Theorem. Let $\pi : X \to B$ be a proper morphism of varieties, with B a smooth curve. If the general fiber F of f is rationally connected, then f has a section.

Corollary. Let $X \rightsquigarrow Y$ be a dominant map of varieties, with general fiber F. If Y and F are rationally connected, then X is.

Preliminaries.

We will assume throughout that X is a smooth, connected projective variety, that $\pi : X \longrightarrow B$ is a nonconstant morphism to a smooth curve B, and that for general $b \in B$ the fiber $X_b = \pi^{-1}(b)$ is rationally connected.

We may also take $B \cong \mathbb{P}^1$; the general case will follow from this.

Finally, for simplicity we will assume the fiber dimension of π is at least 3.

Definition. An n-pointed *stable map* $(C; p_1, \ldots, p_n; f : C \to X)$ consists of

- a nodal curve C;
- n distinct ordered smooth points $p_1, \ldots, p_n \in C$; and
- a map $f : C \to X$

such that $\#\mathrm{Aut}(f) < \infty$.

For any $\beta \in N_1(X)$, the moduli space of all such maps with C of genus g and $f_*[C] = \beta$ is denoted $\overline{M}_{g,n}(X, \beta)$.

Note that we have a map

$$\phi : \overline{M}_{g,n}(X, \beta) \longrightarrow \overline{M}_{g,n}(B, d)$$

where $\pi_*\beta = d \cdot [B]$ (and we write d for $d \cdot [B]$).

Notation. Let C be a curve, E a locally free sheaf on C, $p \in C$ a smooth point and $\xi \subset E_p$ a one-dimensional subspace of the fiber E_p.

We will denote by $E(\xi)$ the sheaf of rational sections of E having at most a simple pole at p in the direction of ξ and regular elsewhere.

Lemma. Fix C, E and an integer n. There exists an integer N such that if $p_1, \ldots, p_N \in C$ are general points, $\xi_i \subset E_{p_i}$ general one-dimensional subspaces, and we set

$$E' = E(\xi_1 + \cdots + \xi_N)$$

then for $q_1, \ldots, q_n \in C$ arbitrary, we have

$$H^1(C, E'(-q_1 - \cdots - q_n)) = 0.$$

Proof. For some m, we have

$$H^1(C, E(p_1 + \cdots + p_m)) = 0.$$

Now just take $N = \mathrm{rank}(E) \cdot (m+n+g)$ and specialize to the case where the ξ_i span the fibers of E at $m+n+g$ points.

Normal bundles to nodal curves.

Suppose $C = D \cup D' \subset X$ is a nodal curve with D and D' smooth and $p = D \cap D'$ a node of C. Let

$$\xi \subset (N_{D/X})_p$$

be the one-dimensional subspace given by $T_p D'$.

We have an inclusion of sheaves

$$0 \to N_{D/X} \to (N_{C/X})|_D$$

identifying $(N_{C/X})|_D$ with the sheaf $N_{D/X}(\xi)$ of sections of $N_{D/X}$ having a pole at p in the direction ξ.

A first-order deformation $\sigma \in H^0(C, N_{C/X})$ of C in X smooths the node p iff $\sigma|_D \notin H^0(D, N_{D/X})$.

On with the argument!

Given the basic setup $\pi : X \to B$, our goal will be to construct a curve $C \subset X$ such that by deforming C in X, we can move the branch points of the projection

$$\pi|_C : C \to B$$

independently—in other words, such that the map

$$\phi : \overline{M}_{g,0}(X, \beta) \longrightarrow \overline{M}_{g,0}(B, d)$$

is locally dominant at the point $[C]$.

Such a curve will be called *flexible*.

Since the space of branched covers of $\mathbb{P}^1$ of degree d and genus g is irreducible, and its closure in $\overline{M}_{g,0}(\mathbb{P}^1, d)$ contains points $f : C \to \mathbb{P}^1$ at the boundary consisting of d copies of $\mathbb{P}^1$ each mapping isomorphically to the target, we can degenerate a flexible curve to a union of d sections of $\pi : X \to B \cong \mathbb{P}^1$.

Note: this is the only point at which we will use the hypothesis that $B \cong \mathbb{P}^1$. We could avoid this by invoking the (less well known) fact that the space of branched covers of B is irreducible whenever g is large relative to d and the genus of B.

When is a curve flexible?

Suppose $C \subset X$ is a smooth curve, with $\pi|_C : C \to B$ simply ramified at points $p_1, \ldots, p_b \in C$, and suppose that the points p_i are smooth points of the fiber of π, so that the differential

$$\pi_* : (N_{C/X})_{p_i} \to T_{\pi(p_i)}\mathbb{P}^1$$

is surjective. Then if

$$H^1(C, N_{C/X}(-p_1 - \cdots - p_b)) = 0,$$

it follows that

$$H^0(C, N_{C/X}) \twoheadrightarrow \oplus_i (N_{C/X})_{p_i}$$

and so deformations of $C \subset X$ dominate deformations of its branch divisor.

Now suppose $C_0 \subset X$ is any smooth curve, disjoint from the singular locus of the map π.

Let $p_1, \ldots, p_N \in C_0$ be general points of C_0 and

$$\xi_i \in (N_{C_0/X})_{p_i} = T_{p_i} X_{p_i}$$

general normal directions to C_0 at the points p_i. Since X_{p_i} is rationally connected, we can find a smooth rational curve $C_i \subset X_{p_i}$ such that

- $C_i \cap C = \{p_i\}$
- $T_{p_i} C_i = \xi_i$; and
- $N_{C_i/X}$ is generated by global sections

Let

$$C = C_0 \cup \bigcup_i C_i$$

By our Lemma, for $N \gg 0$, we see that N_C is generated by global sections, so that C can be deformed to a smooth curve C', still of genus g; and moreover, if R is the ramification divisor

$$H^1(C, N_{C/X}(-R)) = 0$$

so the same is true for C'.

So what's the problem?

Just one: the requirement that C be disjoint from the singular locus of π. If the singular locus of π has codimension 2 or more in X, we can just take C a general complete intersection in X and we're done. But if the singular locus of π has codimension 1—in other words, if a fiber of π has multiple components—this is a problem.

This is serious: if a fiber X_q has a multiple component, then any point of $C \cap X_q$ is necessarily a ramification point of $\pi|_C : C \to B$, and the corresponding branch point q of $\pi|_C$ cannot be moved under deformation.

And, of course, if π has an everywhere nonreduced fiber, there can't be any section of π.

To deal with this, we need a second construction. Let $C \subset X$ be any smooth curve, $\Delta \subset B$ the branch divisor of $\pi|_C$, $b \in B$ not in Δ and $p, q \in C \cap X_b$. Choose a rational curve $D \subset X_b$ such that

$$C \cap D = \{p, q\}.$$

After adding a collection of rational curves C_i in fiber of π meeting C once, we can deform the result to a smooth curve C', with the property that:

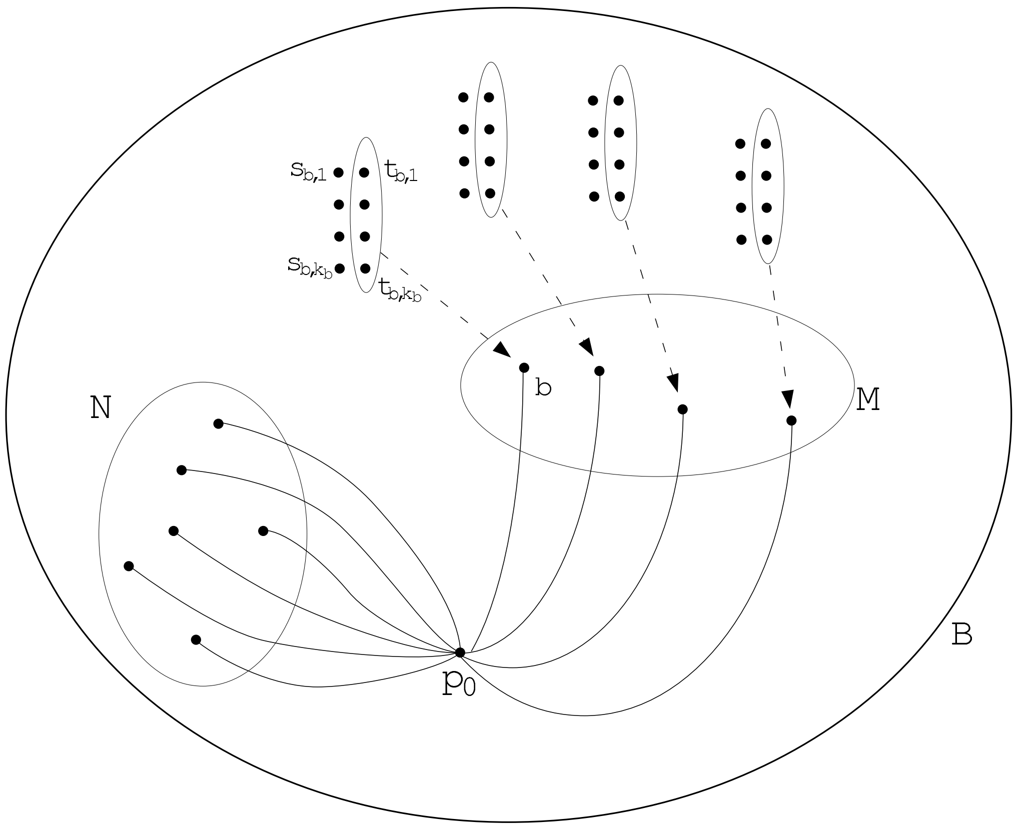
The branch divisor Δ' of $\pi|_{C'}$ is the union of a small deformation of Δ with a pair of points near b, each having monodromy exchanging the sheets containing p and q.

In other words: given a smooth curve $C \subset X$, we can introduce m new pairs of branch points of $\pi|_C$, each with assigned monodromy. We can also ensure that the deformations resulting curve $C' \subset X$ move the branch points of $\pi|_{C'}$ freely.

Now: let $C \subset X$ be a general complete intersection. Let $M \subset B$ be the locus of fibers with multiple components. For each $b \in M$, let σ_b be the monodromy of $\pi|_C$ around b, and express σ_b as a product of transpositions:

$$\sigma_b = \tau_{b,1}\tau_{b,2} \cdots \tau_{b,k_b}$$

Next, for each $b \in M$ and $\alpha = 1, \ldots, k_b$ we create two new branch points with monodromy $\tau_{b,\alpha}$; and for each b we let one of each of these pairs tend to b.



The limiting stable curve will then have no monodromy around $b \in M$; that is, any component of the limit flat over B will be branched away from M.

For arbitrary B:

Just express B as a branched cover $g : B \to \mathbb{P}^1$ of $\mathbb{P}^1$, and take the "norm" of π under g: that is, the variety Y over $\mathbb{P}^1$ whose fiber over a general $p \in \mathbb{P}^1$ is the product

$$Y_b = \prod_{q \in g^{-1}(p)} X_q.$$

By the result for $\mathbb{P}^1$, the map $Y \to \mathbb{P}^1$ has a section, and hence so does $X \to B$. (de Jong)

A converse

Taken literally, the converse of the theorem is nonsense: families with any kind of fibers may have sections. But it's still reasonable to ask whether the theorem holds for any larger, geometrically defined class of varieties.

For example, Serre asked if it held for families of $\mathcal{O}$-*acyclic* varieties—that is, varieties X with $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

We do have the

Theorem (Graber, Mazur, Starr, -). Let $\pi : X \to B$ be any morphism. If, for any irreducible curve $C \subset B$ the restriction

$$\pi_C : X_C = X \times_B C \to C$$

has a section, then X contains a subvariety Z dominating B whose general fiber Z_b is rationally connected.

We can apply this to the universal family over the moduli space of of polarized Enriques surfaces to conclude the

Corollary There exist one-parameter families of Enriques surfaces without rational sections.

G. Lafon has actually constructed families of Enriques surfaces with everywhere nonreduced fibers.