Local geometric Langlands correspondence and representations of affine Kac-Moody algebras

(Overview of the work of Beilinson and Drinfeld)

Lecture 3

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1. D-algebras.

In this lecture X will be a smooth curve over $\mathbb{C}$. A (unital) D-algebra on X is a quasi-coherent sheaf of algebras $\mathcal{A}$, endowed with an additional structure of left D-module, such that the multiplication map

$$\mathcal{A} \underset{\mathcal{O}_X}{\otimes} \mathcal{A} \to \mathcal{A}$$

is compatible with the D-module structure, and the map $\mathcal{O}_X \to \mathcal{A}$, given by the unit, respects the D-module structure.

In other words, if t is a local coordinate on X, we have an operator ∂_t acting on the sections of $\mathcal{A}$, satisfying the Leibniz rule with respect to the product on $\mathcal{A}$, and its action on $\mathcal{O} \hookrightarrow \mathcal{A}$ is the standard one.

In what follows, we will assume that $\mathcal{A}$ is flat (=torsion-free) as an $\mathcal{O}$-module.

Today we will be interested in commutative D-algebras. Their geometric meaning is that their spectra are, by definition, affine D-schemes over X, i.e., schemes over X, endowed with a connection.

Let us give the most basic example of a D-scheme (D-algebra)–the jet scheme into a vector space.

Let V be a finite-dimensional vector space. Consider the commutative D-algebra

$$Jets(V) := \mathsf{Sym}_{\mathcal{O}_X}(\mathsf{D}_X \otimes V^*),$$

where D_X denotes the sheaf of differential operators on X and V^* is the dual vector space to V.

By construction,

$$Hom_{\mathsf{D\text{-}alg}}(Jets(V), \mathcal{A}) \simeq V \otimes \Gamma(X, \mathcal{A}).$$

2. Horizontal sections.

Given a D-algebra $\mathcal{A}$ on a curve X we can attach to it the "space" of horizontal sections, denoted $H_\nabla(X, \mathcal{A})$: (For us, a "space" is by definition a functor on the category of $\mathbb{C}$-algebras.)

Given an algebra R, we set

$$H_\nabla(X, \mathcal{A})(R) = Hom_{\text{D-alg}}(\mathcal{A}, R \otimes \mathcal{O}_X).$$

Lemma 1. *The functor $H_\nabla(X, \mathcal{A})$ is always ind-representable. If X is comlete, then it is representable.*

For example, for $\mathcal{A} = Jets(V)$, we have

$$H_\nabla(X, \mathcal{A}) \simeq V \otimes \Gamma(X, \mathcal{O}_X),$$

where the latter is an infinite-dimensional vector space, regarded as an ind-scheme.

Let $x \in X$ be a point, and let $\mathcal{D}$ (resp., $\mathcal{D}^\times$) be the formal (resp., formal punctured) disc around this point. In what follows, we will denote by $R\widehat{\otimes}\mathcal{O}_{\mathcal{D}}$ (resp., $R\widehat{\otimes}\mathcal{O}_{\mathcal{D}^\times}$) the corresponding completed tensor products. I.e., if t is a local coordinate near x, then

$$R\widehat{\otimes}\mathcal{O}_{\mathcal{D}} \simeq R[[t]],\ R\widehat{\otimes}\mathcal{O}_{\mathcal{D}^\times} \simeq R((t)).$$

We define the functors of horizontal sections of $\mathcal{A}$ over $\mathcal{D}$ and $\mathcal{D}^\times$, respectively, by

$$H_\nabla(\mathcal{D},\mathcal{A})(R) := Hom_{\mathsf{D\text{-}alg}}(\mathcal{A}, R\widehat{\otimes}\mathcal{O}_{\mathcal{D}})$$

and

$$H_\nabla(\mathcal{D}^\times,\mathcal{A})(R) := Hom_{\mathsf{D\text{-}alg}}(\mathcal{A}, R\widehat{\otimes}\mathcal{O}_{\mathcal{D}}^\times).$$

As in the previous lemma, one shows that the functor $H_\nabla(\mathcal{D},\mathcal{A})$ is representable by an affine scheme and $H_\nabla(\mathcal{D}^\times,\mathcal{A})$ is ind-representable.

Lemma 2.

(1) $H_\nabla(\mathcal{D},\mathcal{A}) \simeq Spec(\mathcal{A}_x)$.

(2) *We have a commutative diagram, where the arrows are closed embeddings:*

$$\begin{array}{ccc} H_\nabla(X,\mathcal{A}) & \longrightarrow & H_\nabla(\mathcal{D},\mathcal{A}) \\ \downarrow & & \downarrow \\ H_\nabla(X-x,\mathcal{A}) & \longrightarrow & H_\nabla(\mathcal{D}^\times,\mathcal{A}). \end{array}$$

The geometric meaning of point (1) is that horizontal sections $\mathcal{D} \to Spec(\mathcal{A})$ are in a bijection with just sections of $Spec(\mathcal{A})$ over x, i.e., a point in the fiber can be uniquely extended to a horizontal section on the formal neighbourhood.

The geometric meaning of point (2) is that a horizontal section of $Spec(\mathcal{A})$ over a curve is unquely determined by its restriction to the formal (resp., formal punctured) disc around any given point.

Returning to the example of $\mathcal{A} = Jets(V)$, we obtain that

$$Jets(V)_x \simeq H_{\nabla}(\mathcal{D}, Jets(V)) \simeq \widehat{\mathcal{O}}_x \otimes V,$$

and

$$H_{\nabla}(\mathcal{D}^{\times}, Jets(V)) \simeq \widehat{\mathcal{K}}_x \otimes V,$$

where $\widehat{\mathcal{O}}_x$ and $\widehat{\mathcal{K}}_x$ are the completed local ring and field at the point x, respectively.

The first of these isomorphisms is the source of the name "jets".

3. Back to the critical level: $\mathfrak{z}$ as a D-algebra.

Recall the $\hat{\mathfrak{g}}_{crit}$-module $\mathbb{V}_{crit}$, and the commutative algebra

$$\mathfrak{z} \simeq End(\mathbb{V}_{crit}) \simeq \mathbb{V}_{crit}^{\mathfrak{g}[[t]]}.$$

We will now show that $\mathfrak{z}$ can be realized as the fiber at $x \in X$ of some D-algebra, which we will denote by $\mathfrak{z}_X$.

Recall first of all that $\mathbb{V}_{crit}$ could be realized as $\Gamma(Gr_G, \delta_{1,Gr_G} \otimes \mathcal{L}_{crit})$, where $Gr_G = G((t))/G[[t]]$.

Given a curve X, there exists a scheme $Gr_{G,X}$ over X, whose fiber at any given $x \in X$ identifies with Gr_G, once we identify $\widehat{\mathcal{O}}_x \simeq \mathbb{C}[[t]]$.

Namely, $Gr_{G,X}$ classifies the data of a point $x \in X$ and a principal G-bundle on X with a trivialization off this point. We will denote by π the projection $Gr_{G,X} \to X$.

By construction, $Gr_{G,X}$ carries a connection along X, i.e., it is a D-scheme. Moreover, this connection lifts onto the line bundle $\mathcal{L}_{crit,X}$.

By taking the D-module $\delta_{1_X,Gr_{G,X}}$ on $Gr_{G,X}$, we can consider the quasi-coherent sheaf

$$\mathbb{V}_{crit,X} := \pi_*(\delta_{1_X,Gr_{G,X}} \otimes \mathcal{L}_{crit,X}),$$

which will be a D-module on X.

The fiber of $\mathbb{V}_{crit,X}$ at x can be identified with the vector space underlying the representation $\mathbb{V}_{crit}$ of $\widehat{\mathfrak{g}}_{crit}$. Globally, $\mathbb{V}_{crit,X}$ carries an action of an appropriate sheaf of Kac-Moody algebras.

It makes sense to take $End_{\widehat{\mathfrak{g}}_{crit}}(\mathbb{V}_{crit,X})$, which will again be a D-module on X. It carries a structure of an associative D-algebra, but one can show that it is in fact commutative.

This is our $\mathfrak{z}_X$. By construction, its fiber at any $x \in X$ maps to $\mathfrak{z}$, and one shows that this map is an isomorphism, as required.

Lemma 3. *There exists a natural map*

$$Spec(\mathfrak{Z}) \to H_{\nabla}(\mathcal{D}^{\times}, \mathfrak{z}_X);$$

moreover this map is an isomorphism.

4. $\mathfrak{z}_X$ and connections.

We will now explain the relation between $\mathfrak{Z}$ and $\check{G}$-connections on the formal punctured disc.

Along with the D-module $\delta_{1_X, Gr_{G,X}}$ on $Gr_{G,X}$, for any $V \in \text{Rep}(\check{G})$ one can consider the corresponding D-module $\mathcal{F}_{V,X}$. The direct image

$$\pi_*(\mathcal{F}_{V,X} \otimes \mathcal{L}_{crit,X})$$

will be a D-module on X, and it will carry an action of the above sheaf of Kac-Moody algebras. Set

$$\mathcal{V}_X := Hom_{\hat{\mathfrak{g}}_{crit}}(\mathbb{V}_{crit,X}, \pi_*(\mathcal{F}_{V,X} \otimes \mathcal{L}_{crit,X})).$$

This will be a locally free $\mathfrak{z}_X$-module, endowed with a connection along X. Generalizing the set-up of the previous lecture, we obtain that the functor

$$V \mapsto \mathcal{V}_X$$

defines a $\check{G}$-torsor over the D-scheme $Spec(\mathfrak{z}_X)$, endowed with a connection along X.

Thus, given a point of $H_\nabla(U, \mathfrak{z}_X)$, where U is X (resp., $X - x$, $\mathcal{D}$, $\mathcal{D}^\times$), which is the same as a horizontal homomorphism $\mathfrak{z}_X|_U \to \mathcal{O}_U$, we obtain a $\check{G}$-torsor over U with a connection.

In particular, for $U = \mathcal{D}^\times$, we obtain the desired map

$$H_\nabla(\mathcal{D}^\times, \mathfrak{z}_X) \to LocSys(\mathcal{D}^\times)_{\check{G}}.$$

5. The Beilinson-Drinfeld construction of Hecke eigensheaves.

Assume now that X is complete. Let σ_{glob} be a $\check{G}$-local system on $X - x$. Let σ_{loc} be the restriction of σ_{glob} to $\mathcal{D}^\times$, which is a point of $LocSys(\mathcal{D}^\times)_{\check{G}}$

Suppose that there exists an element $\chi_{glob} \in H_\nabla(X-x, \mathfrak{z}_X)$, such that σ_{glob} is its image under the map

$$H_\nabla(X - x, \mathfrak{z}_X) \to LocSys(X - s)_{\check{G}}.$$

Let χ be the image of χ_{glob} under the map

$$H_\nabla(X - x, \mathfrak{z}_X) \to H_\nabla(\mathcal{D}^\times, \mathfrak{z}_X),$$

cf. Lemma 2(2).

We can think of χ as a character of $\mathfrak{Z}$, and let $\widehat{\mathfrak{g}}_{crit}$-mod$_\chi$ be the sub-category of $\widehat{\mathfrak{g}}_{crit}$-mod consisting of modules with this central character.

Let us recall from the first lecture that we are supposed to have a functor

$$\mathcal{C}_{\sigma_{loc}} \to Hecke(\sigma_{glob}, x),$$

and an equivalence

$$\mathcal{C}_{\sigma} \simeq \widehat{\mathfrak{g}}_{crit}\text{-mod}_{\chi}.$$

Altogether, we are supposed to have a functor

$$\widehat{\mathfrak{g}}_{crit}\text{-mod}_{\chi} \to Hecke(\sigma_{glob}, x).$$

The construction of such a functor has been carried out in the work of Beilinson and Drinfeld.

5. A localization pattern.

Let $\mathcal{Y}$ be a scheme, acted on by the group $G((t))$. Following Beilinson and Bernstein, we have the localization functor

$$Loc : \mathfrak{g}((t))\text{-mod} \to D - mod(\mathcal{Y}),$$

constructed by

$$M \mapsto \mathsf{D}_{\mathcal{Y}} \underset{U(\mathfrak{g}((t)))}{\otimes} M.$$

Suppose that the action of $G((t))$ on $\mathcal{Y}$ is infinitesimally transitive, i.e., $\mathfrak{g}((t))$ maps surjectively onto the tangent space to $\mathcal{Y}$ at every point. Then we can describe explicitly the fibers of $Loc(M)$.

Namely, for $y \in \mathcal{Y}$, let $st(y) \subset \mathfrak{g}((t))$ be its stabilizer. We have:

$$Loc(M)_y \simeq (M)_{st(y)}.$$

More generally, this construction applies to the category $\widehat{\mathfrak{g}}_\kappa$-mod, where $\widehat{\mathfrak{g}}_\kappa$ is a central extension of $\mathfrak{g}((t))$, which acts on a line bundle $\mathcal{L}_{\mathcal{Y}}$, lifting the action of $\mathfrak{g}((t))$ on $\mathcal{Y}$.

We apply this construction to $\mathcal{Y} = Bun_G(x)$. The functor Loc gives rise to a functor

$$\widehat{\mathfrak{g}}_{crit}\text{-mod} \to \text{D-mod}(Bun_G(x)).$$

One can describe explicitly the fibers of $Loc(M)$ for $M \in \widehat{\mathfrak{g}}_{crit}$-mod:

Namely, a point of $Bun_G(x)$ defines a twisted form of the algebra $\mathfrak{g} \otimes \Gamma(X - x, \mathcal{O}_X)$, denoted $\mathfrak{g}_{out}$, together with its embedding into $\widehat{\mathfrak{g}}_{crit}$. Then the fiber of $Loc(M)$ at the above point of $Bun_G(x)$ is given by the space of coinvariants $(M)_{\mathfrak{g}_{out}}$.

Thus, we obtain the functor

$$\widehat{\mathfrak{g}}_{crit}\text{-mod}_\chi \to \text{D-mod}(Bun_G(x)).$$

However, the relation

$$\pi_*(\mathcal{F}_{V,X} \otimes \mathcal{L}_{crit,X}) \simeq \mathbb{V}_{crit,X} \underset{\mathfrak{z}_X}{\otimes} \mathcal{V}_X$$

implies that this functor naturally factors through $Hecke(\sigma_{glob}, x)$.