A GENERALISATION OF THE HORI-VAFA CONJECTURE

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References:

"*Two proofs of a conjecture of Hori and Vafa*", math.AG/0304403 (Duke Math. J, 2005)

"*Gromov-Witten invariants for abelian and nonabelian quotients*", math.AG/0407254

Outline

1) GW-invariants, twisted GW-invariants.

2) J-functions, Quantum Lefschetz.

3) Abelian and non-abelian GIT quotients

4) The Grassmannian and the Hori-Vafa Conjecture.

5) General conjectures.

6) The J-function of a generalized flag manifold.

1. GW-invariants, twisted GW-invariants

Y smooth, projective variety over $\mathbb{C}$, $\quad d \in H_2(Y,\mathbb{Z})$

$\overline{M}_{g,n}(Y,d)$ = moduli space (stack) of stable maps

A point $[C,\{p_i\},f] \in \overline{M}_{g,n}(Y,d)$ is given by

- a (conn., proj.) nodal curve C of (arithmetic) genus g, with n marked points $p_1,...,p_n \in C^{\text{nonsing}}$
- a map $f : C \to Y$ with $f_*[C] = d$.

stable = finite automorphism group

Have evaluation maps

$ev_j : \overline{M}_{g,n}(Y,d) \to Y, \quad ev_j([C,\{p_i\},f]) = f(p_j),$

and "tautological" line bundles $\mathcal{L}_j$ on $\overline{M}_{g,n}(Y,d)$:

fiber of $\mathcal{L}_j$ over $[C,\{p_i\},f]$ is $T^*_{p_j}C$.

$$\psi_j := c_1(\mathcal{L}_j).$$

Definition. *The GW-invariants of Y are*

$$\langle \tau_{a_1}\gamma_1, \ldots, \tau_{a_n}\gamma_n\rangle_{g,d} := \int_{[\overline{M}_{g,n}(Y,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} (\psi_i^{a_i} \cdot ev_i^* \gamma_i)$$

where $\gamma_i \in H^{2*}(Y)$ are (homogeneous) cohomology classes, a_i are nonnegative integers and $[\overline{M}_{g,n}(Y,d)]^{\mathrm{vir}}$ is the virtual fundamental class.

If all $a_i = 0$, the invariants $\langle \gamma_1, \ldots, \gamma_n\rangle_{g,d}$ are called *primary*, and intuitively should "count" curves in Y subject to incidence conditions. Otherwise, they are called *gravitational descendents.*

Today: $g = 0$, will drop from notation.

Twisted GW-invariants (cf. Coates-Givental).

Let V be a vector bundle on Y. The diagram

$$\begin{array}{ccc} \overline{M}_{0,n+1}(Y,d) & \xrightarrow{e=ev_{n+1}} & Y \\ \pi \downarrow & & \\ \overline{M}_{0,n}(Y,d) & & \end{array}$$

determines

$$V_{n,d} = [R^0\pi_* e^* V] - [R^1\pi_* e^* V]$$

in the K-group of vector bundles of $\overline{M}_{0,n}(Y,d)$. It has virtual rank (given by Riemann-Roch formula):

$$\mathrm{vrk}(V_{n,d}) := \mathrm{rk}(V) + \int_d c_1(V).$$

and "top Chern class"

$$c_{\mathrm{top}}(V_{n,d}) = c_{\mathrm{vrk}}(V_{n,d})$$

Define *GW-invariants of Y twisted by the Euler class of V* by

$$\langle \tau_{a_1}(\gamma_1), \cdots, \tau_{a_n}(\gamma_n) \rangle_{d,V} :=$$

$$\int_{[\overline{M}_{0,n}(Y,d)]^{\mathrm{virt}}} \prod_{i=1}^{n} (\psi_i^{a_i} ev_i^*(\gamma_i)) c_{\mathrm{top}}(V_{n,d}).$$

Example: If V is generated by global sections and $Z = Z(s) \subset Y$ is the zero locus of a "transversal" section s of V, then $\langle \dots \rangle_{d,V}$ are (untwisted) GW-invariants of Z: in this case $V_{n,d}$ is an honest vector bundle and

$$i_*[\overline{M}_{0,n}(Z,d)]^{\mathrm{virt}} = [\overline{M}_{0,n}(Y,d)]^{\mathrm{virt}} \cap c_{\mathrm{top}}(V_{n,d}),$$

with $i : Z \hookrightarrow Y$ the inclusion.

2. J-functions, Quantum Lefschetz.

Let $J_{Y,d} \in H^*(Y)[\hbar^{-1}]$ defined by

$$\int_Y \gamma \cdot J_{Y,d} = \sum_{a=0}^{\infty} \hbar^{-a-2} \langle \tau_a(\gamma) \rangle_d$$

for all $\gamma \in H^{2*}(Y)$.

Denote $(t_0, \mathbf{t})$ a general element of

$$H^0(Y, \mathbb{C}) \oplus H^2(Y, \mathbb{C})$$

Define Givental's J-function of Y by

$$J_Y(t_0, \mathbf{t}) = e^{(t_0 + \mathbf{t})/\hbar} \sum_d e^{\int_d \mathbf{t}} J_{Y,d}$$

(generating function for the 1-pt. invariants)

Also have V-twisted J-fcn.:

$$J_{Y,d,V} \in H^*(Y)[\hbar^{-1}]$$

$$\int_Y \gamma \cdot J_{Y,d,V} \cdot c_{\mathrm{top}}(V) = \sum_{a=0}^{\infty} \hbar^{-a-2} \langle \tau_a(\gamma) \rangle_{d,V}$$

$$J_{Y,V}(t_0, \mathbf{t}) = e^{(t_0+\mathbf{t})/\hbar} \sum_d e^{\int_d \mathbf{t}} J_{Y,d,V}$$

Quantum Lefschetz. $Z \subset Y$ complete intersection, i.e. Z is the zero locus of a section of a *decomposable* vector bundle $V = \oplus_{i=1}^m M_i$, with M_i nef *line* bundles. There is a *universal* formula expressing $J_{Y,V}$ - hence (most of) J_Z - in terms of J_Y. Precisely:

For a curve class $d \in H_2(Y)$ put $f_i = \int_d c_1(M_i)$ and

$$\chi_d(V) := \prod_{i=1}^{m} \prod_{l=1}^{f_i} (c_1(M_i) + l\hbar).$$

Theorem (... Y.P. Lee, Coates-Givental). $J_{Y,V}$ *is obtained from*

$$I_{Y,V} := e^{(t_0+\mathbf{t})/\hbar} \sum_d e^{\int_d \mathbf{t}} \chi_d(V) J_{Y,d}$$

by explicit change of variables ("mirror transformation"). If $c_1(Z)$ *is positive enough, then* $J_{Y,V} = I_{Y,V}$*.*

Example. $Y = \mathbb{P}^{n-1}$, H the hyperplane class.

$$J_{\mathbb{P}^{n-1}} = e^{(t_0+tH)/\hbar} \sum_{d \geq 0} e^{dt} \frac{1}{\prod_{l=1}^{d} (H + l\hbar)^n} \quad \text{(Givental)}$$

Take $n = 5$, $V = \mathcal{O}(5)$.

$$I_{\mathbb{P}^4, \mathcal{O}(5)} = e^{(t_0+tH)/\hbar} \sum_{d \geq 0} e^{dt} \frac{\prod_{k=1}^{5d} (5H + l\hbar)}{\prod_{l=1}^{d} (H + l\hbar)^5}$$

In this case

Q. Lefschetz $\leftrightarrow$ mirror formula of Candelas et.al.

Some remarks:

• If V is *indecomposable*, no "universal" correcting class $\chi_d(V)$ is known.

• The only class of Y's for which J was known in general is the class of smooth toric varieties (due to Givental).

• Q. Lefschetz should be thought of as some kind of highly nontrivial functorial property of J-functions. Another, much easier, functoriality is $J_{Y_1 \times Y_2} = J_{Y_1} J_{Y_2}$, e.g.

$$J_{(\mathbb{P}^{n-1})^r} = e^{(t_0 + t_1 H_1 + \cdots + t_r H_r)/\hbar} \times$$

$$\times \sum_{d_i \geq 0} e^{d_1 t_1 + \cdots + d_r t_r} \frac{1}{\prod_{i=1}^{r} \prod_{l=1}^{d_i} (H_i + l\hbar)^n}$$

3. GIT quotients

X – smooth, projective variety over $\mathbb{C}$ (main case to keep in mind: $X = \mathbb{P}^N$).

G – reductive algebraic group acting on X.

T – fixed max. torus in G.

Will assume that (for a linearized ample line bundle) $X^{ss}(T) = X^s(T)$, $X^{ss}(G) = X^s(G)$, and that T and G act freely on the stable points, so that the GIT quotients $X//G = X^s(G)/G$ and $X//T = X^s(T)/T$ are smooth projective varieties.

Ellingsrud–Strømme, Martin, Kirwan: Cohomology of $X//G$ and that of $X//T$ are closely related.

R – root system for G, T

$W = N(T)/T$ the Weyl group; acts on $X//T$.

A root $\alpha \in R$ gives a T-rep $\mathbb{C}_\alpha$, hence a line bundle $L_\alpha = \mathbb{C}_\alpha \times_T X^s(T)$ on $X//T$. Put

$$E := \oplus_{\alpha \in R} L_\alpha$$

Theorem (Martin's Integration Formula).

$$\int_{X//G} \gamma = \frac{1}{|W|} \int_{X//T} \tilde{\gamma}\, c_{\text{top}}(E).$$

Here $\gamma \in H^*(X//G)$, $\tilde{\gamma}$ is a *lift* to $H^*(X//T)^W$ (i.e., γ and $\tilde{\gamma}$ come from the same G-equivariant cohomology class on X via the Kirwan maps.)

Example: $X := \mathbb{P}(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)) = \mathbb{P}(Mat_{r\times n}(\mathbb{C}))$.

$G = GL_r(\mathbb{C})$ acts by left multiplication.

$X//G = G(r,n)$, Grassmannian of r-planes in $\mathbb{C}^n$. It has universal sequence

$$0 \to \mathcal{S} \to \mathcal{O}^n \to \mathcal{Q} \to 0$$

of vector bundles, with $\mathrm{rank}(\mathcal{S}) = r$.

Denote $H_1, \ldots, H_r$ the Chern roots of $\mathcal{S}^*$. Then

$$H^*(G(r,n)) \cong \mathbb{C}[H_1, \ldots, H_r]^{S_r}/(h_{n-r+1}, \ldots, h_n)$$

h_j – the jth complete symmetric function of $H_1, \ldots, H_r$.

Let $T \subset G$ denote the subgroup of diagonal invertible $r \times r$ matrices; then $X//T = (\mathbb{P}^{n-1})^r$.

Put $H_i := pr_i^*(H) \in H^*((\mathbb{P}^{n-1})^r)$, with $H \in H^*(\mathbb{P}^{n-1})$ the hyperplane class.

$$H^*((\mathbb{P}^{n-1})^r) \cong \mathbb{C}[H_1, \ldots, H_r]/(H_1^n, \ldots, H_r^n)$$

The lift of a cohomology class in $G(r, n)$ represented by a symmetric polynomial in the H_i's is the class represented by the same polynomial in the cohomology of $(\mathbb{P}^{n-1})^r$.

The other objects we introduced are explicitly

$R = \{(i, j) \mid 1 \leq i, j \leq r, i \neq j\}$,

$L_{(i,j)} = pr_i^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \otimes pr_j^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$,

$c_{\text{top}}(E) = \prod_{i \neq j}(H_i - H_j)$

4. The Grassmannian and HV Conjecture.

Theorem (B,–,K).

$$J_{G(r,n)} = e^{(t_0+t(H_1+\cdots+H_r))/\hbar} \sum_{d\geq 0} e^{dt} J_d,$$

where

$$J_d = \sum_{d_1+\ldots+d_r=d} \frac{(-1)^{(r-1)d}\prod_{i<j}(H_i - H_j + (d_i - d_j)\hbar)}{\prod_{i<j}(H_i - H_j)\prod_{i=1}^{r}\prod_{l=1}^{d_i}(H_i + l\hbar)^n}$$

It is very easy to see that $J_{G(r,n)}$ is obtained (up to an overall invertible factor) from $J_{(\mathbb{P}^{n-1})^r}$ by applying to it the "Vandermonde operator"

$$\mathcal{D}_\Delta = \prod_{i<j}\left(\hbar\frac{\partial}{\partial t_i} - \hbar\frac{\partial}{\partial t_j}\right),$$

then "symmetrizing" by setting $t_i = t + \pi(r-1)\sqrt{-1}$ and dividing by the class $\prod_{i<j}(H_i - H_j)$. This is the original conjecture of Hori and Vafa.

Another way of seeing the formula as relating $J_{G(r,n)}$ and $J_{(\mathbb{P}^{n-1})^r}$ is as an analogue of quantum Lefschetz: Recall that on $(\mathbb{P}^{n-1})^r$ we have the bundle

$$E = \oplus_{i \neq j} L_{(i,j)}.$$

Now $L_{(i,j)} = pr_i^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \otimes pr_j^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ is not nef: for a curve class $(d_1, \ldots, d_r)$ we have

$$f_{ij} = \int_{(d_1,\ldots,d_r)} c_1(L_{(i,j)}) = d_i - d_j.$$

Nevertheless, we can still define

$$\chi_{(d_1,\ldots,d_r)}(E) := \prod_{i,j} \frac{\prod_{l=-\infty}^{f_{ij}} (c_1(L_{(i,j)}) + l\hbar)}{\prod_{l=-\infty}^{0} (c_1(L_{(i,j)}) + l\hbar)},$$

and our Theorem says in this notation

$$J_{G(r,n),d} = \sum_{d_1+\cdots+d_r=d} \chi_{(d_i)}(E) J_{(\mathbb{P}^{n-1})^r,(d_i)}.$$

5. General conjectures

Let $X//G$ and $X//T$ be GIT quotients as before, with the bundle $E = \oplus_\alpha L_\alpha$ on $X//T$ etc.

For curve classes $d \in H_2(X//G)$ and $\widetilde{d} \in H_2(X//T)$ write $\widetilde{d} \mapsto d$ if

$$\int_d H = \int_{\widetilde{d}} \widetilde{H}$$

for every divisor class $H \in H^2(X//G)$ with lift $\widetilde{H} \in H^2(X//T)^W$.

Conjecture 1.

$$\langle \tau_{a_1}(\gamma_1), \tau_{a_2}(\gamma_2), \ldots, \tau_{a_n}(\gamma_n) \rangle_d^{X//G} =$$

$$\frac{1}{|W|} \sum_{\tilde{d} \to d} \langle \tau_{a_1}(\tilde{\gamma}_1), \tau_{a_2}(\tilde{\gamma}_2), \ldots, \tau_{a_n}(\tilde{\gamma}_n) \rangle_{\tilde{d},E}^{X//T}$$

By work of Coates-Givental, Conj.1 implies

Conjecture 2. $(t_0, \mathbf{t}) \in H^0(X//G, \mathbb{C}) \oplus H^2(X//G, \mathbb{C})$. *Define*

$$I_{X//G} = e^{(t_0+\mathbf{t})/\hbar} \sum_d e^{\int_d \mathbf{t}} \sum_{\tilde{d}\to d} \chi_{\tilde{d}}(E) J_{X//T,\tilde{d}}$$

Then there is an explicit change of variables

$$(t_0, \mathbf{t}) \to f(t_0, \mathbf{t})$$

s.t.

$$J_{X//G}(t_0, \mathbf{t}) = I_{X//G}(f(t_0, \mathbf{t})).$$

If $c_1(X//T)$ is positive enough, then no change of variables is needed for the equality of J and I.

There are useful extensions of these conjectures involving an additional twisting:

Let V be a finite dim'l vector space with linear G action, i.e., a G-representation. It can be viewed also as a T-rep. It induces vector bundles V_G on $X//G$ and V_T on $X//T$. Note that V_T is decomposable, since any T-rep. is completely reducible. Write $V_T = \oplus_i M_i$ and assume each M_i is a nef line bundle.

Conjecture 1'.

$$\langle \tau_{a_1}(\gamma_1), \tau_{a_2}(\gamma_2), \ldots, \tau_{a_n}(\gamma_n) \rangle_{d, V_G}^{X//G} =$$

$$\frac{1}{|W|} \sum_{\tilde{d} \to d} \langle \tau_{a_1}(\tilde{\gamma}_1), \tau_{a_2}(\tilde{\gamma}_2), \ldots, \tau_{a_n}(\tilde{\gamma}_n) \rangle_{\tilde{d}, E \oplus V_T}^{X//T}$$

Conjecture 2'. *Put*

$$I_{X//G,V} = e^{(t_0+\mathbf{t})/\hbar} \sum_d e^{\int_d \mathbf{t}} \sum_{\tilde{d}\to d} \chi_{\tilde{d}}(E)\chi_{\tilde{d}}(V_T) J_{X//T,\tilde{d}}$$

Then there is an explicit change of variables

$$(t_0, \mathbf{t}) \to f(t_0, \mathbf{t})$$

s.t.

$$J_{X//G,V_G}(t_0, \mathbf{t}) = I_{X//G,V}(f(t_0, \mathbf{t})).$$

If $c_1(X//T) - \sum_i c_1(M_i)$ *is positive enough, then no change of variables is needed.*

Theorem (B,–,K). *Conjecture 1' holds for 1-point invariants and* $X//G = G(s_1, n) \times \cdots \times G(s_l, n)$*,* $X//T = \prod_{i=1}^{l} (\mathbb{P}^{n-1})^{s_i}$*. Conjecture 2' also holds for these* $X//G$ *and* $X//T$*.*

The proof is done by localization on moduli spaces of 1-pointed stable maps for the actions of the torus $T' := ((\mathbb{C}^*)^n)^l$.

6. J-FCNS OF ISOTROPIC FLAG MFLDS

The above theorem gives closed formulas for J-functions of zero loci of sections of homogeneous vector bundles on $\prod_{i=1}^{l} G(s_i, n)$. Important examples are

- isotropic (partial) flag mflds. of types A, B, C, D.

• the moduli space of rk. 2 vector bundles with fixed determinant of odd degree on a (hyperelliptic) curve of genus $g \geq 2$.

Two examples will show how straightforward is to obtain these formulas.

Example 1: The Lagrangian Grassmannian.

Consider the standard symplectic form ω on $\mathbb{C}^{2n}$. The Lagrangian Grassmannian LG_n parametrizes maximal (i.e., n-dimensional) subspaces in $\mathbb{C}^{2n}$ which are isotropic for ω. Let S be the universal subbundle on the regular Grassmannian $G(n, 2n)$. The form ω induces a section of the bundle $\Lambda^2 S^*$ and LG_n is the zero locus of this section.

So in this case $V_G = \Lambda^2 S^*$ and it is immediate that the corresponding bundle V_T on $(\mathbb{P}^{2n-1})^n$ is

$$\bigoplus_{1\leq i<j\leq n} \mathcal{O}(H_i + H_j).$$

Hence we get

Theorem.

$$J_{d,LG_n} = \sum_{d_1+\ldots+d_n=d} \left(\prod_{n\geq i>j\geq 1} \frac{\prod_{k=0}^{d_i+d_j}(H_i + H_j + k\hbar)}{(H_i + H_j)} \right)$$

$$\left(\frac{(-1)^{(n-1)d} \prod_{n\geq i>j\geq 1}(H_i - H_j + (d_i - d_j)\hbar)}{\prod_{n\geq i>j\geq 1}(H_i - H_j) \prod_{i=1}^{n} \prod_{k=1}^{d_i}(H_i + k\hbar)^{2n}} \right)$$

Example 2: Type A partial flag varieties.

$F := Fl(s_1, \ldots, s_l, n = s_{l+1})$ parametrizes flags of subspaces

$$\mathbb{C}^{s_1} \subset \cdots \subset \mathbb{C}^{s_l} \subset \mathbb{C}^n$$

On each Grassmannian $G(s_i, n)$ consider the universal sequence

$$0 \to \mathcal{S}_i \to \mathcal{O}^n \to \mathcal{Q}_i \to 0$$

On the product $\prod_{i=1}^{l} G(s_i, n)$ take the vector bundle

$$\oplus_{i=1}^{l-1} Hom(S_i, Q_{i+1}) = \oplus_{i=1}^{l-1} (S_i^* \otimes Q_{i+1})$$

It has a natural section σ coming from composing $0 \to \mathcal{S}_i \to \mathcal{O}^n$ with $\mathcal{O}^n \to \mathcal{Q}_{i+1} \to 0$ and F is the zero locus of σ.

When restricted to F, the bundles S_i give the universal sequence of subbundles on F and the Chern classes of their duals generate the cohomology ring of F.

For each $1 \leq i \leq l+1$ let

$$H_{i,j}, \quad j = 1, ..., s_i$$

be the Chern roots of S_i^*. (in particular, all $H_{l+1,j} = 0$.)

Also use $H_{i,j}$ to denote the hyperplane classes on the corresponding abelian quotient

$$\mathbb{P} := (\mathbb{P}^{n-1})^{s_1} \times \cdots \times (\mathbb{P}^{n-1})^{s_l}$$

A curve class on F is given by $(d_1, \ldots, d_l)$, $d_i \geq 0$, while a curve class on $\mathbb{P}$ is

$$(d_{1,1}, \ldots, d_{1,s_1}, \ldots, d_{l,1} \ldots, d_{l,s_l})$$

Theorem. *For curve classes* $\vec{d} = (d_1, \ldots, d_l)$ *on* F, $J^F_{\vec{d}}$ *is given by*

$$\sum_{\sum d_{i,j}=d_i} \prod_{i=1}^{l} \left(\prod_{1\leq j\neq j'\leq s_i} \frac{\prod_{k=-\infty}^{d_{i,j}-d_{i,j'}}(H_{i,j}-H_{i,j'}+k\hbar)}{\prod_{k=-\infty}^{0}(H_{i,j}-H_{i,j'}+k\hbar)} \cdot \right.$$

$$\left. \prod_{1\leq j\leq s_i,\ 1\leq j'\leq s_{i+1}} \frac{\prod_{k=-\infty}^{0}(H_{i,j}-H_{i+1,j'}+k\hbar)}{\prod_{k=-\infty}^{d_{i,j}-d_{i+1,j'}}(H_{i,j}-H_{i+1,j'}+k\hbar)} \right)$$

The general case is a combination of the previous two examples: an isotropic flag variety is the zero section on an appropriate product of Grassmannians of a bundle which is the direct sum of Hom bundles and either the second symmetric power, or the second exterior power of appropriate universal bundles.