Geometry of $\mathcal{A}_g$ and Its Compactifications

Samuel Grushevsky
Princeton University

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Abelian varieties (algebraically).

Abelian variety: a projective g-dimensional variety A, with the structure of an abelian group.

Polarization on A: an ample line bundle Θ.

Principal polarization: $h^0(A, \Theta) = 1$.

We study $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g.

Denote $\pi : \mathcal{X}_g \to \mathcal{A}_g$ the **universal family of abelian varieties**: the fiber over $[A]$ is A.

The **Hodge bundle** on $\mathcal{A}_g$ is $L := \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$.

Abelian varieties over $\mathbb{C}$.

Complex abelian variety: $A = \mathbb{C}^g/\Lambda$ for a lattice $\Lambda = \mathbb{Z}^g + \tau\mathbb{Z}^g$, where $\tau \in \mathrm{Mat}_{g\times g}(\mathbb{C})$, $\tau^t = \tau$, $\mathrm{Im}\,\tau > 0$.

Theta function: a function of τ and $z \in \mathbb{C}^g$:

$$\theta(\tau, z) := \sum_{n\in\mathbb{Z}^g} \exp(\pi i(n^t\tau n + 2n^t z)).$$

The theta function is even in z, and automorphic w.r.t. to Λ: if $v \in \mathbb{Z}^g + \tau\mathbb{Z}^g$, then $\theta(\tau, z + v) = \exp(f(v, z))\,\theta(\tau, z)$.

Thus the zero locus of theta is defined in the abelian variety $A_\tau = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Its **theta divisor** Θ is the principal polarization.

$\mathcal{A}_g$ over $\mathbb{C}$, analytically.

The **Siegel upper half-space** $\mathcal{H}_g$: all τ such that $\tau^t = \tau$, $\mathrm{Im}\,\tau > 0$.

$\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \tau := (A\tau + B)(C\tau + D)^{-1}.$$

Then $\mathcal{A}_g = \mathcal{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$; dim $\mathcal{A}_g = \frac{g(g+1)}{2}$.

The Hodge bundle is $L|_{[A_\tau]} = H^{1,0}(A_\tau)$.

$\mathcal{A}_g$ is not compact ($\mathcal{H}_g$ is not compact)

Question: How can one compactify $\mathcal{A}_g$?

First answer. Projective embedding:
represent $\mathcal{A}_g$ explicitly as a quasi-projective variety, i.e. embed $\mathcal{A}_g$ into $\mathbb{P}^N$ and take the closure.

For this, one needs a very ample line bundle. L is ample, but not very ample.

A **ρ-valued modular form**, for $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Z})$, and a representation $\rho : \Gamma \to \mathrm{End}\, V$, is a function $F : \mathcal{H}_g \to V$ such that

$$\forall \gamma \in \Gamma, \tau \in \mathcal{H}_g \qquad F(\gamma \circ \tau) = \rho(\gamma) \circ F(\tau).$$

A (scalar) weight k modular form: for the linear representation $\rho(\gamma) = \det(C\tau + D)^k$.

L is the bundle of modular forms of weight 1.

Theta constants.

Choose $a, b \in (\mathbb{Z}/n)^g$. The **level n theta constant** is

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) := \text{const } \theta(\tau, a\tau + b).$$

This is a modular form of weight 1/2 w.r.t. a finite index normal subgroup $\Gamma(2n, 4n) \subset \mathrm{Sp}(2g,\mathbb{Z})$. The corresponding **level cover** is

$$\mathcal{H}_g/\Gamma(2n, 4n) =: \mathcal{A}_g(2n, 4n) \to \mathcal{A}_g.$$

Define a map

$$\begin{aligned} Th_n : \mathcal{A}_g(2n, 4n) &\dashrightarrow \mathbb{P}^{n^{2g}-1} \\ [\tau] &\mapsto \left\{ \theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau, 0) \right\}_{\text{all } a,b \in (\mathbb{Z}/n)^g} \end{aligned}$$

Theorem (Igusa, Mumford)
Th_n is an embedding for all $n > 1$.

Taking higher level theta functions allows one to embed a single abelian variety into a projective space. $Th_n(\tau)$ is the image of the origin under such an embedding.

Instead, one can consider the Gauss map at the origin, i.e. the map to a Grassmannian:

$$\begin{aligned} \Phi_n : &\mathcal{A}_g(2n, 4n) \dashrightarrow G(g, n^{2g}) \\ &\tau \mapsto \left\{ \mathrm{grad}\, \theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau, z)|_{z=0} \right\}_{\text{all } a,b,\in (\mathbb{Z}/n)^g} \end{aligned}$$

Theorem (Salvati Manni, G.- '04)
Φ_n is an embedding for all $n > 1$.

This is the abelian variety version of the fact that any plane quartic is determined by its bitangents (Caporaso, Sernesi, '00 Lehavi '01) or, more generally, by its odd theta characteristics (Caporaso, Sernesi, '02)

Compactifying $\mathcal{A}_g$.

What points should we add, and what kind of degenerate objects do they parameterize?

$\mathcal{H}_g$ is not compact.

Question: What should be $\lim\limits_{t\to\infty} \begin{pmatrix} it & w \\ w^t & \tau \end{pmatrix}$?

Answer 1: can take as the limit $[\tau]$, so we add $\mathcal{A}_{g-1}$ as a boundary component. For

$$\begin{pmatrix} i\infty & i\infty & w_1 \\ i\infty & i\infty & w_2 \\ w_1^t & w_2^t & \tau \end{pmatrix} \quad \text{we then take} \quad [\tau] \in \mathcal{A}_{g-2}.$$

The **Satake**, or **minimal** compactification, is

$$\mathcal{A}_g^S := \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0.$$

It is highly singular; boundary points do not represent actual degenerations of abelian varieties. Th_n embeds $\mathcal{A}_g^S$ into $\mathbb{P}^{n^{2g}-1}$.

Partial compactification of $\mathcal{A}_g$.

Answer 2: as $\lim\limits_{t\to\infty} \begin{pmatrix} it & w \\ w^t & \tau \end{pmatrix}$ we take (τ, w).

The vector w is only defined up to $\mathbb{Z}^{g-1}\tau$, i.e. $w \in A_\tau$. Thus we get the **partial compactification**

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{X}_{g-1}.$$

$\mathcal{A}_g^*$ is the blowup of (the partial Satake) $\mathcal{A}_g \sqcup \mathcal{A}_{g-1}$ along the boundary.

The boundary is codimension one; its points represent **semiabelian** varieties: compactify

$$1 \to \mathbb{C}^* \to G \to B \to 0, \qquad [B] \in \mathcal{A}_{g-1},$$

to a $\mathbb{P}^1$-bundle $\tilde{G}$, by adding 0 and ∞ sections, which are identified with a shift by w.

The principal polarization on a semiabelian variety is the blowup over $\Theta_{A_\tau} \cap t_w \Theta_{A_\tau}$ of a section of $\tilde{G}$.

Toroidal compactifications of $\mathcal{A}_g$.

How can we get a compactification with boundary normal crossing divisors? What should be

$$\lim_{t_1, t_2 \to \infty} \begin{pmatrix} it_1 & x & z_1 \\ x & it_2 & z_2 \\ z_1^t & z_2^t & \tau \end{pmatrix}?$$

Can certainly get $(\tau, z_1, z_2) \in \mathcal{X}_{g-2}^{\times 2\,(\text{fiberwise})}$.

Problem: How do we keep track of x, which may also go to $i\infty$?
We need to make a choice — a so-called fan decomposition. Some common choices are:

The **Perfect cone,** aka **first Voronoi** compactification $\overline{\mathcal{A}_g}^P$: the monoidal blowup of Satake along the boundary

The **Second Voronoi** compactification $\overline{\mathcal{A}_g}^V$: the one to which $\overline{\mathcal{M}_g}$ maps.

All toroidal compactifications $\overline{\mathcal{A}_g}$ admit a morphism to $\mathcal{A}_g^S$. The preimage of $\mathcal{A}_{g-i} \subset \mathcal{A}_g^S$ is a $(\mathbb{C}^*)^{\frac{i(i-1)}{2}}$ torsor over $\mathcal{X}_{g-i}^{\times i}$.

What kind of objects do the boundary points of $\overline{\mathcal{A}_g}$ parameterize? Should be some kind of semiabelian varieties $1 \to (\mathbb{C}^*)^i \to G \to B \to 0$ for $[B] \in \mathcal{A}_{g-i}$. What is the polarization?

Theorem (Alexeev '99) The second Voronoi compactification $\overline{\mathcal{A}_g}^V$ is (a component of) a functorial compactification; the boundary points represent geometric objects. $\overline{\mathcal{A}_g}^V$ is projective.

Open problem: Which compactification does Φ_n induce — it embeds $\mathcal{A}_g^*(2n, 4n)$?

Homology and Chow rings of $\mathcal{A}_g$.

For $\mathcal{A}_1$ and $\mathcal{A}_2$ known classically.

$H^i_{\mathbb{Q}}(\mathcal{A}_3)$ and $H^i_{\mathbb{Q}}(\overline{\mathcal{A}_3})$ computed: (Hain '02)
The dimensions are 1,0,1,0,1,0,2 and 1,0,1,0,1,0,2,0,1,0,1,0,1, respectively.

$CH^*(\overline{\mathcal{A}_3})$ computed: (van der Geer '97)
The dimensions are 1,2,4,6,4,2,1.

Theorem (van der Geer '96).
The subring of $CH^*_{\mathbb{Q}}(\overline{\mathcal{A}_g})$ generated by the **Hodge classes** $\lambda_i := c_i(L)$ has only one relation:

$$(1+\lambda_1+\ldots+\lambda_g)(1-\lambda_1+\ldots+(-1)^g\lambda_g)=1.$$

For $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ have one more relation: $\lambda_g=0$.

(The torsion of $\lambda_g \in CH^*_{\mathbb{Z}}(\mathcal{A}_g)$, and subvarieties representing it were studied by Ekedahl, van der Geer '03,'04.)

Intersection theory of divisors on $\overline{\mathcal{A}_g}$.

Corollary (van der Geer '97).
The intersection numbers of divisors on $\overline{\mathcal{A}_3}$ are

L^6	L^5D	L^4D^2	L^3D^3	L^2D^4	LD^5	D^6
$\frac{1}{181440}$	0	0	$\frac{1}{720}$	0	$-\frac{203}{240}$	$-\frac{4103}{144}$

Theorem (Erdenberger, Hulek, G.- '05) The intersection numbers of divisors on $\overline{\mathcal{A}_4}^P$ are

L^{10}	L^6D^4	L^3D^7	LD^9	D^{10}
$\frac{1}{907200}$	$-\frac{1}{3780}$	$-\frac{1759}{1680}$	$\frac{1636249}{1080}$	$\frac{101449217}{1440}$

,
while all others are zero.

The intersection theory of divisors on $\overline{\mathcal{A}_4}^V$ (i.e. including E) was also determined.

Work in progress: intersection numbers of divisors on $\overline{\mathcal{A}_g}^P$ for all g.

Subvarieties of $\mathcal{A}_g$.

$\lambda_1^{\frac{g(g-1)}{2}+1}=0\in CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ (van der Geer '96); since λ_1 is ample on $\mathcal{A}_g$, there are no complete subvarieties of $\mathcal{A}_g$ of codimension less than g. However, $\lambda_1^{\frac{g(g-1)}{2}}\neq 0$, so...?

Conjecture (Oort)/**Theorem** (Keel, Sadun '02).
Over $\mathbb{C}$, there does not exist a complete subvariety of $\mathcal{A}_g$ of codimension g.

Open problems: What is the cohomological dimension of $\mathcal{A}_g$? What is the maximal dimension of a complete subvariety (since $\partial\mathcal{A}_g^S$ is codimension g, we know that there exist complete subvarieties of dimension $g-1$).

Andreotti-Mayer loci.

We can construct some (non-complete) loci inside $\mathcal{A}_g$: Jacobians, Pryms, $\ldots$, but how do we get a stratification?

The **Andreotti-Mayer locus** $N_k \subset \mathcal{A}_g$: those abelian varieties for which $\dim \operatorname{Sing} \Theta = k$.

Theorem (Andreotti, Mayer) N_{g-4} contains the Jacobian locus as an irreducible component; N_{g-3} contains the hyperelliptic locus.

Theorem (Ciliberto, van der Geer '99). For all $k \leq g-3$ we have $\operatorname{codim} N_k \geq k+2$.

Is this a reasonable bound for codimension?

Conjecture. Within the locus of simple abelian varieties, $\operatorname{codim} N_k \geq \frac{(k+1)(k+2)}{2}$.

Open problem: Is it possible that $N_k = N_{k+1}$?

For reducible abelian varieties $A = A_1 \times A_2$ have

$$\Theta_A = (A_1 \times \Theta_{A_2}) \bigcup (\Theta_{A_1} \times A_2)$$

and thus

$$\operatorname{Sing} \Theta_A \supset \Theta_{A_1} \times \Theta_{A_2}.$$

Conjecture (Arbarello, De Concini) / **Theorem** (Ein, Lazarsfeld '96) N_{g-2} is equal to the locus of reducible abelian varieties.

Multiplicity of the theta divisor.

Let $\operatorname{Sing}_k \Theta := \{x \in A \mid \operatorname{mult}_x \Theta \geq k\}$.

Theorem (Kollár '95) The pair (A, Θ) is log canonical; thus $\operatorname{codim}_A \operatorname{Sing}_k \Theta \geq k$.

The **multiplicity locus** S_k: those abelian varieties for which $\operatorname{Sing}_k \Theta$ is non-empty. Note that $S_1 = \mathcal{A}_g$, $S_2 = N_0$, $S_{g+1} = \emptyset$.

Theorem (Smith, Varley '96)
$S_g = \{\text{products of } g \text{ elliptic curves}\}$.

This is a special case of
Theorem (Ein, Lazarsfeld '96) If for $k \geq 2$ we have $\operatorname{codim}_A \operatorname{Sing}_k \Theta = k$, then A is reducible.

Open problems:
What is the dimension of S_k?
What is the maximal k such that S_k contains irreducible varieties?

Birational geometry of $\mathcal{A}_g$.

$\operatorname{Pic}(\mathcal{A}_g) = \operatorname{Pic}(\mathcal{A}_g^S) = \mathbb{Q}L$ (Borel). Thus $\operatorname{Pic}(\mathcal{A}_g^*) = \mathbb{Q}L \oplus \mathbb{Q}D$.

The canonical class is

$$K_{\mathcal{A}_g^*} = (g+1)L - D.$$

The **slope** of a divisor $E = aL - bD \in \mathrm{Pic}(\mathcal{A}_g^*)$ is $s(E) = a/b$: for modular forms this is the weight divided by the generic vanishing order on the boundary.

Theorem (Tai) Any section of $mK_{\mathcal{A}_g^*}$ extends to $\overline{\mathcal{A}_g}^P$, so for studying the birational geometry $\mathcal{A}_g^*$ is good enough.

The nef cone of $\mathcal{A}_g^*$.

L is ample on $\mathcal{A}_g^S$ and thus nef on $\mathcal{A}_g^*$. Moreover, $D|_D = -2\Theta$ (when we identify $\partial\mathcal{A}_g^* = \mathcal{X}_{g-1}$), and thus $-D$ is relatively ample wrt $\mathcal{A}_g^* \to \mathcal{A}_g^S$.

We have two obvious curve classes in $\mathcal{A}_g^*$: $\overline{\mathcal{A}_1} \times pt$, and any curve in the boundary projecting to a point in $\mathcal{A}_g^S$.

Theorem (Hulek '97)
a) The cone of curves on $\mathcal{A}_g^*$ is generated by these two curves; thus the nef cone of $\mathcal{A}_g^*$ is

$$\{aL - bD | a \geq 12b \geq 0\}.$$

b) For the genus 3 toroidal compactifications $\overline{\mathcal{A}_3}^P = \overline{\mathcal{A}_3}^V$ the same result holds.

The nef cones of $\overline{\mathcal{A}_4}$.

In general $\overline{\mathcal{A}_g}^P \neq \overline{\mathcal{A}_g}^V$. In fact $\mathrm{Pic}(\overline{\mathcal{A}_g}^P) = \mathbb{Q}L \oplus \mathbb{Q}D$, while $\mathrm{Pic}(\overline{\mathcal{A}_g}^V)$ is larger.

However, there exists a morphism $\overline{\mathcal{A}_4}^V \to \overline{\mathcal{A}_4}^P$, the exceptional divisor E of which is the other generator of $\mathrm{Pic}(\overline{\mathcal{A}_4}^V)$.

Theorem (Hulek, Sankaran '02) The nef cone of $\overline{A_4}^P$ is also equal to

$$\{aL - bD | a \geq 12b \geq 0\},$$

while the nef cone of $\overline{A_4}^V$ is

$$\{aL - bD - cE | a \geq 12b \geq 24c \geq 0\},$$

In general the nef cone of $\overline{\mathcal{A}_g}^V$ is hard to describe, since there are many generators.

Canonical model of $\mathcal{A}_g$.

Theorem (Shepherd-Barron '05) In any genus the nef cone of $\overline{\mathcal{A}_g}^P$ is the same as that of $\mathcal{A}_g^*$, i.e. is $\{aL - bD | a \geq 12b \geq 0\}$.

Corollary (Shepherd-Barron '05) $\overline{\mathcal{A}_g}^P$ is the canonical model of $\mathcal{A}_g$ for $g \geq 12$, since $K_{\overline{\mathcal{A}_g}^P} = (g+1)D - L$ is then ample.

Idea of the proof. Assume that $12L - D$ is not nef, i.e. there exists a curve in $\overline{\mathcal{A}_g}^P$ that it intersects negatively. This curve must lie entirely in $\partial\overline{\mathcal{A}_g}^P$. By using the torus action on the stratum of $\overline{\mathcal{A}_g}^P$ over $\mathcal{A}_{g-i} \subset \mathcal{A}_g^S$, show that if there is a curve there, then there is such a curve that is fixed by the torus, and there is such a curve over $\mathcal{A}_{g-i-1}$. At the end get a curve over $\mathcal{A}_0$, because there the torus action is transitive. □

Kodaira dimension of $\mathcal{A}_g$.

L is big and nef; thus if $K_{\mathcal{A}_g^*} = cL$+effective for $c > 0$, $\overline{\mathcal{A}_g}^P$ is of general type. How do we get effective divisors of small slope?

Product of all theta constants of level two gives slope $s(\theta_{\text{null}}) = 8 + \frac{1}{2^{g-3}}$.

For the Andreotti-Mayer divisor the slope is:

$$s(N_0) = 6 + \frac{12}{g+1} \qquad \text{(Mumford)}.$$

Theorem (Freitag, Tai, Mumford) $\mathcal{A}_g$ is of general type for $g \geq 7$
Theorem (Clemens) $\mathcal{A}_4$ is unirational.
Theorem (Donagi; Mori, Mukai; Verra) $\mathcal{A}_5$ is unirational.

Question: What about $\mathcal{A}_6$?

New (effective) geometric divisors on $\mathcal{A}_g \otimes \mathbb{C}$.

Constructing a divisor: Choose globally on $\mathcal{A}_g(m)$ some d distinct points of order m, and consider the theta divisor shifted by these d points (call the translates $D_1 \dots D_d$), and consider the locus $T_{m,d} \subset \mathcal{A}_g(m)$ of abelian varieties for which the intersection $D_1 \cap \dots \cap D_d$ is singular. Then project this to $\mathcal{A}_g$ to get a locus there.

Proposition (Lehavi, G.- '05). For $d \leq g+1,\ m \geq 3$, the locus $T_{m,d}$ is a divisor.

Technical confession: when lifted to the universal family of level abelian varieties (and when we identify the fibers of $\mathcal{X}_g(m)$ and $\mathcal{X}_g$), translates become sections of $m\Theta$.

Note $T_{1,1} = N_0$. Other interesting cases are:

1) $T_{m,g-1}$: the intersection $\cap D_i$ is generically a smooth curve, so we get a rational map from $\mathcal{A}_g(m)$ to a moduli space of curves

2) $T_{m,g}$: we are looking at the divisor of those abelian varieties for which some of the $g!$ intersection points of $\cap D_i$ coincide (become multiple).

3) $T_{m,g+1}$: the abelian varieties for which the intersection of $g+1$ translated theta divisors is non-empty.

Computing the class of $T_{m,g}$ in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_g^*)$

We need to compute the number of vertical singularities over a test curve

$$\begin{array}{ccccc} \mathcal{M} = \cap \mathcal{D}_i & \hookrightarrow & \mathcal{A} & \subset & \mathcal{X}_g(m) \\ \downarrow p_{\mathcal{M}} & & \downarrow p_{\mathcal{A}} & & \downarrow \\ C & = & C & \subset & \mathcal{A}_g^*(m) \end{array}$$

where $\mathcal{D}_i$ is the universal family of translated theta divisors (they are all numerically equivalent).

By Grothendieck-Riemann-Roch, aka Riemann-Hurwitz, this is

$$\mathrm{ch}(p_{\mathcal{M}!}\mathcal{O}_{\mathcal{M}}(K_{\mathcal{M}/C})) = \frac{1}{2} p_{\mathcal{A}*}\left(g\mathcal{D}^{g+1} - \mathcal{D}^g c_1(T_{\mathcal{A}/C})\right)$$

For $C \subset \mathcal{A}_g$, in the smooth locus, Mumford computed

$$p_{\mathcal{A}*}(\Theta^{g+1}) = \frac{(g+1)!}{2}\lambda; \quad p_{\mathcal{A}*}(\Theta^g) = g!.$$

Our divisor $\mathcal{D} = m\Theta$.

The universal semiabelian variety

The universal semiabelian variety

$$p : \mathcal{X}_g \longrightarrow \partial\mathcal{A}_g^* \longrightarrow \mathcal{A}_{g-1} \subset \mathcal{A}_g^S$$

is described as follows.

Take $[B] \in \mathcal{A}_{g-1}$. Then $p^{-1}([B])$ is the birigidified Poincaré bundle on $B \times B$:

First, we take a $\mathbb{C}$-bundle $P \to B \times B$ such that for any point $b \in B$ we have

$$P|_{\{0\}\times B} = \mathcal{O}; \quad P|_{B\times\{b\}} = \mathcal{O}[b]$$

(so it is numerically trivial on all horizontal and vertical fibers).

$p^{-1}(B)$ is obtained by taking the projectivization $\overline{P} := \mathbb{P}(\mathcal{O} \oplus P)$, and gluing the 0-section to the ∞-section with a shift:

$$(z, b, 0) \sim (z + b, b, \infty).$$

The structure for $\mathcal{X}_g(m) \to \mathcal{A}_g^*(m)$ is more involved (m copies of $\overline{P}$ glued).

The slopes

Using this description, one can compute the class pushforwards for a test curve $C \subset \partial\mathcal{A}_g^*(m)$.

Proposition (Lehavi, G.- '05). The pushforwards are

$$p_{\mathcal{A}*}(\mathcal{D}^{g+1}) = \frac{m^{g+1}(g+1)!}{6}\Theta_B$$

$$p_{\mathcal{A}*}(\mathcal{D}^g c_1(T_{\mathcal{A}/C}) = \frac{m^{g+1}g!(1-1/m^2)}{3}\Theta_B.$$

Theorem (Lehavi, G.- '05).

$$s(T_{m,g}) \le 6\,\frac{g^2+g+2}{g^2+g-4(1-1/m^2)}$$

The slopes we thus get are as follows ($N_0^* = N_0 - 2\theta_{\mathrm{null}}$, used by Mumford):

g	$s(K_{\mathcal{A}_g})$	$s(N_0^*)$	$s(T_{3,g})$
4	5	8	8.03
5	6	7.71	7.26
6	7	7.53	6.87
7	8	7.40	6.64
...			
∞	∞	6	6

Corollary (Lehavi, G.- '05). $\mathcal{A}_6$ is of general type (thus determining the Kodaira dimension for the last unknown case)

Theorem (essentially due to Tai) The minimal slope of effective divisors on $\mathcal{A}_g^*$ goes to zero as g goes to infinity.

$\mathcal{A}_g$ in finite characteristic.

The nef cone and the canonical model results still hold.

There **exists a stratification** of $\mathcal{A}_g \otimes \mathbb{F}_p$, by the type of the group scheme $A[p]$ (Ekedahl, Oort); the strata are quasi-affine.

$\mathcal{A}_g \otimes \mathbb{F}_p$ **has** a complete codimension g subvariety: abelian varieties that do not have points of order p.

Let k be algebraically closed, with $\operatorname{char} k = p$.
The p**-rank** of $A \in \mathcal{A}_g \otimes \mathbb{F}_p$ is f such that $\sharp A[p](k) = p^f$.
Let V_f be the locus of abelian varieties of p-rank at most f. Then (van der Geer '96)

$$[V_f] = (p-1)(p^2-1)\cdots(p^{g-f}-1)\lambda_{g-f},$$

so the Hodge classes are effectively represented.
