Definition of NÉRON MODEL

$k = \bar{k}$ B nonsingular curve / k (connected) $K = k(B)$

basic case: $B = \operatorname{Spec} R$ R a D.V.R.

$\operatorname{Spec} K \in B$ the "generic point"

A_K = abelian variety / K $A_K \longrightarrow \operatorname{Spec} K$

or

a torsor under such an abelian variety

The Néron model $N(A_K)$ of A_K is a smooth scheme / B having A_K as generic fiber and satisfying the Néron Mapping Property

N.M.P. $\forall\, Z \longrightarrow B$ smooth

$\forall\, \varphi_K : Z_K \longrightarrow A_K$

$\exists !\ \varphi : Z \longrightarrow N(A_K)$ extending φ_K

Thm (Néron '60): $N(A_K)$ exists.

Properties:

1. UNIQUENESS
2. SEPARATION (! NO PROPERNESS! in general)
3. LOCAL $\longleftrightarrow$ GLOBAL (hence $B = \mathrm{Spec} R$)
4. The group/torsor structure of A_K naturally extends to $N(A_K)$

- BAD FUNCTORIAL PROPERTIES

- $$A_K(K) \underset{1-1}{\longleftrightarrow} N(A_K)(R)$$

OUR SET UP

(Raynaud 70)

$\mathcal{X}_K$ smooth proj. curve / K
connected

$A_K = \mathrm{Pic}^0 \mathcal{X}_K$ the Jacobian of $\mathcal{X}_K$

$f : \mathcal{X} \longrightarrow B$ the rel. minimal model of $\mathcal{X}_K$ over B

$X :=$ the special fiber (a curve / k possibly singular non integral)

Our case $X = \bigcup_{i=1}^{\gamma} C_i$ X stable curve

$\mathrm{Pic}^0_f \longrightarrow B$ the (relative) degree 0 PICARD SCHEME ($\mathrm{Pic}^0_{\mathcal{X}/B}$)

↑ a smooth model of $\mathrm{Pic}^0 \mathcal{X}_K$ but NOT SEPARATED

Special fiber of Pic^0_f is

$$Pic^0 X = \coprod_{\substack{\underline{d} \in \mathbb{Z}^\gamma \\ |\underline{d}|=0}} Pic^{\underline{d}} X$$

$Pic^{\underline{d}} X \cong Pic^{\underline{d}'} X \quad \forall \underline{d}, \underline{d}' \in \mathbb{Z}^\gamma$

$$\mathcal{O}_{\mathcal{X}_K} \rightsquigarrow \left\{ \mathcal{O}_{\mathcal{X}}\left(\sum_1^\gamma m_i C_i\right)\Big/_X \right\}_{m_i \in \mathbb{Z}} =: Tw_f X$$

"f-TWISTERS"

<u>$N(Pic^0 \mathcal{X}_K)$ is the largest separated quotient of $Pic^0 f$</u>.

Special fiber of $N(Pic^0 \mathcal{X}_K)$ is

$$Pic^0 X \big/ Tw_f X \cong \coprod_{\delta \in \Delta_X} Pic^{\underline{d}^\delta} X$$

$$\Delta_X = \frac{\{\underline{d} \in \mathbb{Z}^\gamma : |\underline{d}| = 0\}}{\{\text{multidegrees of all twisters}\}} \leftarrow$$

A FINITE group

A COMBINATORIAL INVARIANT of X

Notation : $$N_g^0 := N(\mathrm{Pic}^0 \mathcal{X}_K)$$

RK: its special fiber does NOT depend on g

$\Rightarrow$
$$N_X^0 := \left(N_g^0\right)_k \qquad \square$$

Similarly $\forall\, d \in \mathbb{Z}$, $A_K = \mathrm{Pic}^d \mathcal{X}_K$ a torsor under $\mathrm{Pic}^0 \mathcal{X}_K$ ....

$$N_g^d := N(\mathrm{Pic}^d \mathcal{X}_K) \qquad N_X^d := \left(N_g^d\right)_k$$

Q: Do the N_X^d's glue together over $\overline{\mathcal{M}}_g$?

d, $g \geq 3$ fixed

ANSWER

Thm (OS). Assume $(d-g+1, 2g-2) = 1$

1) $\exists$ a modular D-M stack $\mathcal{P}_{d,g}$ with a strongly representable natural map

$$\mathcal{P}_{d,g} \longrightarrow \overline{\mathcal{M}}_g$$

s.t. $\forall f : \mathcal{X} \longrightarrow B$ ($\mathcal{X}$ regular, X stable) there is a natural isomorphism of schemes

$$N_g^d \cong \mathcal{P}_{d,g} \times_{\overline{\mathcal{M}}_g} B$$

2) $\mathcal{P}_{d,g}$ is completed by a modular D-M stack $\overline{\mathcal{P}_{d,g}} \longrightarrow \overline{\mathcal{M}_g}$ strongly representable $/ \overline{\mathcal{M}_g}$

STACK FREE VERSION (Weaker!)

$\widetilde{\text{Thm}}$: Assume $(d-g+1, 2g-2) = 1$

There exists a smooth scheme

$$P_g^d \longrightarrow \overline{M}_g^{\,0}$$

(which is modular (i.e. a FINE moduli scheme))

such that $\forall\, f: \mathcal{X} \longrightarrow B$
family of automorphism-free stable curves with $\mathcal{X}$ regular, there is a natural isomorph

$$N_g^d \cong P_g^d \times_{\overline{M}_g^{\,0}} B \,.$$

INGREDIENTS of PROOF

1) (Scheme version) P_g^d = a suitable subscheme in a compactification of the Universal Picard over $\overline{M}_g$ (PhD thesis 93)

Modularity = P_g^d parametrizes line bundles on stable curves

2) (Stackification)

Abramovich-Vistoli (01)
Edidin (00)
others

⟸ Deligne-Mumford
Artin Vistoli

3) $(d-g+1, 2g-2) = 1$ [Mestrano-Ramanan 85
is nec. and suff. for

(3.a) POINCARÉ LINE BUNDLE (MR)

(3.b) Néron N.M.P.

(3.c) Stackification

Further motivations

① COMPARING COMPLETIONS

a) Different compactifications of Picard

[$d = g-1$ Alexeev 04]

b) Completions of Picard with completions of remarkable subfunctors e.g. SPIN, PRYM

[~ Casagrande Cornalba 04]

Lots of mysteries!

② TOWARDS BRILL-NOETHER THEORY OF STABLE CURVES

Recall: C smooth proj. curve $d>0$

the d-th **ABEL MAP** of C:

$$C^d \xrightarrow{\alpha_C^d} \operatorname{Pic}^d C$$

$$(p_1, \dots, p_d) \longmapsto \mathcal{O}_C\left(\sum_1^d p_i\right)$$

α_C^d is the moduli map of a suitable line bundle on $C^d \times C$

Brill-Noether varieties (rough)

$$W_d^0(C) = \operatorname{Im} \alpha_C^d$$

$$\cup$$

$$W_d^r(C) = \{L : \dim (\alpha_C^d)^{-1}(L) \geq r\}$$

What if X is singular (i.e. a reducible stable curve)?

If X is INTEGRAL $\longrightarrow$ [Altman – Kleiman, Esteves – Gagné – Kleiman (00)]

Let $X \hookrightarrow \mathcal{X} \xrightarrow{f} B$ $\qquad$ $\mathcal{X}, \mathcal{X}_K$ regular

$$
\begin{array}{ccccc}
 & & \mathcal{X}_K^d & \xrightarrow{\alpha_K^d} & \mathrm{Pic}^d \mathcal{X}_K \\
 & & \downarrow & & \downarrow \\
\mathrm{Smooth}(f)^d =: & & \mathring{\mathcal{X}}_B^d & \xrightarrow[\exists!\, \alpha_f^d]{} & N_f^d \quad \leftarrow \text{N.M.P.} \\
 & & \downarrow & & \downarrow \\
B & \xleftarrow{f^d} & \mathcal{X}_B^d & & \overline{N}_f^d \quad \leftarrow \text{Thm 2)} \\
 & & \uparrow & \nearrow_{\overline{\alpha}_f^d} & \\
 & & \mathcal{Y} & &
\end{array}
$$

$d = 1$ (Edixhoven $\geq$ 98)

Thm (OS) 1) α'_g extends to a regular map

$$\bar{\alpha}'_g : \mathcal{H} \longrightarrow \overline{N}'_g$$

2) $\bar{\alpha}'_g / X$ does NOT depend on g

(i.e. $\bar{\alpha}'_X : X \longrightarrow \overline{N}'_X$)

3) α'_g is an IMMERSION $\Longleftrightarrow$ the normalization of X at its SEPARATING nodes contains no rational ~~curve~~ connected comps.

— • —— IN PROGRESS —— • —