Geometric Transitions, CY Integrable Systems, and Open GW Invariants

2005 Summer Inst.
Seattle, July 25

* Geometric transitions + int. sys
  hep-th/0506196

  Diaconescu, Dijkgraaf, –, Hofman, Pantev

* Geometric transitions + MHS.
  hep-th/050619?

  Diaconescu, –, Grassi, Pantev

* Hitchin systems + twisted complexes
  in prep

  Diaconescu, –, Pantev

Geometric transitions

$$X_\mu \rightsquigarrow X_0 \longleftarrow \widetilde{X}$$

X_μ : family of (complex strs. on) CY's
X_0 : a singular CY in the family
$\widetilde{X}$: its small resolution, still CY, contains some exceptional 2 cycles $\mathbb{P}^1 \approx S^2$.

Large N duality :

closed strings on X_μ $\quad\quad$ open strings on $\widetilde{X}$

This involves:

$$\lim_{N\to\infty} \mathcal{M}_N ,$$

where $\mathcal{M}_N$ is a quantum moduli space of N branes on $\widetilde{X}$.

Dualities:

$$
\begin{tikzcd}
\text{IIB closed} & X_\mu \arrow[r, squiggly] \arrow[d, leftrightarrow, "\text{MS}"'] \arrow[rr, bend left, leftrightarrow, "\text{large } N"] & X_0 & \tilde{X} \arrow[l] \arrow[d, leftrightarrow, "\text{OMS}"] & \text{IIB open} \\
\text{IIA closed} & \tilde{Y} \arrow[r] \arrow[rr, bend right, leftrightarrow, "\text{large } N"'] & Y_0 & Y_v \arrow[l, squiggly] & \text{IIA open}
\end{tikzcd}
$$

LHS : Mirror symmetry

IIB on X_μ period integrals on cycles in X_μ	$\Longleftrightarrow$	IIA on $\tilde{Y}$ closed string GW invariants on $\tilde{Y}$

RHS: Open Mirror Symmetry (MS w/ branes)

integrals on chains in $\tilde{X}$ bounding the vanishing cycles	$\Longleftrightarrow$	Open string GW invariants, count holo maps from R.S. w/ boundary to Y_v, boundary $\to L_v$

Mirror of the exceptional 2-cycles $\mathbb{P}^1 \approx S^2 \subset \tilde{X}$ are SLAG vanishing 3-cycles $L_v \subset Y_v$.

Large N duality interchanges left & right. Sometimes, allows "calculation" of open GW invariants.

[D+V] relate B-model topological strings on local CYs, via large N duality, to <u>matrix models</u>.

Typical picture:

$$\begin{tikzcd} X_a = X \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathbb{C}_x \arrow[r, "W'"] & \mathbb{C}_z \end{tikzcd}$$

$\boxed{Z \subset \mathbb{C}^4 : -y^2 + uv + z^2 = 0}$

$\boxed{X \subset \mathbb{C}^4 : -y^2 + uv + W'(x)^2 = 0}$

$W = W_a(x) = \Sigma_{i=0}^{n+1} a_i x^i =$ superpotential

X : CY3, singular at n points $\Big\Vert\ u = v = y = 0,\ W'(x) = 0.$

$\Omega_X = \frac{du\, dx\, dy}{u} = \ldots$ holo 3form.

When $a = 0$, X is singular along a curve

$C \simeq \mathbb{C}_x : \{u = v = y = 0\}.$

$\tilde{X}$ has a family of holomorphic spheres $\simeq \mathbb{P}^1$

When $a \neq 0$, have transversally holomorphic family of (non-holomorphic) S^2's

Integrate Ω_X on these S^2's $\Rightarrow$ holo 1-form ω on $\mathbb{C}_x$.

$W(x) := \int_{x_0}^{x} \omega$: <u>classical superpotential.</u>

[DV] picture:

combine <u>superpotential</u> deformations, X_a with <u>smoothing</u> deformations, X_μ:

$$X_{a,\mu} : -y^2 + uv + W'(x)^2 + f_\mu(x) = 0$$

$$f_\mu(x) = \Sigma_{i=0}^{n-1} \mu_i x^i$$

From matrix models, they get a quantized superpotential $W = W_{a,\mu}(\tilde{x})$

$\tilde{x}$: coordinate on the hyperelliptic curve

$$\tilde{C}_{a,\mu} : y^2 = W'(x)^2 + f_\mu(x)$$

Coefficients of W w.r.t. <u>special coordinates</u> $\Rightarrow$ <u>open string GW invariants</u> on mirror Y_v.

Large N duality $\Rightarrow$ expansion in μ

(actually, in special coordinates equiv't to μ)

0-th order:

$$\lim_{\mu \to 0} \int_{\Gamma_\mu} \Omega_{X_\mu} = \int_{\Gamma} \Omega_X \quad \text{(Clemens-Schmid)}$$

Γ : 3-chain in X, $\partial \Gamma = \Sigma$ (exceptional $\mathbb{P}^1$'s)

Γ_0 : its image in X_0, a 3-cycle.

Γ_μ : its deformation to 3-cycle in X_μ.

<u>Natural interpretation of W:</u>

it is a "normal function", i.e. a section of the family of <u>intermediate Jacobians</u> $J(X_\mu)$.

Want: behavior of $J(X_\mu)$ near the transition, $\mu \to 0$.

a la [DV], we'll study this for $J(X_{a,\mu})$ near $a = \mu = 0$.

$$\begin{tikzcd} \tilde{S} \arrow[d] & C & \tilde{m} \arrow[d] & & \\ S & C & m & c & L \end{tikzcd}$$

A simple <u>compact</u> example: (cf. [KMP])

$X_{a,\mu} = Q \cap R$ $\quad$ Q = quadric $\subset \mathbb{P}^5$, $\quad$ R = quartic $\subset \mathbb{P}^5$

$a = \mu = 0 \Leftrightarrow \operatorname{rank}(Q) = 3 \Leftrightarrow \operatorname{Sing}(Q) = \mathbb{P}^2 \subset \mathbb{P}^5$

$\Rightarrow \operatorname{Sing}(X_{a,\mu})$ = plane quartic $\underset{g=3}{C} \subset \mathbb{P}^2$ $\Rightarrow$ X has 1-param family of $\mathbb{P}^1$'s

$\mu = 0 \Leftrightarrow \operatorname{rank}(Q) \leq 4 \Leftrightarrow \operatorname{Sing}(Q) \supset \mathbb{P}^1 \subset \mathbb{P}^5$

$\Rightarrow \operatorname{Sing}(X_{a,\mu}) \supset$ 4 points $\Leftrightarrow a \in H^0(C, K_C)$.

$\operatorname{rank}(Q) = 5, 6 \Rightarrow X$ (generically) n.s.

rank(Q)	# param's Q	$h^{2,1}$
3	14	83
4	17	86
6	20	89

$$\underset{83}{S} \overset{a}{\subset} \underset{86}{M} \overset{\mu}{\subset} \underset{89}{L}$$

a-parameters = 86 − 83 = 17 − 14 = 3

μ-parameters = 89 − 86 = 20 − 17 = 3.

* Main geometric prediction of large-N duality: special geom. on L can be reconstructed from sheaf theory on CY's in $\widetilde{M}$.
* More detailed: L is foliated π S, spec. geom. on leaf L_x through $x \in S$ is determined by a moduli problem in $D^b(\widetilde{X})$.
* Linearization: $\Gamma \subset SL_2(\mathbb{C})$ corresponds to the singularity type. V: rank 2, Γ-equiv't VB on C, $\det V \cong K_C$.
  Linearized $X := Tot(V)/\Gamma$.
  $\widetilde{X}$ = blowup contains ruled surface F.
* Linearized foliation: $\widetilde{M}$ is a VB of rank = rank(G) over S, L is a VB of rank $= (g-1) \cdot \dim G$ over S, $M = \widetilde{M}/W$ is singular along S.

Integrable system:

$$\begin{array}{ccc} T & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

everything is algebraic (or: analytic),

T : complex tori.

(X, σ) : holo. symplectic variety.

$\pi : X \to B$ holomorphic Lagrangian fibration.

$\sigma|_T \equiv 0$

$\dim T = \dim B = \frac{1}{2} \dim X$.

Example 1:

X compact Riemann surface

Hodge: $H^1(X,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$

$\Rightarrow$ Jacobian $J(X) = H^1(X,\mathcal{O}) / (H^{1,0} + H^1(X,\mathbb{Z}))$

is an algebraic torus.

Example 2:

X compact Kahler 3fold

Hodge: $H^3(X,\mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$

Intermediate Jacobian

$$J(X) = H^3(X,\mathbb{C}) / (H^{3,0} + H^{2,1} + H^3(X,\mathbb{Z}))$$

is a polarized, non-algebraic torus.

E.g. X = Calabi-Yau 3 fold,

(i.e. $\Omega^3_X \approx \mathcal{O}$; $H^1(X,\mathbb{C}) = 0$)

$\Rightarrow$ signature of period is $(1, h^{21})$.

Q: When is a family of complex tori Lagrangian?

$B \underset{\text{open}}{\subset} V^*$, $\quad V \approx \mathbb{C}^g$.

$\pi : X \to B$ family of complex tori, with

<u>period</u> <u>map</u>:

$$p : B \longrightarrow (\mathrm{Sym}^2 V)_{n.d}$$

<u>The cubic condition</u> [D, Markman]

TFAE:

① $\exists$ complex symplectic σ on X s.t.

$\pi : (X, \sigma) \to B$ is Lagrangian

σ induces identity:
$$\begin{array}{ccc} T_{X/B} & \longrightarrow & \pi^* T^*_B \\ \wr\wr & & \wr\wr \\ \pi^* V & & \pi^* V \end{array}$$

② $p : B \to \mathrm{Sym}^2 V$ is (locally in B) the Hessian of a holomorphic function on B: "prepotential"

③ $dp_b \in \mathrm{Hom}(T_B, \mathrm{Sym}^2 V) \approx V \otimes \mathrm{Sym}^2 V$
actually lives in: $\qquad \cup$
$\qquad\qquad\qquad\qquad \mathrm{Sym}^3 V$.

$\boxed{\text{Calabi-Yau integrable system}}$:

X : CY3, $\Omega^3_X = 0$

$\mathcal{M}$ = moduli space = {complex structures on X}/isom

$$T_{[X]}\mathcal{M} = H^1(T_X) \approx H^1(\Omega^2_X) = H^{2,1}$$

(Bogomolov, Tian, Todorov : unobstructed)

$\widetilde{\mathcal{M}} \to \mathcal{M}$: natural $\mathbb{C}^*$-bundle

(choose : holo. volume form ω)

$\mathcal{J} \to \mathcal{M}$: universal int. Jacobian

$\widetilde{\mathcal{J}}' \to \widetilde{\mathcal{M}}'$: pullback.

$$\begin{tikzcd} \widetilde{\mathcal{J}}' \arrow[r] \arrow[d] & \mathcal{J} \arrow[d] \\ \widetilde{\mathcal{M}}' \arrow[r] & \mathcal{M} \end{tikzcd}$$

[D+M, '94] : $\widetilde{\mathcal{J}} \to \widetilde{\mathcal{M}}$ is an analytically integrable system.

* fibers $\mathcal{J}(X)$ are Lagrangian
* the image of any Abel-Jacobi map is isotropic.

The cubic = **Yukawa's**:

$$\otimes^3 H^1(T_X) \to H^3(\Lambda^3 T_X) = H^3(\Omega_X^{-3}) \xrightarrow{\cdot\omega^2} H^3(\Omega^3_X) \xrightarrow{\int} \mathbb{C}$$

$\mathfrak{X} \to B$ family of CY3's

$\forall t \in B$, $C_t = C_t^+ - C_t^-$: a 1-cycle in X_t, homologous to 0.

$\Rightarrow$ $\boxed{\text{Abel-Jacobi map}}$ = "normal function"

$$AJ : B \to J(\mathfrak{X}/B)$$
$$t \mapsto \int_{\Gamma_t} \quad \in H^3(X_t, \mathbb{C}) / \ldots\ldots = J(X_t)$$

where Γ_t is a 3-chain in X_t, $\partial \Gamma_t = C_t$.

* Independent of choices.

Various extensions:

* C not null-homologous: replace $J(X)$ by <u>Deligne cohomology group</u>.

* Special case $\mathfrak{X} = X \times B$:

$$AJ : \mathcal{H}ilb^{\circ}(X) \longrightarrow J(X)$$

* $\exists$ "<u>transversally</u> <u>holomorphic</u>" version: C is a real surface (non holomorphic) but it "varies holomorphically".

Other examples (from algebraic geom.)

S: complex symplectic surface

$C \subset S$: a holo. curve

$\Rightarrow$ short exact sequence

$$(*) \quad \boxed{0 \to T_C \to T_S|_C \to N_{C/S} \to 0}$$

S symplectic $\Rightarrow N_C = \omega_C$ = canonical bundle

The SES $(*)$ determines an extension class in:

$$\mathrm{Ext}^1(N_{C/S}, T_C) = H^1(N_{C/S}^{-1} \otimes T_C)$$

$$= H^1(T_C^{\otimes 2})$$

$$= H^0(\omega_C^{\otimes 3})^* \longrightarrow \mathrm{Sym}^3 H^0(C, \omega_C)^*$$

$\Rightarrow$ A.I.S.

Base $= H^0(C, \omega_C) = H^0(C, N_{C/S}) \sim$ deformations of C in S

Fiber over C is $\mathcal{J}(C)$.

E.g. $S = K3$ (or T^4): $\boxed{\text{Mukai's I.S.}}$

Related to: symplectic structure on moduli spaces of vector bundles or coherent sheaves on $K3$.

Another example:

$B =$ curve (= compact R.S.)

$S := T^*B$, holomorphically symplectic

$$\begin{array}{ccl} T^*B & \supset & C = \text{"spectral curve"} \\ & \searrow \; \swarrow & \\ & B & \left(\begin{array}{l}\text{n-sheeted} \\ \text{branched cover}\end{array}\right) \end{array}$$

$\Rightarrow$ $\boxed{\text{Hitchin's I.S.}}$

Base $= \{C\} = H^0(S, \mathcal{O}(C)) \approx \bigoplus_{i=1}^{n} H^0(B, K_B^{\otimes i})$

Fiber over $C = \mathcal{J}(C)$.

Total space = {Higgs bundles (V, φ) on B}

V: rank n vector bundle on B

$\varphi : V \to V \otimes \omega_B$: Higgs field

Variants:

* meromorphic Higgs bundles $\rightsquigarrow$ Markman's Poisson I.S.
  ($\varphi : V \to V \otimes \omega_B(D)$ for fixed D)
* Replace the LB by a $\mathbb{C}^*$-bundle $\Rightarrow$ Sklyanin's.
* Replace the LB by an elliptic fibration $\Rightarrow$ moduli spaces of bundles on elliptic fibr'n.
* Replace vector bundles by principal G-bundles

. . . . . .

$\boxed{\text{Hitchin system:}}$ (for group G)

B: a curve

G: a reductive group

<u>Total space</u>: $\{$ Higgs bundles (V, φ)

V: G-bundle on B

$\varphi \in \Gamma(B, \mathrm{ad}\, V \otimes K_B)$ $\}$

Base $= \{ C \to B$ spectral cover$\}$

$= \oplus_{i=1}^{r} \Gamma(B, K_B^{\otimes d_i})$

$\{d_i\}$ = degrees of invariant polynomials for G.

Fiber over $[C]$ is a Prym variety

Prym (C/B),

roughly $J(C)/J(B)$.

Relevant cases for A_1 singularities:

$$G = \left| \begin{array}{l} SL_2(\mathbb{C}) \\ \mathbb{P}GL_2 = SL_2/(\pm) \end{array} \right.$$

Spectral curves: $\boxed{w^2 = \beta}$

β: quadratic differential

w: multi-valued

Our setup:

$X_{0,0}$: CY3 with curve C of singularities (say, of type G, e.g. simplest: A_1.)

$X_{a,0}$: CY3's with finite number n of singularities which can be resolved: $\widetilde{X}_{a,0} \to X_{a,0}$.

$X_{a,\mu}$: smoothing of $X_{a,0}$.

$$S \subset \mathcal{M} \subset L, \qquad S \hookrightarrow \widetilde{\mathcal{M}} \to \mathcal{M}$$

We want to understand $CYIS(L)$ near $a = \mu = 0$.

Claim: to first order,

$$\boxed{CYIS(L) \approx CYIS(S) \times \text{Hitchin}(C, G).}$$

In fact: $\exists$ family of IS's parametrized by $t \in \mathbb{C}$, s.t. for $t \neq 0$ get $CYIS(L)$, for $t = 0$ get $CYIS(S) \times \text{Hitchin}(C, G)$.

2D analogue: [D, Ein, Lazarsfeld]

S = K3 surface
$\cup$
C = curve, $D \in |nC|$

Mukai's I.S. for line bundles on $D \subset S$
degenerates to:
Hitchin's I.S. for C, group $G = SL(n, \mathbb{C})$.

nilpotent cone in Mukai = sheaves supported on original C

is an affine twist of:
nilpotent cone in Hitchin $= \{(V, \varphi) \mid \varphi$ is nilpo$\}$

Data for the affine twist $\Leftrightarrow$ extension class encoded in n-th order neighborhood of C in S.

__Idea__: $\exists$ __degeneration of S to $N_{C/S} = T^*C$__ induces the degeneration of Mukai to Hitchin.

The deformation to normal cone for CY_3's:

$$\boxed{\begin{array}{ccc} X & \longrightarrow & X_{0,0} \\ \cup & & \cup \\ F & \longrightarrow & \mathcal{E} \end{array}}$$ ruled surface

$N_{F\backslash X} = Tot(K_F)$

$H^0(C, K_C) \approx H^1(F, K_F) \to H^{2,1}(X).$

$H^{1,0}(C) \xrightarrow{\sim} H^{1,0}(F) \longrightarrow H^{2,0}(X)$

$$\boxed{\begin{array}{ccccc} \widetilde{S} & \subset & \widetilde{m} & & \\ \| & & \downarrow & & \\ S & \subset & m & \subset & L \end{array}}$$

$$\boxed{\begin{array}{lll} N_{S\backslash \widetilde{m}} & = & H^0(\mathcal{E}, K_{\mathcal{E}}) \otimes (\text{weights of } G) \\ \downarrow & & \\ N_{S\backslash L} & = & \bigoplus_{i=2}^{n} H^0(\mathcal{E}, K_{\mathcal{E}}^{\otimes i}) = \text{Hitchin base} \end{array}}$$

The map is non-linear.

Hitchin base $\leftrightarrow$ spectral covers $\widetilde{C} \xrightarrow{n} C$

Image of $N_{S\backslash \widetilde{m}}$ $\leftrightarrow$ completely reducible covers, $\widetilde{C} = \cup_{i=1}^{n} C_i$.

Outline: geometric proof

Identify Hitchin base:

$$B = \mathrm{Maps}(C, (\underline{t} \otimes K_C)/W)$$

$\underline{t}/W$ parametrizes deformations of the surface $\mathbb{C}^2/\Gamma$, $\Gamma \subset SL_2(\mathbb{C})$. This is $\mathbb{C}^*$-equivariant.

So get family $\boxed{\mathcal{X} \to B}$ of <u>open</u> CYs, each fibered over C with:

fibers = deformations of $\mathbb{C}^2/\Gamma$.

B also parametrizes the W-Galois <u>cameral</u> covers: $\tilde{C}_b \to C$, and for each rep ρ of G, corresponding <u>spectral</u> covers: $\overline{C}_{b,\rho} \to C$.

$$\boxed{\begin{array}{c}\tilde{\mathcal{C}} \\ \downarrow \\ B \times C\end{array}}$$

Everything pulls back from $(\underline{t} \otimes K_C)/W$ and (locally in C) from $\underline{t}/W$.

* Hitchin fibers are generalized Prym's, modelled on $H^1(C, \widetilde{C} \overset{W}{\times} \Lambda_{roots})$.

* Int. Jacs are complex tori modelled on $H^3(X, \mathbb{Z})$ (or: H_3)

* Both cohomologies can be computed by Leray, boils down to two local systems over B, both pullback from $\underline{t}/W$: $\underline{t} \longrightarrow \underline{t}/W$ vs. W-orbits in $(\Lambda \times \underline{t})/W \longrightarrow \underline{t}/W$.

Holomorphic CS + twisted Higgs complexes

* wrap N topological B-branes & N antibranes on exceptional curves of $\widetilde{X}_{\widetilde{m}}$

$$Q^+ = \Sigma_{a=1}^{N} \mathcal{O}_{F_{p_a}} \qquad Q^- = \Sigma_{b=1}^{N} \mathcal{O}_{F_{q_b}}$$

complex: $Q = Q^+ \oplus Q^-[-1]$

gives boundary topological B-model.

* Offshell string states:

$$\boxed{A = \oplus_{k=0}^{3} \oplus_{m,n\in\mathbb{Z}} \Omega^{0,k}_{\widetilde{X}} (E_m^\vee \otimes E_n)}$$

where $E = E_m \hookrightarrow E_{m+1} \to \ldots$ is a locally free resolution of Q.

* $\widetilde{X}$ is total space of a LB over $F \Longrightarrow$ convert bundles on $\widetilde{X}$ to Higgs bundles on F: $A = \Omega_F \otimes \mathrm{End}(Q)$,

$$\boxed{\begin{aligned}\Omega_F &= \oplus_{p=0}^{2} \oplus_{q=0}^{1} \Omega_F^{0,p} \otimes \Lambda^q N_{F/\widetilde{X}} \\ &= \oplus_{p=0}^{2} \oplus_{q=0}^{1} \Omega_F^{2q,p}\end{aligned}}$$

* Holomorphic CS action:

$$\phi = \phi^{0,1} + \phi^{2,0}$$

for ghost number $p+q=1$ fields:

$$S_{CS} = \int_F \mathrm{Tr}\left(\tfrac{1}{2}\phi\bar\partial\phi + \tfrac{1}{3}\phi^3\right)$$

$$= \int_F \mathrm{Tr}\left(\phi^{2,0} \wedge F^{0,2}\right)$$

$F^{0,2}$ = (0,2) part of curvature of deformed connection $A + \phi^{0,1}$

* Extends to open strings specified by a complex E, via construction of twisted complexes / Bondal-Kapranov, Lazaroiu

DG category of VBs on F with $\Lambda^0 N$-valued maps

shift extension

twisted complexes (MC)