? Hassett answers this, too:

Theorem (Hassett). For $[X] \in \mathcal{C}_d$, the special Hodge structure of X is isomorphic to the primitive Hodge structure of a K3 surface if and only if

- 4 does not divide d;
- 9 does not divide d; and
- The only primes other than 2 and 3 dividing d are congruent to 1 (mod 3).

Of course, if we are going to obtain X from $\mathbb{P}^4$ by a series of blow-ups and blow-downs, it's not enough that the Hodge structure of X be (a summand of) the Hodge structure of a surface S; there also has to be a family of rational curves on X parametrized by S. This also occurs:

Theorem (Hassett). For infinitely many d, if X is the cubic fourfold corresponding to a general point of $\mathcal{C}_d$ then the Fano variety of lines on X is isomorphic to the symmetric square S_2 of a K3 surface S.

Finally, Hassett has shown that there is an infinite series of families of rational cubic fourfolds. Each family has codimension 2 in the moduli space of cubic fourfolds, and they are all contained in $\mathcal{C}_8$.

In sum: the rationality of cubic fourfolds remains a deep mystery. But if I had to bet, I'd guess the locus of rational cubic fourfolds formed a countable union of proper subvarieties of the moduli space $\mathcal{M}$.