# DENSITY ESTIMATES FOR A VARIATIONAL MODEL DRIVEN BY THE GAGLIARDO NORM

OVIDIU SAVIN AND ENRICO VALDINOCI

ABSTRACT. We prove density estimates for level sets of minimizers of the energy
$$\varepsilon^{2s}\|u\|_{H^s(\Omega)}^2 + \int_\Omega W(u)\,dx,$$
with $s \in (0,1)$, where $\|u\|_{H^s(\Omega)}$ denotes the total contribution from $\Omega$ in the $H^s$ norm of $u$, and $W$ is a double-well potential.

As a consequence we obtain, as $\varepsilon \to 0^+$, the uniform convergence of the level sets of $u$ to either a $H^s$-nonlocal minimal surface if $s \in (0, \frac{1}{2})$, or to a classical minimal surface if $s \in [\frac{1}{2}, 1)$.

## 1. INTRODUCTION

A classical model for the energy of a two-phase fluid of density $u$ lying in a bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geqslant 2$, is given by the Ginzburg-Landau energy functional
$$\int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\,dx.$$
The function $W : \mathbb{R} \to [0, +\infty)$ is a double well potential with two zeros (minima) at the densities of the stable phases, which we assume for simplicity to be $+1$ and $-1$. The kinetic energy is given by the Dirichlet integral
$$\frac{\varepsilon^2}{2}\int |\nabla u|^2 dx$$

which takes into account interactions at small scales between the fluid particles. The typical energy minimizer has two regions where $u$ is close to $+1$ and $-1$ which are separated by a "phase transition" which lies in an $\varepsilon$ neighborhood of the 0 level set $\{u = 0\}$.

In this paper we consider a different model in which the kinetic term is replaced by the $H^s$ norm of $u$,
$$\varepsilon^{2s}\|u\|_{H^s}^2 \quad \text{with } s \in (0,1).$$
This means that the interactions at small scales have nonlocal character. In this case the boundary data for $u$ is defined in $\mathscr{C}\Omega$, that is the complement of $\Omega$. Similar models driven by a fractional, Gagliardo-type norm were considered in [12, 13]; see also [1, 14, 15] and references therein for a onedimensional related system that models phase transitions on an interval.

From the physical point of view, the importance of these type of models relies in their attempt to capture, via the nonlocal term, the features arising from the long-range particle interactions, and it is of course desirable to understand if and how the nonlocal aspect influences the interfaces and to have good estimates on their width. Our results are a first attempt to give some answers to this questions. Indeed, we show that the level sets of the minimizers for this nonlocal energy satisfy a uniform density property. For the Allen-Cahn-Ginzburg-Landau energy such density estimates were proved in [7]. As a consequence we obtain that when $s \in (0, 1/2)$ the phase transition converges locally uniformly as $\varepsilon \to 0^+$ to a $H^s$-nonlocal minimal surface (see [8] for the precise definition), and when $s \in [1/2, 1)$ the phase transition converges locally uniformly to a classical minimal surface.

We define
$$X := \big\{ u \in L^\infty(\mathbb{R}^n) \ \text{s.t.} \ \|u\|_{L^\infty(\mathbb{R}^n)} \leqslant 1 \big\},$$
the space of admissible functions – when dealing with a minimization problem in $\Omega$, we prescribe $u \in X$ with boundary data $u_o$ outside $\Omega$ (i.e., $u = u_o$ in $\mathscr{C}\Omega$), and we say that a sequence $u_n \in X$ converges to $u$ in $X$ if $u_n$ converges to $u$ in $L^1(\Omega)$.

We define also
$$\mathscr{K}(u; \Omega) := \frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \int_\Omega \int_{\mathscr{C}\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,$$
the $\Omega$ contribution in the $H^s$ norm of $u$
$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,$$
i.e we omit the set where $(x, y) \in \mathscr{C}\Omega \times \mathscr{C}\Omega$ since all $u \in X$ are fixed outside $\Omega$.

The energy functional $J_\varepsilon$ in $\Omega$ is defined as
$$J_\varepsilon(u; \Omega) := \varepsilon^{2s} \mathscr{K}(u; \Omega) + \int_\Omega W(u) \, dx.$$

Throughout the paper we assume that $W : [-1, 1] \to [0, \infty)$,

(1.1) $\qquad W \in C^2([-1, 1]), \quad W(\pm 1) = 0, \quad W > 0 \quad \text{in } (-1, 1)$
$$W'(\pm 1) = 0, \quad \text{and} \quad W''(\pm 1) > 0.$$

We say that $u$ is a minimizer[1] for $J_\varepsilon$ in $\Omega$ if
$$J_\varepsilon(u; \Omega) \leqslant J_\varepsilon(v; \Omega)$$
for any $v$ which coincides with $u$ in $\mathscr{C}\Omega$.

We remark is that if $u$ minimizes $J_\varepsilon$ in $\Omega$ then it minimizes $J_\varepsilon$ in any subdomain $\Omega' \subset \Omega$ since
$$\mathscr{K}(u; \Omega) = \mathscr{K}(u; \Omega')$$
$$+ \frac{1}{2} \int_{\Omega - \Omega'} \int_{\Omega - \Omega'} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Omega - \Omega'} \int_{\mathscr{C}\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,$$
and the latter two integral terms do not depend on the values of $u$ in $\Omega'$.

If $u$ is a minimizer for $J_\varepsilon$ in all bounded open sets $\Omega$, we say, simply, that $u$ is a minimizer[2]. The behavior of $J_\varepsilon$ as $\varepsilon \to 0^+$ is quite different as $s \in (0, 1/2)$, $s = 1/2$ and $s \in (1/2, 1)$. We showed in [21] that in each case $J_\varepsilon$ must be multiplied by

---

[1]A similar notion of minimizer will hold, later on, for the suitably rescaled versions of $J_\varepsilon$, namely $\mathscr{F}_\varepsilon$ and $\mathscr{E}$.

[2]Sometimes, in the literature, minimizers are called "local", or "class A", minimizers.

an appropriate constant depending on $\varepsilon$ in order to obtain the $\Gamma$-convergence to a limiting functional. More precisely, given any $\varepsilon > 0$, we define the functional $\mathscr{F}_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ as

$$(1.2) \qquad \mathscr{F}_\varepsilon(u) = \mathscr{F}_\varepsilon(u; \Omega) := \begin{cases} \varepsilon^{-2s} J_\varepsilon(u; \Omega) & \text{if } s \in (0, 1/2), \\ |\varepsilon \log \varepsilon|^{-1} J_\varepsilon(u; \Omega) & \text{if } s = 1/2, \\ \varepsilon^{-1} J_\varepsilon(u; \Omega) & \text{if } s \in (1/2, 1). \end{cases}$$

In the case when $s \in (0, 1/2)$, the limiting functional $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$(1.3) \qquad \mathscr{F}(u) := \begin{cases} \mathscr{K}(u; \Omega) & \text{if } u|_\Omega = \chi_E - \chi_{\mathscr{C}E}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

In this case, $\mathscr{F}$ agrees with the nonlocal area functional of $\partial E$ in $\Omega$ that was studied in [8, 9, 2]. Remarkably, such nonlocal area functional is well defined exactly when $s \in (0, 1/2)$.

In the case when $s \in [1/2, 1)$ the limiting functional $\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$(1.4) \qquad \mathscr{F}(u) := \begin{cases} c_\star \operatorname{Per}(E; \Omega) & \text{if } u|_\Omega = \chi_E - \chi_{\mathscr{C}E}, \text{ for some set } E \subset \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

where $c_\star$ is a constant depending on $n$, $s$ and $W$.

We recall the $\Gamma$-convergence results in [21]:

**Theorem 1.1.** *Let $s \in (0, 1)$. Then, $\mathscr{F}_\varepsilon$ $\Gamma$-converges to $\mathscr{F}$, i.e.,*

(i) *for any $u_\varepsilon$ converging to $u$ in $X$,*

$$\mathscr{F}(u) \leqslant \liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon),$$

(ii) *for any $u \in X$ there exists $u_\varepsilon$ converging to $u$ in $X$ such that*

$$\mathscr{F}(u) \geqslant \limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon).$$

**Theorem 1.2.** *If $\mathscr{F}_\varepsilon(u_\varepsilon; \Omega)$ is uniformly bounded for a sequence of $\varepsilon \to 0^+$, then there exists a convergent subsequence*

$$u_\varepsilon \to u_* := \chi_E - \chi_{\mathscr{C}E} \quad \text{in } L^1(\Omega).$$

*Moreover, if $u_\varepsilon$ minimizes $\mathscr{F}_\varepsilon$ in $\Omega$,*

(i) *if $s \in (0, \frac{1}{2})$ and $u_\varepsilon$ converges weakly to $u_o$ in $\mathscr{C}\Omega$, then $u_*$ minimizes $\mathscr{F}$ in (1.3) among all the functions that agree with $u_o$ in $\mathscr{C}\Omega$;*

(ii) *if $s \in [1/2, 1)$, then $u_*$ minimizes $\mathscr{F}$ in (1.4).*

Theorem 1.1 may be seen as a nonlocal analogue of the celebrated $\Gamma$-convergence result of [17] (see also [16, 3, 18] for further extensions). In this framework, we recall that a very important issue, besides $\Gamma$-convergence, is the "geometric" convergence of the level sets of minimizers to the limit surface. This topic has been widely studied in the case of local functionals by using appropriate density estimates (see [7], and also [11] and references therein for several other applications). The idea of these density estimates is to give an optimal bound on the measure occupied by the level sets of a minimizer in a ball.

Our results give a nonlocal counterpart of these density estimates for minimizers of $J_\varepsilon$ (or $\mathscr{F}_\varepsilon$). For this, it is convenient to scale space by a factor of $\varepsilon^{-1}$ so that

the dependence of $J_\varepsilon$ on $\varepsilon$ disappears. To be more precise, if $u$ minimizes $J_\varepsilon$ in $\Omega$, then the rescaled function

$$u_\varepsilon(x) := u(\varepsilon x)$$

minimizes $\mathscr{E}$ in $\Omega_\varepsilon := \Omega/\varepsilon$, where

$$\mathscr{E}(v; \tilde{\Omega}) := J_1(v; \tilde{\Omega}) = \mathscr{K}(v; \tilde{\Omega}) + \int_{\tilde{\Omega}} W(v)\, dx.$$

Our first result gives a uniform bound for the energy $\mathscr{E}$ of a minimizer in $B_R$ for large $R$.

**Theorem 1.3.** *Let $u$ be a minimizer of $\mathscr{E}$ in $B_{R+2}$ with $R \geqslant 1$. Then*

$$(1.5) \qquad \mathscr{E}(u; B_R) \leqslant \begin{cases} \overline{C}\, R^{n-2s} & \text{if } s \in (0, 1/2), \\ \overline{C}\, R^{n-1} \log R & \text{if } s = 1/2, \\ \overline{C}\, R^{n-1} & \text{if } s \in (1/2, 1), \end{cases}$$

*where $\overline{C}$ is a positive constant depending on $n$, $s$, and $W$.*

Theorem 1.3 can be stated in terms of minimizers $u$ of $\mathscr{F}_\varepsilon$ in $B_{1+2\varepsilon}$ as

$$\mathscr{F}_\varepsilon(u; B_1) \leqslant \overline{C}.$$

Then, we have the following density estimate on the level sets of minimizers:

**Theorem 1.4.** *Let $u$ be a minimizer of $\mathscr{E}$ in $B_R$. Then for any $\theta_1$, $\theta_2 \in (-1, 1)$ such that*

$$(1.6) \qquad\qquad\qquad\qquad u(0) > \theta_1,$$

*we have that*

$$(1.7) \qquad\qquad\qquad \left| \{u > \theta_2\} \cap B_R \right| \geqslant \overline{c}\, R^n$$

*if $R \geqslant \overline{R}(\theta_1, \theta_2)$. The constant $\overline{c} > 0$ depends only on $n$, $s$ and $W$ and $\overline{R}(\theta_1, \theta_2)$ is a large constant that depends also on $\theta_1$ and $\theta_2$.*

By scaling, Theorem 1.4 gives the uniform density estimate for minimizers $u$ of $\mathscr{F}_\varepsilon$ (or $J_\varepsilon$) in $B_r$: if $u(0) > \theta_1$ then

$$\left| \{u > \theta_2\} \cap B_r \right| \geqslant \overline{c}\, r^n \qquad \text{if } r \geqslant \overline{R}\varepsilon.$$

*Remark* 1.5. Our assumptions on $W$ are not the most general. For example in the Theorem 1.3 it suffices to say that $W$ is bounded and $W(\pm 1) = 0$. Also in Theorem 1.4 it suffices to assume that there exists a small constant $c > 0$ such that

$$(1.8) \qquad \begin{aligned} &W(t) \geqslant W(r) + c(1+r)(t-r) + c\,(t-r)^2 \text{ when } -1 \leqslant r \leqslant t \leqslant -1+c \\ &\text{and } W(r) - W(t) \leqslant (1+r)/c \text{ when } -1 \leqslant r \leqslant t \leqslant +1. \end{aligned}$$

Of course, (1.8) is warranted by our assumptions in (1.1), but we would like to stress that less smooth, or even discontinuous, potentials, may be dealt with using (1.8).

The proof of Theorem 1.4 is contained in Section 3 and it requires a careful analysis of the measure theoretic properties of the minimizers and several nontrivial modifications of the original proof of [7], together with some iteration techniques of [8]. In particular, the construction of a new barrier function is needed in order to keep track of the densities of the level sets in larger and larger balls. Also, the

proof of (1.7) is somewhat delicate and it requires[3] the following estimate for the double integral

$$(1.9) \qquad L(A, D) := \int_A \int_D \frac{1}{|x - y|^{n+2s}} dx dy.$$

**Theorem 1.6.** *Let $s \in (0, 1)$. Let $A$ and $B$ be disjoint measurable subsets of $\mathbb{R}^n$ and let $D := \mathscr{C}(A \cup B)$. Then, there exists $c \in (0, 1)$, possibly depending on $n$ and $s$, for which the following estimates hold:*

- *if $|B| \leqslant c |A|$ and $|A| > 0$, then*

$$(1.10) \qquad L(A, D) \geqslant \begin{cases} c |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ c |A|^{(n-1)/n} \log(|A|/|B|) & \text{if } s = 1/2, \\ c |A|^{(n-2s)/n} \left(|B|/|A|\right)^{1-2s} & \text{if } s \in (1/2, 1), \end{cases}$$

- *if $|B| > c |A|$ then*

$$(1.11) \qquad L(A, D) \geqslant c |A|^{(n-2s)/n} \left(|B|/|A|\right)^{-2s/n}.$$

Theorems 1.3 and 1.4 have, of course, physical relevance, since they give optimal bounds on the energy of the limit interface, and on the measure of the level sets of the minimizers – i.e., roughly speaking, on the probability of finding a given phase in a certain portion of the mediumm.

Also, due to the work of [7], density estimates as the ones in Theorem 1.4 are known to have useful scaling properties and to play a crucial role in the geometric analysis of the level sets of the rescaled minimizers, especially in relation with the asymptotic interface. For instance, we point out the following consequence of Theorem 1.1 and Theorem 1.4.

**Corollary 1.7.** *Suppose that $u$ is a global minimizer of $\mathscr{E}$ in $\mathbb{R}^n$, i.e minimizes $\mathscr{E}$ in any bounded domain $\Omega \subset \mathbb{R}^n$. Let*

$$u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$

*Then*

(i) *$u_\varepsilon$ is a global minimizer for $\mathscr{F}_\varepsilon$,*
(ii) *$u_\varepsilon$ converges, up to subsequences, in $L^1_{\text{loc}}(\mathbb{R}^n)$ to some $u_\star = \chi_E - \chi_{\mathscr{C}E}$ and $u_\star$ is a global minimizer of $\mathscr{F}$ (see (1.3), (1.4)),*
(iii) *given any $\theta \in (0, 1)$, the set $\{|u_\varepsilon| \leqslant \theta\}$ converges to $\partial E$ locally uniformly, that is, for any $R > 0$ and any $\delta > 0$ there exists $\varepsilon_o \in (0, 1]$, possibly depending on $R$ and $\delta$, such that, if $\varepsilon \in (0, \varepsilon_o]$ then*

$$(1.12) \qquad \{|u_\varepsilon| \leqslant \theta\} \cap B_R \subseteq \bigcup_{p \in \partial E} B_\delta(p).$$

The minimizer $u$ above satisfies the Euler-Lagrange equation

$$(1.13) \qquad (-\Delta)^s u(x) + W'(u(x)) = 0,$$

where

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy$$

---

[3]When $s \in (0, 1/2)$ there is also an alternative approach based on the fractional Sobolev inequality. We will perform this different proof in [22].

and the integral is understood in the principal value sense. As usual, $(-\Delta)^s$ is (up to a normalizing multiplicative constant, depending on $n$ and $s$) the fractional power of the positive operator $-\Delta$.

Corollary 1.7 follows immediately: (i) from the scaling properties of $\mathscr{E}$, (ii) from Theorem 1.2, and (iii) is a consequence of the density estimates for the level sets of $u_\varepsilon$ and the $L^1_{\mathrm{loc}}$-convergence to $u_\star$ (see [7] for further details).

The minimizing property of $u_\star$ of Corollary 1.7(ii) says that when $s \in (0, 1/2)$ the limit interface $\partial E$ is a nonlocal minimal surface in the setting of [8], and when $s \in [1/2, 1)$, $\partial E$ is a classical minimal surface. This is interesting also because any regularity or rigidity property proved for $\partial E$ may reflect into similar ones for the minimizers of $\mathscr{F}_\varepsilon$ (see, e.g., [20]). In particular, (1.13) may be seen as a semilinear equation driven by the fractional Laplacian. Some rigidity properties for this kind of equations have been recently obtained, for instance, in [6, 23], but many fundamental questions on this subject are still open.

The paper is organized as follows. In Section 2 we prove Theorem 1.3. In Section 3 we prove Theorem 1.4 by treating the cases $s \in (0, 1/2)$ and $s \in [1/2, 1)$ separately. Theorem 1.6, together with a localized version of it (i.e., Proposition 4.3), is proved in Section 4. Often in the proofs, when there is no possibility of confusion, we denote the constants by $C$ and $c$ although they may change from line to line.

## 2. Proof of Theorem 1.3

We use the following notation
$$u(A, B) = \int_A \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.$$

Since
$$\mathscr{E}(u, B_R) = \mathscr{K}(u, B_R) + \int_{B_R} W(u) \, dx$$
$$\leqslant \frac{1}{2} u(B_{R+1}, B_{R+1}) + u(B_R, \mathscr{C}B_{R+1}) + \int_{B_R} W(u) \, dx,$$

it suffices to bound each term on the right by the quantity that appears in (1.5).

We define
$$(2.1) \qquad \psi(x) = -1 + 2\min\{(|x| - R - 1)^+, 1\}$$

so that $\psi = -1$ in $B_{R+1}$ and $\psi = 1$ in $\mathscr{C}B_{R+2}$.

First we show that $\mathscr{E}(\psi, B_{R+2})$ satisfies the bounds in (1.5). Let
$$d(x) := \max\{R - |x|, \quad 1\}$$

and notice that
$$|\psi(x) - \psi(y)| \leqslant \begin{cases} 2d(x)^{-1}|x - y| & \text{if } |x - y| < d(x), \\ 2 & \text{if } |x - y| \geqslant d(x). \end{cases}$$

We obtain
$$\int_{\mathbb{R}^n} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{n+2s}} \, dy$$
$$\leqslant \omega_{n-1} \int_0^{d(x)} \frac{(2r/d(x))^2}{r^{n+2s}} r^{n-1} \, dr + \omega_{n-1} \int_{d(x)}^\infty \frac{4}{r^{n+2s}} r^{n-1} \, dr \leqslant C d(x)^{-2s}.$$

Now we integrate this inequality for all $x \in B_{R+2}$ and obtain that $\mathscr{K}(\psi, B_{R+2})$ (therefore $\mathscr{E}(\psi, B_{R+2})$) satisfies the energy bound of the theorem.

Let $v = \min\{u, \psi\}$ and denote $A := \{v \leqslant u\} \cap B_{R+2}$. Clearly $B_{R+1} \subset A$ and $u = v$ in $\mathscr{C}A$. We write that $u$ is a minimizer for $\mathscr{E}$ in $B_{R+2}$, and therefore in $A$:

(2.2)
$$\frac{1}{2}u(A, A) + u(A, \mathscr{C}A) + \int_A W(u)\,dx$$
$$\leqslant \frac{1}{2}v(A, A) + v(A, \mathscr{C}A) + \int_A W(v)\,dx.$$

If $x \in A$ and $y \in \mathscr{C}A$ then $v(x) = \psi(x) \leqslant u(x)$, $v(y) = u(y) \leqslant \psi(y)$ thus
$$|v(x) - v(y)| \leqslant \max\{|u(x) - u(y)|, |\psi(x) - \psi(y)|\},$$

which gives
$$v(A, \mathscr{C}A) \leqslant u(A, \mathscr{C}A) + \psi(A, \mathscr{C}A).$$

We use this in the energy inequality (2.2), we simplify $u(A, \mathscr{C}A)$ on both sides, and we obtain
$$\frac{1}{2}u(A, A) + \int_A W(u)\,dx$$
$$\leqslant \frac{1}{2}\psi(A, A) + \psi(A, \mathscr{C}A) + \int_A W(v)\,dx = \mathscr{E}(\psi, A) \leqslant \mathscr{E}(\psi, B_{R+2}).$$

Since $B_{R+1} \subset A$ we obtain the desired bounds for $u(B_{R+1}, B_{R+1})$ and $\int_{B_R} W(u)\,dx$.

On the other hand $u(B_R, \mathscr{C}B_{R+1})$ also satisfies a similar bound since
$$\int_{\mathscr{C}B_{R+1}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\,dy \leqslant C \int_{d(x)}^{\infty} r^{-1-2s}dr \leqslant Cd(x)^{-2s} \quad \text{for all } x \in B_R,$$

and then we integrate in $x \in B_R$.

## 3. Proof of Theorem 1.4

3.1. **Preliminary computations.** A minimizer $u$ of $\mathscr{E}$ in $B_R$ with $R \geqslant 2$ satisfies the Euler-Lagrange equation in (1.13), hence,
$$\|u\|_{C^\alpha(B_1)} \leqslant C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_2)}) \leqslant C,$$

for some small $\alpha > 0$ (see, e.g., Lemma ????????? in [19]).

This shows that, by relabeling $\theta_1$, we can replace (1.6) by

(3.1)
$$\left|\{u > \theta_1\} \cap B_{R_o}\right| \geqslant \mu$$

for some constants $R_o > 0$ and $\mu > 0$, possibly depending on $n$, $s$ and $W$ – and, in fact, the proof of Theorem 1.4 will make use only of (3.1) rather than (1.6).

The strategy of the proof is, roughly speaking, to use the minimality of $u$ in order to obtain an estimate of $|\{u > \theta_2\} \cap B_{2\rho}|$ in terms of $|\{u > \theta_2\} \cap B_\rho|$. Then the conclusion will follow by iterating (3.1).

First, we construct the following useful barrier:

**Lemma 3.1.** *Let $n \geqslant 1$. Given any $\tau > 0$, there exists $C \geqslant 1$, possibly depending on $n$, $s$ and $\tau$, such that the following holds: for any $R \geqslant C$, there exists a rotationally symmetric function*

(3.2)
$$w \in C(\mathbb{R}^n, [-1 + CR^{-2s}, 1]),$$

*with*

(3.3)
$$w = 1 \text{ in } \mathscr{C}B_R,$$

*such that*

$$(3.4) \qquad -(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{w(y) - w(x)}{|x - y|^{n+2s}} \, dy \leqslant \tau\big(1 + w(x)\big)$$

*and*

$$(3.5) \qquad \frac{1}{C}\big(R + 1 - |x|\big)^{-2s} \leqslant 1 + w(x) \leqslant C\big(R + 1 - |x|\big)^{-2s}$$

*for any $x \in B_R$.*

*Proof.* We fix a large $r \geqslant 1$, to be conveniently chosen with respect to $R$ and $\tau$ in the sequel.

For $t \in (0, +\infty)$ and $x \in \mathbb{R}^n$, we define

$$
\begin{aligned}
g(t) &:= t^{-2s}, \\
h(t) &:= \begin{cases} \min\Big\{1, \, g(t) - g(r/2) - g'(r/2)\big(t - (r/2)\big)\Big\} & \text{if } t \leqslant r/2, \\ \qquad\qquad\qquad 0 & \text{if } t \geqslant r/2, \end{cases} \\
v(x) &:= \begin{cases} h(r - |x|) & \text{if } x \in B_r, \\ 1 & \text{if } x \in \mathscr{C}B_r. \end{cases}
\end{aligned}
$$

Such a $v$, up to a proper rescaling, will provide the existence of the desired function $w$. To check this, we first notice that

$$(3.6) \qquad \begin{aligned} &\text{if } t \leqslant r/2 \text{ and } h(t) < 1 \text{ then} \\ &\quad h(t) = g(t) - g(r/2) - g'(r/2)\big(t - (r/2)\big) \\ &\qquad \geqslant g(t) - g(r/2) - |g'(r/2)| \, (r/2) \\ &\qquad \geqslant t^{-2s} - 16 r^{-2s}. \end{aligned}$$

Moreover, $v$ is continuous, radially symmetric, radially nondecreasing and $0 \leqslant v \leqslant 1$, due to the convexity of $g$.

Also, we claim that

$$(3.7) \qquad \text{for any } x \in B_r, \, \|D^2 v\|_{L^\infty(B_{(r-|x|)/2}(x))} \leqslant 2^8 (r - |x|)^{-2(1+s)}.$$

To prove (3.7), we observe that $v = 0$ in $B_{r/2}$ and so $D^2 v = 0$ in $B_{r/2}$. Then, we take $y \in B_{(r-|x|)/2}(x) \cap (\mathscr{C}B_{r/2})$ and we observe that

$$|y| \leqslant |y - x| + |x| \leqslant \frac{r - |x|}{2} + |x| = \frac{r + |x|}{2},$$

hence

$$r - |y| \geqslant r - \frac{r + |x|}{2} = \frac{r - |x|}{2}.$$

In particular,

$$|r - |y|| = r - |y| \leqslant |y|$$

and, as a consequence,

$$|D^2 v(y)| \leqslant \max\left\{|g''(r - |y|)|, \, \frac{|g'(r - |y|)|}{|r - |y||}\right\}$$

$$\leqslant 2s(1 + 2s)\,(r - |y|)^{-2(1+s)} \leqslant \frac{2^8}{(r - |x|)^{-2(1+s)}},$$

proving (3.7).

From Lemma **??** in [19] and (3.7), we obtain that

for any $x \in B_r$,

$$\left| \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x - y|^{n+2s}} \, dy \right|$$

(3.8)
$$\leqslant C_0 \left[ \|D^2 v\|_{L^\infty(B_{(r-|x|)/2}(x))} \left( (r - |x|)/2 \right)^{2(1-s)} + 2 \left( (r - |x|)/2 \right)^{-2s} \right]$$

$$\leqslant C_1 \left[ (r - |x|)^{-4s} + (r - |x|)^{-2s} \right]$$

$$\leqslant C_2 (r - |x|)^{-2s},$$

for suitable $C_0, C_1, C_2 > 0$.

Now, we claim that

(3.9)
$$\{v < 1\} \subseteq B_{r - (1/2)}.$$

Indeed, we take $x \in \{v < 1\}$ and we define $t_x := r - |x|$, so $h(t_x) < 1$. Hence either $t_x > r/2$ or $0 < t_x \leqslant r/2$ with $h(t_x) < 1$. In the first case, we would have that $|x| \leqslant r/2 < r - 1$ if $r$ is large enough and (3.9) would hold, therefore we focus on the second case. But then, recalling (3.6), for large $r$,

$$1 > t_x^{-2s} - 16 r^{-2s} \geqslant t_x^{-2s} - (2^{2s} - 1).$$

That is, $t_x \geqslant 1/2$, proving (3.9).

A straightforward consequence of (3.9) is that

(3.10)
$$\|D^2 v\|_{L^\infty(\{v < 1\})} \leqslant C_3,$$

for a suitable $C_3 > 0$.

Now, we set

$$\Xi(x) := \begin{cases} \nabla v(x) & \text{if } x \in \{v < 1\}, \\ 0 & \text{if } x \in \{v = 1\} \end{cases}$$

and we claim that

(3.11)
$$v(y) - v(x) - \Xi(x) \cdot (y - x) \leqslant C_3 |x - y|^2,$$

for any $x \in B_r$ and any $y \in \mathbb{R}^n$.

To prove the claim above, we observe that, if $x \in \{v = 1\}$, then the left hand side of (3.11) is nonpositive, so (3.11) holds true. Also, (3.11) follows from (3.10) if both $x$ and $y$ lie in $\{v < 1\}$, so it only remains to prove (3.11) when $v(x) < 1$ and $v(y) = 1$. In such a case, we define $v^\sharp$ to be a smooth, radial extension of $v$ outside $\{v < 1\}$ such that $1 \leqslant v^\sharp \leqslant 2$ outside $\{v < 1\}$ and $\|D^2 v^\sharp\|_{L^\infty(\{v < 1\})} \leqslant C_3$. Then,

$$v(y) - v(x) - \Xi(x) \cdot (y - x) = 1 - v^\sharp(x) - \nabla v^\sharp(x) \cdot (y - x)$$
$$\leqslant v^\sharp(y) - v^\sharp(x) - \nabla v^\sharp(x) \cdot (y - x) \leqslant C_3 |x - y|^2,$$

proving (3.11) in this case too.

Thus, thanks to (3.11), we may use estimate (**??**) in Lemma **??** of [19] and obtain that

(3.12) $\quad$ for any $x \in B_r$, $\quad \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x - y|^{n+2s}} \, dy \leqslant C_4$

for a suitable $C_4 > 0$.

Now, we point out that

$$(3.13) \qquad \min\left\{1, t^{-2s}\right\} \leqslant h(t) + 16r^{-2s}.$$

Indeed, if $t \leqslant r/2$, then (3.13) is a consequence of (3.6), while if $t > r/2$ we have that $t^{-2s} \leqslant 8r^{-2s} < 1$, which implies (3.13).

As a consequence of (3.13), we have that

$$(3.14) \qquad \min\left\{1, (r - |x|)^{-2s}\right\} \leqslant v(x) + 16r^{-2s}.$$

From (3.8), (3.12) and (3.14), we conclude that

$$(3.15) \qquad \text{for any } x \in B_r, \qquad \int_{\mathbb{R}^n} \frac{v(y) - v(x)}{|x - y|^{n+2s}}\, dy$$
$$\leqslant C_5 \min\left\{1, (r - |x|)^{-2s}\right\}$$
$$\leqslant C_5\left(v(x) + 16r^{-2s}\right),$$

for a suitable $C_5 > 0$.

Moreover, for any $t \in [0, r/2]$,

$$h(t) \leqslant g(t) + |g'(r/2)|\,(r/2) \leqslant t^{-2s} + 8r^{-2s}$$

and so

$$(3.16) \qquad \text{for any } x \in B_r,\ v(x) \leqslant (r - |x|)^{-2s} + 8r^{-2s}.$$

Now we define

$$C_o \ := \ \left(\frac{C_5}{\tau}\right)^{1/2s},$$
$$\beta \ := \ 32r^{-2s}$$
$$w(x) \ := \ (2 - \beta)v\left(\frac{x}{C_o}\right) + \beta - 1.$$

and we take $r := R/C_o$. Notice that $r$ is large if so is $R$, possibly in dependence of $\tau$ (thus, from now on, the constants are also allowed to depend on $\tau$). Also, $w$ is radially non decreasing, and $w = 1$ in $\mathscr{C}B_R$. In particular,

$$(3.17) \qquad 1 + w(x) \leqslant 2.$$

Moreover,

$$(3.18) \qquad w(x) = \beta - 1 \text{ for any } x \in B_{R/2}$$

and, from (3.16), for any $x \in B_R$,

$$1 + w(x) \leqslant 2v(x/C_o) + \beta \leqslant 2C_o^{2s}(R - |x|)^{-2s} + 8r^{-2s} + \beta.$$

That is, for a suitable $C_6 > 0$,

$$(3.19) \qquad 1 + w(x) \leqslant C_6(R - |x|)^{-2s} \qquad \text{for any } x \in B_R \setminus B_{R/2}.$$

By (3.18) and (3.19), we obtain that

$$(3.20) \qquad 1 + w(x) \leqslant C_7(R - |x|)^{-2s} \qquad \text{for any } x \in B_R,$$

for a suitable $C_7 \geqslant 1$.

Now, we claim that

$$(3.21) \qquad 1 + w(x) \leqslant \left(2^{2s} + 2^{1-2s}\right)C_7\,(R + 1 - |x|)^{-2s} \qquad \text{for any } x \in B_R.$$

Indeed, if $|x| \leqslant R - 1$, we have that $R - |x| \geqslant (R + 1 - |x|)/2$, therefore (3.21) is a consequence of (3.20). If, on the other hand, $|x| \geqslant R - 1$, we have that $R + 1 - |x| \leqslant 2$, thus we use (3.17) to obtain

$$1 + w(x) \leqslant 2 = 2^{1-2s} 2^{-2s} \leqslant 2^{1-2s} (R + 1 - |x|)^{-2s},$$

which gives (3.21).

Then, (3.21) implies the upper bound in (3.5), and the lower bound may be obtained analogously, using (3.6).

Moreover, recalling (3.15), for any $x \in B_R$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{w(y) - w(x)}{|x - y|^{n+2s}}\, dy &= (2 - \beta)\, C_o^{-2s} \int_{\mathbb{R}^n} \frac{v(y) - v(x/C_o)}{|(x/C_o) - y|^{n+2s}}\, dy \\
&\leqslant (2 - \beta)\, C_o^{-2s} C_5 \Big( v(x/C_o) + 16 r^{-2s} \Big) \\
&\leqslant C_o^{-2s} C_5 \Big( (2 - \beta) v(x/C_o) + 32 r^{-2s} \Big) \\
&= \tau \big( 1 + w(x) \big).
\end{aligned}
$$

This proves (3.4) and it completes the proof of Lemma 3.1. $\qquad\square$

Now, we give an elementary, general estimate:

**Lemma 3.2.** *Let $\sigma$, $\mu \in (0, +\infty)$, $\nu \in (\sigma, +\infty)$ and $\gamma$, $R_o$, $C \in (1, +\infty)$.*

*Let $V : (0, +\infty) \to (0, +\infty)$ be a nondecreasing function. For any $r \in [R_o, +\infty)$, let*

$$\alpha(r) := \min \left\{ 1,\ \frac{\log V(r)}{\log r} \right\}.$$

*Suppose that*

$$(3.22) \qquad\qquad V(R_o) \geqslant \mu$$

*and*

$$(3.23) \qquad r^\sigma\, \alpha(r)\, V(r)^{(\nu - \sigma)/\nu} \leqslant C V(\gamma r), \qquad \text{for any } r \in [R_o, +\infty).$$

*Then, there exist $c \in (0, 1)$ and $R_\star \in [R_o, +\infty)$, possibly depending on $\mu$, $\nu$, $\gamma$, $R_o$ and $C$, such that*

$$V(r) \geqslant c r^\nu, \qquad \text{for any } r \in [R_\star, +\infty).$$

*Proof.* Let $j_1$ be the smallest natural number for which $\gamma^{j_1} \geqslant R_o$. Notice that such a definition is well posed since $\gamma > 1$.

Let

$$(3.24) \qquad c := \min \left\{ \frac{\mu}{\gamma^{\nu j_1}},\ \left( \frac{1}{C \gamma^\nu} \right)^{\nu/\sigma},\ \left( \frac{\nu}{2 C \gamma^\nu} \right)^{\nu/\sigma} \right\}.$$

Let $j_2$ be the smallest integer for which

$$(3.25) \qquad\qquad \frac{|\log c|}{j_2 \log \gamma} \leqslant \frac{\nu}{2}.$$

Let $j_\star := j_1 + j_2$. For any $j \in \mathbb{N} \cap [j_\star, +\infty)$, we define $v_j := V(\gamma^j)$. We claim that

$$(3.26) \qquad\qquad v_j \geqslant c \gamma^{\nu j}, \qquad \text{for any } r \in [R_o, +\infty).$$

The proof of (3.26) is by induction over $j$. First of all,

$$v_{j_\star} \geqslant V(\gamma^{j_1}) \geqslant V(R_o) \geqslant \mu \geqslant c\gamma^{\nu j_1},$$

thanks to (3.24).

Then, we suppose that (3.26) holds for some $j \geqslant j_\star$ and we prove it for $j + 1$, via the following argument.

Since we assumed that (3.26) holds $j$,

$$
\begin{aligned}
\alpha(\gamma^j) \quad &= \quad \min\left\{1, \frac{\log(v_j)}{\log \gamma^j}\right\} \\
&\geqslant \quad \min\left\{1, \frac{\log(c\gamma^{\nu j})}{\log \gamma^j}\right\} \\
&= \quad \min\left\{1, \nu - \frac{|\log c|}{j\log \gamma}\right\} \\
&\geqslant \quad \min\left\{1, \frac{\nu}{2}\right\},
\end{aligned}
$$

thanks to (3.25).

Therefore, using (3.23) with $r := \gamma^j$ and the assumption that (3.26) holds for $j$, we conclude that

$$
\begin{aligned}
v_{j+1} \quad &= \quad V(\gamma^{j+1}) \\
&\geqslant \quad \frac{\gamma^{\sigma j}}{C}\, \alpha(\gamma^j)\, v_j^{(\nu-\sigma)/\nu} \\
&\geqslant \quad \min\left\{\frac{1}{C}, \frac{\nu}{2C}\right\} c^{(\nu-\sigma)/\nu}\gamma^{\nu j}.
\end{aligned}
$$

Recalling (3.24), we see that this last quantity is greater or equal than $c\gamma^{(j+1)\nu}$, thus completing the induction argument which proves (3.26).

From (3.26), the desired result plainly follows.                    $\square$

*Remark* 3.3. In the sequel, we will use Lemma 3.2 with $V(R) := |\{u > \theta_\star\} \cap B_R|$, with $\theta_\star \leqslant \theta_1$. In this way, condition (3.22) is warranted by (3.1).

Now, we make some useful computations, valid for any $s \in (0, 1)$. We fix $K \geqslant 2(R_o + 1)$, to be taken suitably large in the sequel, where $R_o$ is the one given by the statement of Theorem 1.4, and $R > 2K$. Given any measurable $w : \mathbb{R}^n \to [-1, 1]$ such that

(3.27)                              $w = 1$ in $\mathscr{C}B_R,$

we define

(3.28)                          $v(x) := \min\{u(x), w(x)\}$

and $D := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathscr{C}B_R \times \mathscr{C}B_R)$. Notice that

$$\mathscr{K}(u; B_R) = \frac{1}{2} \iint_D \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy$$

and that

(3.29)                              $v = u$ in $\mathscr{C}B_R.$

So, by a simple algebraic computation, we have that

$$
\mathscr{K}(u-v; B_R) + \mathscr{K}(v; B_R) - \mathscr{K}(u; B_R)
$$
$$
= -\iint_D \frac{\big((u-v)(x) - (u-v)(y)\big)\big(v(y)-v(x)\big)}{|x-y|^{n+2s}}\, dx\, dy
$$
$$
= -\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big((u-v)(x) - (u-v)(y)\big)\big(v(y)-v(x)\big)}{|x-y|^{n+2s}}\, dx\, dy
$$
$$
= -2\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big(u(x)-v(x)\big)\big(v(y)-v(x)\big)}{|x-y|^{n+2s}}\, dx\, dy
$$
$$
= 2\int_{\{u>v=w\}} \big(u(x)-v(x)\big)\left[\int_{\mathbb{R}^n} \frac{v(y)-w(x)}{|x-y|^{n+2s}}\, dy\right] dx
$$
$$
\leqslant 2\int_{B_R \cap \{u>w\}} \big(u(x)-v(x)\big)\left[\int_{\mathbb{R}^n} \frac{w(y)-w(x)}{|x-y|^{n+2s}}\, dy\right] dx
$$
$$
= -2\int_{B_R \cap \{u>w\}} \big(u(x)-v(x)\big)(-\Delta)^s w(x)\, dx.
$$

As a consequence, using once more (3.29) and the minimality of $u$, we conclude that

(3.30)
$$
\mathscr{K}(u-v; B_R)
$$
$$
\leqslant \mathscr{E}_R(u) - \mathscr{E}_R(v) + \int_{B_R} W(v) - W(u)\, dx
$$
$$
- 2\int_{B_R \cap \{u>w\}} \big(u(x)-v(x)\big)(-\Delta)^s w(x)\, dx
$$
$$
\leqslant \int_{B_R \cap \{u>w\}} W(w) - W(u)\, dx
$$
$$
- 2\int_{B_R \cap \{u>w\}} \big(u(x)-w(x)\big)(-\Delta)^s w(x)\, dx.
$$

Now, we fix $\theta_1$, $\theta_2 \in (-1,1)$ as in the statement of Theorem 1.4, and we take

(3.31)
$$
\theta_\star \leqslant \min\{\theta_1,\, \theta_2,\, -1+c\},
$$

with $c$ as in (1.8). We define

(3.32)
$$
A(R) := c\int_{B_R \cap \{w<u\leqslant\theta_\star\}} (u-w)^2\, dx.
$$

From the behavior of $W$ near its minima (see (1.8)), we deduce that

$$
\int_{B_R \cap \{u>w\}} W(w) - W(u)\, dx
$$
$$
\leqslant -c\int_{B_R \cap \{w<u\leqslant\theta_\star\}} (1+w)(u-w)\, dx
$$
$$
+ \frac{1}{c}\int_{B_R \cap \{u>\max\{w,\theta_\star\}\}} (1+w)\, dx - A(R).
$$

This and (3.30) give that

$$
\begin{aligned}
\mathscr{K}(u-v; B_R) \leqslant -c \int_{B_R \cap \{w < u \leqslant \theta_\star\}} & (1+w)(u-w)\, dx \\
+ \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_\star\}\}} & (1+w)\, dx \\
- 2 \int_{B_R \cap \{u > w\}} & \big(u(x) - w(x)\big)\, (-\Delta)^s w(x)\, dx \\
- A(R). &
\end{aligned}
$$
(3.33)

While (3.33) is valid for any $w$ satisfying (3.27), we now choose $w$ in a convenient way. That is, we define $\tau := c/4$ and we take $w$ to be the function constructed in Lemma 3.1. With this choice, (3.33) and Lemma 3.1 give that

$$
\begin{aligned}
& \mathscr{K}(u-v; B_R) + \frac{c}{2} \int_{B_R \cap \{w < u \leqslant \theta_\star\}} (1+w)(u-w)\, dx \\
& \leqslant -\frac{c}{2} \int_{B_R \cap \{w < u \leqslant \theta_\star\}} (1+w)(u-w)\, dx \\
& \quad + \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_\star\}\}} (1+w)\, dx \\
& \quad + 2\tau \int_{B_R \cap \{u > w\}} \big(u(x) - w(x)\big) \big(1 + w(x)\big)\, dx \; - A(R) \\
& \leqslant \frac{1}{c} \int_{B_R \cap \{u > \max\{w, \theta_\star\}\}} (1+w)\, dx \; - A(R) \\
& \leqslant C \int_{B_R \cap \{u > \theta_\star\}} (R + 1 - |x|)^{-2s}\, dx \; - A(R),
\end{aligned}
$$
(3.34)

for a suitable $C > 0$.

Now, we define

$$
V(R) := \big|\{u > \theta_\star\} \cap B_R\big|.
$$
(3.35)

Hence, using the Coarea formula, we deduce from (3.34) that

$$
\begin{aligned}
& A(R) + \mathscr{K}(u-v; B_R) + \frac{c}{2} \int_{B_R \cap \{w < u \leqslant \theta_\star\}} (1+w)(u-w)\, dx \\
& \leqslant C \int_0^R (R + 1 - t)^{-2s} \left( \int_{\partial B_t} \chi_{\{u > \theta_\star\}}(x)\, d\mathscr{H}^{n-1}(x) \right) dt \\
& = C \int_0^R (R + 1 - t)^{-2s} V'(t)\, dt.
\end{aligned}
$$
(3.36)

3.2. **Completion of the proof of Theorem 1.4.** Now we use the estimates of Theorem 1.6. For convenience, the proof of Theorem 1.6 itself is postponed to Section 4.

Given a measurable set $A \subseteq \mathbb{R}^n$, we define

$$
\ell(A) := \begin{cases} |A|^{(1-2s)/n} & \text{if } s \in (0, 1/2), \\ \log|A| & \text{if } s = 1/2, \\ 1 & \text{if } s \in (1/2, 1). \end{cases}
$$
(3.37)

Given $s \in (1/2, 1)$ and $\alpha \geqslant 0$, we notice that the map

$$(0, +\infty) \ni t \mapsto \alpha t^{1-2s} + t$$

has minimum at $t = \big(\alpha(2s - 1)\big)^{1/(2s)}$ and therefore

(3.38) $$\inf_{t \in (0,+\infty)} \alpha t^{1-2s} + t \geqslant c_1 \alpha^{1/(2s)},$$

for a suitable $c_1 > 0$, as long as $s \in (1/2, 1)$.

Also, the map

$$(0, +\infty) \ni t \mapsto \alpha^{(n-1)/n} \log \frac{\alpha}{t} + t$$

has minimum at $t = \alpha^{(n-1)/n}$, so

(3.39) $$\inf_{t \in (0,+\infty)} \alpha^{(n-1)/n} \log \frac{\alpha}{t} + t \geqslant \frac{\alpha^{(n-1)/n}}{n} \log \alpha.$$

Moreover, if, given $\kappa > 0$, we consider the map

$$[\kappa, +\infty) \ni t \mapsto \Phi(t) := \frac{t^{1/n}}{1 + |\log t|},$$

we have that $\Phi(t) > 0$ for any $t \in [\kappa, +\infty)$ and

$$\lim_{t \to +\infty} \Phi(t) = +\infty,$$

therefore

(3.40) $$i_\kappa := \inf_{t \in [\kappa,+\infty)} \Phi(t) > 0.$$

Now, we claim that, if $A$ and $B$ are disjoint measurable subsets of $\mathbb{R}^n$ and $D := \mathscr{C}(A \cup B)$, with $|A| \geqslant \kappa > 0$, then there exists $c_0 > 0$, possibly depending on $\kappa$ such that

(3.41) $$L(A, D) + |B| \geqslant c_0 |A|^{(n-1)/n} \ell(A).$$

To prove (3.41), we take $c$ as[4] in Theorem 1.6 and we distinguish two cases. First, if $|B| > c|A|$, we use (3.40) to see that

$$L(A, D) + |B| \geqslant |B| > c|A|$$
$$= \begin{cases} c|A|^{2s/n}|A|^{(n-2s)/n} \geqslant c\kappa^{2s/n} |A|^{(n-2s)/n}, \\ \qquad\qquad \text{if } s \in (0, 1/2), \\[2ex] c|A|^{1/n}|A|^{(n-1)/n} \geqslant c\, i_\kappa \, (1 + |\log|A||)|A|^{(n-1)/n}, \\ \qquad\qquad \text{if } s \in [1/2, 1), \end{cases}$$

---

[4]Of course, no confusion should arise with the $c$ in (1.8) which was previously used in (3.31) – in any case, one could just take $c$ to be the smallest of these two constants to make them equal.

which gives (3.41). Therefore, we may suppose that $|B| \leqslant c|A|$ and use (1.10), (3.38) and (3.39) to conclude that

$$\frac{1}{c}\Big(L(A,D) + |B|\Big) \geqslant \begin{cases} |A|^{(n-2s)/n} + |B| & \text{if } s \in (0, 1/2), \\ |A|^{(n-1)/n} \log(|A|/|B|) + |B| & \text{if } s = 1/2, \\ |A|^{2s(n-1)/n}|B|^{1-2s} + |B| & \text{if } s \in (1/2, 1), \end{cases}$$

$$\geqslant \begin{cases} |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ \Big(|A|^{(n-1)/n} \log|A|\Big)/n & \text{if } s = 1/2, \\ c_1\Big(|A|^{2s(n-1)/n}\Big)^{1/(2s)} & \text{if } s \in (1/2, 1), \end{cases}$$

$$\geqslant c_2 |A|^{(n-1)/n} \ell(A),$$

for some $c_2 > 0$, proving (3.41).

Now, we take a free parameter $K > 1$, that will be chosen conveniently large in what follows. The radius $R$ of Theorem 1.4 will be taken larger than $K$. We observe that, by (3.5),

$$(3.42) \qquad w \leqslant -1 + C(K+1)^{-2s} < -1 + \frac{1+\theta_\star}{2} \text{ in } B_{R-K},$$

as long as $K$ is large enough possibly in dependence of $\theta_\star$ which was fixed in (3.31), and so

$$(3.43) \qquad a_R := B_R \cap \left\{ u - w \geqslant \frac{1+\theta_\star}{4} \right\} \supseteq B_{R-K} \cap \{u > \theta_\star\}.$$

By (3.1), (3.31) and (3.43), when $R$ is large

$$|a_R| \geqslant |\{u > \theta_1\} \cap B_{R_o}| \geqslant \mu.$$

As a consequence, we may apply (3.41) with

$$\kappa := \mu,$$
$$A := a_R,$$
$$B := b_R := B_R \cap \left\{ \frac{1+\theta_\star}{8} < u - w < \frac{1+\theta_\star}{4} \right\}$$
$$\text{and} \qquad D := d_R := \mathscr{C}(A \cup B) = \mathscr{C}B_R \cup \left( B_R \cap \left\{ u - w \leqslant \frac{1+\theta_\star}{8} \right\} \right).$$

We obtain that

$$L(a_R, d_R) + |b_R| \geqslant c_3 |a_R|^{(n-1)/n} \ell(a_R)$$

for a suitable $c_3 > 0$, possibly depending on $\mu$.

Accordingly, recalling the notation in (3.35), (3.37) and (3.43),

$$(3.44) \qquad L(a_R, d_R) + |b_R| \geqslant c_3 V(R-K)^{(n-1)/n} \ell_{R-K},$$

where

$$(3.45) \qquad \ell_R := \ell\Big(B_R \cap \{u > \theta_\star\}\Big) = \begin{cases} (V(R))^{(1-2s)/n} & \text{if } s \in (0, 1/2), \\ \log(V(R)) & \text{if } s = 1/2, \\ 1 & \text{if } s \in (1/2, 1). \end{cases}$$

Notice that

$$(3.46) \qquad \text{the map } R \mapsto \ell_R \text{ is nondecreasing.}$$

Now, we recall (3.28) and (3.29) to see that

$$\text{if } y \in d_R \text{ then } (u-v)(y) \leqslant \frac{1+\theta_\star}{8}.$$

Therefore,

$$\text{if } y \in d_R \text{ and } x \in a_R \text{ then}$$
$$|(u-v)(x) - (u-v)(y)| \geqslant (u-v)(x) - (u-v)(y)$$
$$\geqslant \frac{1+\theta_\star}{4} - \frac{1+\theta_\star}{8} = \frac{1+\theta_\star}{8}.$$

Recalling (1.9), this implies that

$$(3.47) \qquad \begin{aligned} & 2\,\mathscr{K}(u-v; B_R) \\ & \geqslant \int_{a_R} \int_{d_R} \frac{|(u-v)(x) - (u-v)(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ & \geqslant \left(\frac{1+\theta_\star}{8}\right)^2 L\,(a_R, d_R). \end{aligned}$$

Furthermore, by (3.42),

$$(3.48) \qquad b_{R-K} = B_{R-K} \cap \left\{\frac{1+\theta_\star}{8} < u - w < \frac{1+\theta_\star}{4}\right\} \subseteq B_{R-K} \cap \{u \leqslant \theta_\star\}.$$

Also, by (3.32),

$$(3.49) \qquad \begin{aligned} A(R) &= c \int_{B_R \cap \{w < u \leqslant \theta_\star\}} (u-w)^2\, dx \\ &\geqslant c \int_{b_R \cap \{u \leqslant \theta_\star\}} (u-w)^2\, dx \\ &\geqslant c_4\, |b_R \cap \{u \leqslant \theta_\star\}|, \end{aligned}$$

for a suitable $c_4 \in (0,1)$.

Now, we observe that, if $t \in [R-K, R]$,

$$(R+1-t)^{-2s} \geqslant (1+K)^{-2s}$$

and therefore, recalling the notation in (3.35),

$$\begin{aligned} V(R) - V(R-K) &= \int_{R-K}^R V'(t)\, dt \\ &\leqslant \frac{1}{c_5} \int_{R-K}^R (R+1-t)^{-2s} V'(t)\, dt \leqslant \frac{1}{c_5} \int_0^R (R+1-t)^{-2s} V'(t)\, dt, \end{aligned}$$

for a suitable $c_5 \in (0,1)$, depending on $K$, that is now fixed once and for all. Therefore, exploiting (3.48),

$$\begin{aligned} |b_R| &\leqslant |b_R \cap \{u \leqslant \theta_\star\}| + |b_R \cap \{u > \theta_\star\}| \\ &= |b_R \cap \{u \leqslant \theta_\star\}| + |(b_R \setminus b_{R-K}) \cap \{u > \theta_\star\}| \\ &\leqslant |b_R \cap \{u \leqslant \theta_\star\}| + |(B_R \setminus B_{R-K}) \cap \{u > \theta_\star\}| \\ &= |b_R \cap \{u \leqslant \theta_\star\}| + V(R) - V(R-K) \\ &\leqslant |b_R \cap \{u \leqslant \theta_\star\}| + \frac{1}{c_5} \int_0^R (R+1-t)^{-2s} V'(t)\, dt. \end{aligned}$$

With this, we can write (3.49) as

$$A(R) + \frac{c}{2} \int_{B_R \cap \{w < u \leqslant \theta_\star\}} (1 + w)(u - w) \, dx \geqslant A(R)$$

$$\geqslant c_4 \, |b_R| - \frac{c_4}{c_5} \int_0^R (R + 1 - t)^{-2s} V'(t) \, dt.$$

The latter estimate, (3.36) and (3.47) give that

$$C \int_0^R (R + 1 - t)^{-2s} V'(t) \, dt$$

$$\geqslant \quad \mathscr{K}(u - v; B_R) + A(R) + \frac{c}{2} \int_{B_R \cap \{w < u \leqslant \theta_\star\}} (1 + w)(u - w) \, dx$$

$$\geqslant \quad c_6 \Big( L(a_R, d_R) + |b_R| \Big) - \frac{c_4}{c_5} \int_0^R (R + 1 - t)^{-2s} V'(t) \, dt,$$

for suitable $C \in (1, +\infty)$ and $c_6 \in (0, 1)$, possibly depending on $\theta_\star$, that was fixed in (3.31).

Therefore, taking the last term on the other side, recalling (3.44) and possibly renaming $C \geqslant 1$, we conclude that[5]

$$(3.50) \qquad C \int_0^R (R + 1 - t)^{-2s} V'(t) \, dt \geqslant \ell_{R-K} V(R - K)^{(n-1)/n}.$$

Now, we notice that, if $\rho \geqslant 1$,

$$(3.51) \quad \int_t^{(3/2)\rho} (R + 1 - t)^{-2s} \, dR \leqslant \begin{cases} (4^{1-2s}/(1-2s))\rho^{1-2s} & \text{if } s \in (0, 1/2), \\ \log((5/2)\rho) & \text{if } s = 1/2, \\ 1/(2s - 1) & \text{if } s \in (1/2, 1). \end{cases}$$

Therefore, we make use of (3.46) and (3.51) in order to integrate (3.50) in $R \in [\rho, (3/2)\rho]$, with $\rho \geqslant 2K$, and we obtain that

$$\rho \, \ell_{\rho-K} \, V(\rho - K)^{(n-1)/n}$$

$$\leqslant C \int_\rho^{(3/2)\rho} \left( \int_0^R (R + 1 - t)^{-2s} V'(t) \, dt \right) dR$$

$$\leqslant C \int_0^{(3/2)\rho} \left( \int_t^{(3/2)\rho} (R + 1 - t)^{-2s} \, dR \right) V'(t) \, dt$$

$$\leqslant \begin{cases} C' \rho^{1-2s} V((3/2)\rho) & \text{if } s \in (0, 1/2), \\ C' (\log \rho) \, V((3/2)\rho) & \text{if } s = 1/2, \\ C' V((3/2)\rho) & \text{if } s \in (1/2, 1). \end{cases}$$

for some $C, C' \geqslant 1$.

That is, for large $r$, recalling (3.45) and possibly renaming $C > 1$,

$$(3.52) \qquad \begin{cases} r^{2s} V(r)^{(n-2s)/n} \leqslant CV(2r) & \text{if } s \in (0, 1/2), \\ r \dfrac{\log V(r)}{\log r} V(r)^{(n-1)/n} \leqslant CV(2r) & \text{if } s = 1/2, \\ r V(r)^{(n-1)/n} \leqslant CV(2r) & \text{if } s \in (1/2, 1). \end{cases}$$

------

[5]The reader may observe in (3.50) the structurally different iteration between the cases $s \in (0, 1/2)$ and $s \in [1/2, 1)$, which is encoded in $\ell_{R-K}$, according to (3.45).

By (3.52) and Lemma 3.2, applied here with $\sigma := 2s$ when $s \in (0, 1/2)$, or $\sigma := 1$ when $s \in [1/2, 1)$, and $\gamma := 2$ (recall also Remark 3.3), we obtain that $V(R) \geqslant c_o R^n$ for large $R$, for a suitable $c_o \in (0, 1)$.

Therefore, recalling also (3.31),

$$
\begin{aligned}
(3.53) \qquad \big|\{u > \theta_2\} \cap B_R\big| &+ \big|\{\theta_\star < u \leqslant \theta_2\} \cap B_R\big| \\
&= \big|\{u > \theta_\star\} \cap B_R\big| = V(R) \geqslant c_o R^n
\end{aligned}
$$

for large $R$. On the other hand, by (1.5),

$$
\begin{aligned}
(3.54) \qquad \frac{\overline{C} R^n}{\log R} &\geqslant \mathscr{E}(u, B_R) \geqslant \int_{\{\theta_\star < u \leqslant \theta_2\} \cap B_R} W(u(x)) \, dx \\
&\geqslant \inf_{r \in [\theta_\star, \theta_2]} W(r) \, \big|\{\theta_\star < u \leqslant \theta_2\} \cap B_R\big|.
\end{aligned}
$$

By (3.53) and (3.54), we obtain that (1.7) holds true, and this completes the proof of Theorem 1.4.

## 4. Proof of Theorem 1.6

Given $\xi \in \mathrm{S}^{n-1}$, we denote by $\pi_\xi$ the hyperplane normal to $\xi$ passing through the origin, namely

$$
\pi_\xi := \big\{x \in \mathbb{R}^n \text{ s.t. } \xi \cdot x = 0\big\}.
$$

Given $\Omega \subseteq \mathbb{R}^n$, we consider the projection of $\Omega$ along $\xi$, i.e.

$$
\Pi_\xi(\Omega) := \big\{p \in \pi_\xi \text{ s.t. there exists } t \in \mathbb{R} \text{ for which } p + t\xi \in \Omega\big\}.
$$

Next result relates the $n$-dimensional measure of $\Omega$ with the largest possible $(n-1)$-dimensional measure of $\Pi_\xi(\Omega)$ (i.e., pictorially, the measure of an object in a room with the measure of its shadows on the walls and on the floor).

**Lemma 4.1.** *Let $\Omega$ be a measurable subset of $\mathbb{R}^n$. Then,*

  (i) $|\Omega|^{n-1} \leqslant |\Pi_{e_1}(\Omega)| \ldots |\Pi_{e_n}(\Omega)|$,
  (ii) *there exists $k \in \{1, \ldots, n\}$ for which*

$$
|\Pi_{e_k}(\Omega)| \geqslant |\Omega|^{(n-1)/n}.
$$

*Proof.* First of all, we use the generalized Hölder inequality (see, e.g., page 623 of [10]) to observe that, if $\psi_o \geqslant 0$ and $\psi_1, \ldots, \psi_{n-1} \in L^1\big(\mathbb{R}, [0, +\infty]\big)$, then

$$
\begin{aligned}
(4.1) \qquad \int_{\mathbb{R}} & \Big(\psi_o \psi_1(t) \ldots \psi_{n-1}(t)\Big)^{1/(n-1)} dt \\
&= \psi_o^{1/(n-1)} \int_{\mathbb{R}} \Big(\psi_1(t) \ldots \psi_{n-1}(t)\Big)^{1/(n-1)} dt \\
&\leqslant \psi_o^{1/(n-1)} \left(\int_{\mathbb{R}} \psi_1(t) \, dt\right)^{1/(n-1)} \ldots \left(\int_{\mathbb{R}} \psi_{n-1}(t) \, dt\right)^{1/(n-1)} \\
&= \left(\psi_o \int_{\mathbb{R}} \psi_1(t) \, dt \ldots \int_{\mathbb{R}} \psi_{n-1}(t) \, dt\right)^{1/(n-1)}.
\end{aligned}
$$

Now, we introduce some notation. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, for any $i \in \{1, \ldots, n\}$ we define

$$
\hat{x}_i := x - x_i e_i = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).
$$

Also, for any $i \leqslant k \in \{1, \ldots, n\}$, we set

$$\hat{x}_{i;k} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k).$$

Notice that $\hat{x}_i \in \mathbb{R}^n$, $\hat{x}_{i;k} \in \mathbb{R}^{k-1}$ and the above notation means that $\hat{x}_{k;k} := (x_1, \ldots, x_{k-1})$. We also stress the fact that

(4.2)                                $\hat{x}_i$ does not depend on $x_i$.

For short, we also set $\chi := \chi_\Omega$ and $\chi_i := \chi_{\Pi_{e_i}(\Omega)}$. Then, we have that

(4.3)                     $$\int_{\mathbb{R}^{n-1}} \chi_i(\hat{x}_i) d\hat{x}_{i;n} = |\Pi_{e_i}(\Omega)|.$$

Now, we observe that

$$\chi(x) \leqslant \chi_1(\hat{x}_1) \ldots \chi_n(\hat{x}_n)$$

and so

$$\chi(x) = \left( \chi(x) \right)^{1/(n-1)} \leqslant \left( \chi_1(\hat{x}_1) \ldots \chi_n(\hat{x}_n) \right)^{1/(n-1)}.$$

Hence, integrating in $dx_1$ and using (4.1) and (4.2),

$$\int_{\mathbb{R}} \chi(x) \, dx_1 \leqslant \left( \chi_1(\hat{x}_1) \int_{\mathbb{R}} \chi_2(\hat{x}_2) \, dx_1 \ldots \int_{\mathbb{R}} \chi_n(\hat{x}_n) \, dx_1 \right)^{1/(n-1)}.$$

So, integrating in $dx_2$ and using (4.1) and (4.2) once more,

$$\int_{\mathbb{R}^2} \chi(x) \, d(x_1, x_2)$$

$$\leqslant \quad \left( \int_{\mathbb{R}} \chi_1(\hat{x}_1) \, dx_2 \int_{\mathbb{R}} \chi_2(\hat{x}_2) dx_1 \right.$$

$$\left. \int_{\mathbb{R}^2} \chi_3(\hat{x}_3) \, d(x_1, x_2) \ldots \int_{\mathbb{R}^2} \chi_n(\hat{x}_n) \, d(x_1, x_2) \right)^{1/(n-1)}$$

$$= \quad \left( \int_{\mathbb{R}} \chi_1(\hat{x}_1) \, d\hat{x}_{1;2} \int_{\mathbb{R}} \chi_2(\hat{x}_2) d\hat{x}_{2;2} \right.$$

$$\left. \int_{\mathbb{R}^2} \chi_3(\hat{x}_3) \, d(x_1, x_2) \ldots \int_{\mathbb{R}^2} \chi_n(\hat{x}_n) \, d(x_1, x_2) \right)^{1/(n-1)},$$

where we denoted by $d(x_1, x_2)$ the volume element in $\mathbb{R}^2$. By iterating this argument, for any $k \leqslant n$, we conclude that

$$\int_{\mathbb{R}^k} \chi(x) \, d(x_1, \ldots, x_k)$$

$$\leqslant \quad \left( \int_{\mathbb{R}^{k-1}} \chi_1(\hat{x}_1) \, d\hat{x}_{1;k} \ldots \int_{\mathbb{R}^{k-1}} \chi_k(\hat{x}_k) d\hat{x}_{k;k} \right.$$

$$\left. \int_{\mathbb{R}^k} \chi_{k+1}(\hat{x}_{k+1}) \, d(x_1, \ldots, x_k) \ldots \int_{\mathbb{R}^k} \chi_n(\hat{x}_n) \, d(x_1, \ldots, x_k) \right)^{1/(n-1)},$$

and, finally,

$$\int_{\mathbb{R}^n} \chi(x) \, dx \quad \leqslant \quad \left( \int_{\mathbb{R}^{n-1}} \chi_1(\hat{x}_1) \, d\hat{x}_{1;n} \ldots \int_{\mathbb{R}^{n-1}} \chi_n(\hat{x}_n) d\hat{x}_{n;n} \right)^{1/(n-1)}.$$

This and (4.3) imply the claim in (i). Then, (ii) easily follows from (i).                    □

As a curiosity, we remark that the estimates in Lemma 4.1 are optimal, as the example of the cube shows, and that they may be seen as suitably refined versions of the classical isoperimetric and isodiametric inequalities.

The main estimate needed for the proof of Theorem 1.6 is the following:

**Lemma 4.2.** *Let $s \in (0,1)$. Let $A$ and $B$ be disjoint measurable subsets of $\mathbb{R}^n$, with $|A| = 1$. Let $D := \mathscr{C}(A \cup B)$. Then, there exists $\delta \in (0, 1/10)$ depending on $n$ and $s$ such that the following holds: if $|B| \leqslant \delta$, then*

$$
L(A, D) \geqslant \begin{cases} \delta & \text{if } s \in (0, 1/2), \\ \delta \log(1/|B|) & \text{if } s = 1/2, \\ \delta\, |B|^{1-2s} & \text{if } s \in (1/2, 1). \end{cases}
$$

*Proof.* The main step of the proof consists in the following estimate: there exists $\tilde{c} \in (0,1)$, suitably small, depending on $n$ and $s$, such that, for any

$$
r \in \left[ \tilde{C}|B|, \, \tilde{c} \right], \tag{4.4}
$$

with $\tilde{C} := 1/\tilde{c}$, we have

$$
\int_A \left[ \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x - y|^{n+2s}} \right] dx \geqslant \tilde{c}\, r^{1-2s}. \tag{4.5}
$$

In order to prove (4.5), we divide $\mathbb{R}^n$ into a collection $K$ of nonoverlapping cubes $Q$ of size $r$. We define

$$
K_B := \left\{ Q \in K \text{ s.t. } \frac{|Q \cap B|}{|Q|} \geqslant \frac{1}{3} \right\},
$$

$$
K_D := \left\{ Q \in K \setminus K_B \text{ s.t. } \frac{|Q \cap D|}{|Q|} \geqslant \frac{1}{3} \right\},
$$

$$
K_A := K \setminus (K_B \cup K_D).
$$

We also set

$$
Q_B := \bigcup_{Q \in K_B} Q, \qquad Q_D := \bigcup_{Q \in K_D} Q, \qquad Q_A := \bigcup_{Q \in K_A} Q.
$$

We observe that

$$
\delta \geqslant |B| \geqslant \sum_{Q \in K_B} |Q \cap B| \geqslant \frac{1}{3} \sum_{Q \in K_B} |Q| = \frac{1}{3}\, |Q_B|. \tag{4.6}
$$

Moreover,

$$
\text{if } Q \in K_A,\ |Q \cap A| = |Q| - |Q \cap B| - |Q \cap D|
$$

$$
\geqslant |Q| - \frac{1}{3}|Q| - \frac{1}{3}|Q| = \frac{1}{3}|Q|. \tag{4.7}
$$

In particular,

$$
|Q_A| = \sum_{Q \in K_A} |Q| \leqslant 3 \sum_{Q \in K_A} |Q \cap A| \leqslant 3|A| = 3. \tag{4.8}
$$

We also point out that if $x \in Q \subset K_D$ and $\tilde{C} > \sqrt{n}$, then $Q \subseteq B_{\tilde{C}r}(x)$ and so

$$
\int_{Q \cap A} \left[ \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx
$$

(4.9)
$$
\geqslant \int_{Q \cap A} \left[ \int_{(Q \cap D) \setminus B_{\tilde{c}r}(x)} \frac{dy}{|x-y|^{n+2s}} \right] dx
$$

$$
\geqslant \int_{Q \cap A} \frac{|(Q \cap D) \setminus B_{\tilde{c}r}(x)|}{(\sqrt{n}r)^{n+2s}} \, dx
$$

$$
\geqslant \frac{|Q \cap A| \left( r^n/3 - \tilde{c}^n \, |B_1| \, r^n \right)}{(\sqrt{n}r)^{n+2s}} \geqslant cr^{-2s}|Q \cap A|,
$$

provided that $\tilde{c}$ is sufficiently small. Now, two cases may occur. Either

$$
(4.10) \qquad\qquad\qquad |Q_D \cap A| \geqslant r
$$

or not. If (4.10) holds, then we exploit (4.9) to obtain

$$
\int_A \left[ \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx
$$

$$
\geqslant \sum_{Q \in K_D} \int_{Q \cap A} \left[ \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx
$$

$$
\geqslant \sum_{Q \in K_D} cr^{-2s} \, |Q \cap A| \geqslant cr^{-2s}|Q_D \cap A| \geqslant cr^{1-2s}.
$$

That is, if (4.10) holds true, then (4.5) is proved, up to renaming the constants.

Therefore, we can focus on the case in which (4.10) does not hold and suppose from now on that

$$
(4.11) \qquad\qquad\qquad |Q_D \cap A| < r.
$$

Hence, recalling (4.4) and (4.6), and the fact that $\delta < 1/10$, we conclude that

(4.12)
$$
|Q_A| \geqslant |Q_A \cap A| = |A| - |Q_B \cap A| - |Q_D \cap A|
$$

$$
\geqslant 1 - |Q_B| - r \geqslant \frac{1}{2}.
$$

From (4.12) and Lemma 4.1(ii), we have that, up to rotation,

$$
(4.13) \qquad\qquad\qquad |\Pi_{e_n}(Q_A)| \geqslant c_0.
$$

Thus, we organize the cubes of $K$ into subfamily of columns in direction $e_n$: more explicitly, the column containing a cube $Q \in K$ is given by the union of all the cubes of the form $Q + j\eta e_n$, for any $j \in \mathbb{Z}$.

We define $C_A$ to be the union of all the columns that have a cube belonging to $K_A$, and $C_B$ to be the union of all the columns in $C_A$ that have at least one cube belonging to $K_B$[6]. We also let $M_A$ and $M_B$ to be the cardinality of the columns belonging to $C_A$ and to $C_B$, respectively. We remark that

$$
(4.14) \qquad\qquad \text{the number of cubes in } K_A \text{ is finite,}
$$

---

[6]Of course, $C_A$ and $C_B$ may well have some common columns in the intersection.

due to (4.8), and so $M_A$ is finite[7].

Notice that $Q_A \subseteq C_A$, therefore, by (4.13),

$$(4.15) \qquad c_0 \leqslant |\Pi_{e_n}(C_A)| = r^{n-1} M_A.$$

On the other hand, if $C_o$ is a column belonging to $C_B$, then it contains one cube $Q^{(1)}$ belonging to $K_B$, and therefore

$$|C_o \cap B| \geqslant |Q^{(1)} \cap B| \geqslant r^n/3.$$

Consequently,

$$|B| \geqslant |C_B \cap B| \geqslant M_B r^n/3.$$

Accordingly, recalling (4.4) and (4.15),

$$(4.16) \qquad M_B \leqslant \frac{3\,|B|}{r^n} \leqslant \frac{3\tilde{c}r}{r^n} \leqslant \frac{c_0}{2r^{n-1}} \leqslant \frac{M_A}{2},$$

if $\tilde{c} \leqslant c_0/6$. As a consequence of this, using (4.15) once more, we conclude that

$$(4.17) \qquad \begin{array}{c} \text{the number of columns in } C_A \setminus C_B \\[4pt] \text{is at least } M_A - M_B \geqslant \dfrac{M_A}{2} \geqslant \dfrac{c_0}{2r^{n-1}}. \end{array}$$

Now, let $C^\star$ be a column in $C_A \setminus C_B$. Then, recalling (4.14), we see that $C^\star$ must contain only a finite number of cubes belonging to $K_A$, so we may define the cube $Q_o^\star$ as the cube of $C^\star$ belonging to $C_A$ with the highest possible $e_n$-coordinate.

We consider the cube $Q_1^\star := Q_o^\star + 2r e_n$. By construction $Q_1^\star \in K_D$. Notice that if $x \in Q_o^\star$ and $y \in Q_1^\star$ then $|x - y| \geqslant r$, and $|x - y| \leqslant (2 + \sqrt{n})r \leqslant \tilde{C}r$, provided that $\tilde{C}$ is sufficiently large. Therefore, if $x \in Q_o^\star$, then $Q_1^\star \subseteq B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x)$, and

$$\int_{A \cap Q_o^\star} \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dx\,dy}{|x-y|^{n+2s}}$$
$$\geqslant \int_{A \cap Q_o^\star} \int_{D \cap Q_1^\star} \frac{dx\,dy}{|x-y|^{n+2s}}$$
$$\geqslant cr^{-(n+2s)} |A \cap Q_o^\star| |D \cap Q_1^\star|$$
$$\geqslant cr^{n-2s},$$

for a suitable $c \in (0,1)$ (independent of $\tilde{c}$ and $\tilde{C}$, and possibly different line after line). As a consequence,

$$\int_{C^\star \cap A} \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dx\,dy}{|x-y|^{n+2s}} \geqslant cr^{n-2s}.$$

Since this is valid for any column $C^\star$ in $C_A \setminus C_B$, in the light of (4.17) we obtain that

$$\int_A \int_{D \cap (B_{\tilde{C}r}(x) \setminus B_{\tilde{c}r}(x))} \frac{dx\,dy}{|x-y|^{n+2s}} \geqslant cr^{1-n} r^{n-2s} = cr^{1-2s}.$$

This completes the proof of (4.5), by choosing $\tilde{c} \in (0,1)$ small enough.

Since $\tilde{c}$ is now fixed once and for all, we can suppose that $\delta$ in the statement of Lemma 4.2 is smaller than $\tilde{c}^3$, hence $\tilde{c}^3 \geqslant |B|$. So, let $k_0$ be the largest integer so that $\tilde{c}^{2k_0+1} \geqslant |B|$. From (4.4) and (4.5), we deduce that, for any $1 \leqslant k \leqslant k_0$

$$\tilde{C}|B| \leqslant \tilde{r}_k := \tilde{c}^{2k} \leqslant \tilde{c},$$

---

[7]Similarly, exploiting (4.6), one can see that $M_B$ is finite – however, a more precise estimate on $M_B$ will be given in the forthcoming (4.16).

we have that

$$\int_A \left[ \int_{D \cap (B_{\tilde{r}_k / \tilde{c}}(x) \setminus B_{\tilde{c}r_k}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx \geqslant \tilde{c}^{1+2k(1-2s)}.$$

Consequently,

$$\int_A \left[ \int_D \frac{dy}{|x-y|^{n+2s}} \right] dx$$

$$\geqslant \sum_{k=1}^{k_0} \int_A \left[ \int_{D \cap (B_{\tilde{r}_k / \tilde{c}}(x) \setminus B_{\tilde{c}r_k}(x))} \frac{dy}{|x-y|^{n+2s}} \right] dx$$

$$\geqslant \sum_{k=1}^{k_0} \tilde{c}^{1+2k(1-2s)}$$

$$\geqslant \begin{cases} \tilde{c}^{2(1-2s)+1} & \text{if } s \in (0, \, 1/2), \\ k_0 \tilde{c} & \text{if } s = 1/2, \\ \tilde{c}^{2k_0(1-2s)+1} & \text{if } s \in (1/2, \, 1), \end{cases}$$

This implies the desired result, by taking $\delta$ appropriately small with respect to $\tilde{c}$. $\quad\square$

We now complete the proof of Theorem 1.6 by the following argument. If $|B| > c|A|$, then $|A \cup B| \leqslant |A| + |B| < (1 + c^{-1})|B|$. Hence, we make use of the Sobolev-type inequality in (A.1) and we possibly rename $c$, to conclude that

$$(4.18) \qquad \begin{aligned} L(A, D) &= \int_A \int_{\mathscr{C}(A \cup B)} \frac{dx\, dy}{|x-y|^{n+2s}} \\ &\geqslant c\, |A| |A \cup B|^{-2s/n} \geqslant c\, |A| |B|^{-2s/n} \end{aligned}$$

which proves (1.11).

If, on the other hand, if $|B| \leqslant c|A| \neq 0$, we define

$$\tilde{A} := \frac{1}{|A|^{1/n}} A, \ \ \tilde{B} := \frac{1}{|A|^{1/n}} B \text{ and } \ \tilde{D} := \frac{1}{|A|^{1/n}} D = \mathscr{C}(\tilde{A} \cup \tilde{B}).$$

We observe that $|\tilde{A}| = 1$ and $|\tilde{B}| = |B|/|A| \leqslant c$, so we can apply Lemma 4.2 and obtain that

$$\begin{aligned} L(A, D) &= |A|^{(n-2s)/n} L(\tilde{A}, \tilde{D}) \\ &\geqslant \begin{cases} c\, |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ c\, |A|^{(n-1)/n} \log(1/|\tilde{B}|) & \text{if } s = 1/2, \\ c\, |A|^{(n-2s)/n} |\tilde{B}|^{1-2s} & \text{if } s \in (1/2, 1). \end{cases} \\ &= \begin{cases} c\, |A|^{(n-2s)/n} & \text{if } s \in (0, 1/2), \\ c\, |A|^{(n-1)/n} \log(|A|/|B|) & \text{if } s = 1/2, \\ c\, |A|^{(n-2s)/n} (|B|/|A|)^{1-2s} & \text{if } s \in (1/2, 1). \end{cases} \end{aligned}$$

This proves (1.11) and completes the proof of Theorem 1.6.

At the end of this section we prove another localized version of (1.11) in Theorem 1.6 which could be useful in other situations, such as in [21].

**Proposition 4.3.** *Let* $s \in [1/2, 1)$. *Let* $A$, $D$ *be disjoint subsets of a cube* $Q \subset \mathbb{R}^n$ *with*

$$\min\{|A|, |D|\} \geqslant \sigma|Q|,$$

*for some $\sigma > 0$. Let $B = Q \setminus (A \cup D)$. Then,*

$$L(A, D) \geqslant \begin{cases} \delta|Q|^{\frac{n-1}{n}} \log(|Q|/|B|) & \text{if } s = 1/2, \\ \\ \delta|Q|^{\frac{n-2s}{n}} (|Q|/|B|)^{2s-1} & \text{if } s \in (1/2, 1). \end{cases}$$

*for some $\delta > 0$ depending on $\sigma$, $n$ and $s$.*

*Proof.* By rescaling, we can assume that $Q$ is the unit cube.

The proof follows the lines of Lemma 4.2: we show that for each

$$r \in [\tilde{C}|B|, \tilde{c}],$$

we satisfy (4.5) with $\tilde{C} := 1/\tilde{c}$, for some small $\tilde{c}$ depending on $n$, $s$ and $\sigma$.

We divide the unit cube into cubes of size $r$ that we partition into the three sets $K_A$, $K_B$ and $K_D$. The difference now is that $D$ is defined only on the unit cube and therefore the existence of $Q_1^* \in K_D$ at the end of the proof of Lemma 4.2 requires a more careful argument.

As before, we only need to deal with the case (see (4.12))

$$|Q_A| \geqslant \frac{\sigma}{2}, \quad |Q_D| \geqslant \frac{\sigma}{2}.$$

Denote

$$\alpha := |Q_A|, \quad \frac{\sigma}{2} \leqslant \alpha \leqslant 1 - \frac{\sigma}{2}.$$

From Lemma 4.1(ii), we have that, up to rotation,

$$(4.19) \qquad |\Pi_{e_n}(Q_A)| \geqslant \alpha^{\frac{n-1}{n}}.$$

We define $C_A$ to be the union of all the columns that have a cube belonging to $K_A$, $C_B$ to be the columns in $C_A$ that have at least one cube belonging to $K_B$ and $C_D$ to be the columns in $C_A \setminus C_B$ that have at least one cube belonging to $K_D$. We also let $M_A$ $M_B$, and $M_D$ to be the cardinality of the columns belonging to $C_A$, $C_B$ and to $C_D$, respectively.

By (4.19),

$$\alpha^{\frac{n-1}{n}} \leqslant |\Pi_{e_n}(C_A)| = r^{n-1}M_A.$$

and we also have (see (4.16))

$$M_B \leqslant 3\tilde{c}r^{1-n}.$$

On the other hand, the cardinality $m_a$ of the columns in $C_A \setminus (C_B \cup C_D)$ (that contain only cubes from $K_A$) satisfies

$$m_a r^{n-1} \leqslant |Q_A| = \alpha.$$

Thus, if $\tilde{c}$ is sufficiently small,

$$M_D = M_A - m_a - M_B \geqslant r^{1-n}(\alpha^{\frac{n-1}{n}} - \alpha - \tilde{c}/3) \geqslant cr^{1-n}.$$

Let $C^*$ be a column belonging to $C_D$. Then

$$C^* \subset K_A \cup K_D, \quad C^* \cap K_A \neq \varnothing, \quad C^* \cap K_D \neq \varnothing,$$

and we can easily conclude that on this column there must exist two cubes $Q_0^* \subset K_A$, $Q_1^* \subset K_D$ at distance either $2r$ or $3r$ from each other. Then

$$L(A \cap Q_0^\star, D \cap Q_1^\star) \geqslant cr^{n-2s},$$

and the proof continues as before. $\qquad \square$

## APPENDIX A. A SOBOLEV-TYPE INEQUALITY FOR SETS

For completeness, we give here an elementary proof of the Sobolev-type inequality exploited in our paper, see (4.18). For related and more general results see [4, 5] and references therein.

**Lemma A.1.** *Fix $x \in \mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a measurable set with finite measure. Then,*

$$\int_{\mathscr{C}E} \frac{dy}{|x-y|^{n+2s}} \geqslant c(n,s) \, |E|^{-2s/n},$$

*for a suitable constant $c(n,s) > 0$.*

*Proof.* Let

$$\rho := \left( \frac{|E|}{\omega_n} \right)^{1/n}.$$

Then,

$$|\mathscr{C}E \cap B_\rho(x)| = |B_\rho(x)| - |E \cap B_\rho(x)|$$
$$= |E| - |E \cap B_\rho(x)| = |E \cap \mathscr{C}B_\rho(x)|.$$

Therefore,

$$\int_{\mathscr{C}E} \frac{dy}{|x-y|^{n+2s}} = \int_{\mathscr{C}E \cap B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} + \int_{\mathscr{C}E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}$$
$$\geqslant \int_{\mathscr{C}E \cap B_\rho(x)} \frac{dy}{\rho^{n+2s}} + \int_{\mathscr{C}E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}$$
$$= \frac{|\mathscr{C}E \cap B_\rho(x)|}{\rho^{n+2s}} + \int_{\mathscr{C}E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}$$
$$= \frac{|E \cap \mathscr{C}B_\rho(x)|}{\rho^{n+2s}} + \int_{\mathscr{C}E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}$$
$$\geqslant \int_{E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} + \int_{\mathscr{C}E \cap \mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}$$
$$= \int_{\mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}.$$

By using polar coordinate centered at $x$, the desired result easily follows.  $\square$

By integrating in $x$ the estimate of Lemma A.1, we obtain:

**Corollary A.2.** *Let $E$, $F \subset \mathbb{R}^n$ be measurable sets with finite measure. Then,*

$$(A.1) \qquad \int_F \int_{\mathscr{C}E} \frac{dx \, dy}{|x-y|^{n+2s}} \geqslant c(n,s) \, |F| \, |E|^{-2s/n},$$

*for a suitable constant $c(n,s) > 0$.*

*In particular,*

$$(A.2) \qquad \int_E \int_{\mathscr{C}E} \frac{dx \, dy}{|x-y|^{n+2s}} \geqslant c(n,s) \, |E|^{(n-2s)/n}$$

*for any measurable set $E$ with finite measure.*

## References

[1] G. Alberti, G. Bouchitté and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal. 144 (1998), no. 1, 1–46.

[2] L. Ambrosio, G. de Philippis, L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Preprint, 2010.

[3] G. Bouchitté, *Singular perturbations of variational problems arising from a two-phase transition model*, Appl. Math. Optim. 21 (1990), no. 3, 289–314.

[4] J. Bourgain, H. Brezis and P. Mironescu, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, Dedicated to the memory of Thomas H. Wolff, J. Anal. Math., 87 (2002), 77–101.

[5] H. Brezis and P. Mironescu, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. 1 (2001), no. 4, 387–404.

[6] X. Cabré and J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Commun. Pure Appl. Math. 58 (2005), no. 12, 1678–1732.

[7] L. A. Caffarelli and A Córdoba, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. 48 (1995), no. 1, 1–12.

[8] L. A. Caffarelli, J.-M. Roquejoffre and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63 (2010), no. 9, 1111–1144.

[9] L. A. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, To appear in Calc. Var. Partial Differential Equations.

[10] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp.

[11] A. Farina and E. Valdinoci, *Geometry of quasiminimal phase transitions*, Calc. Var. Partial Differential Equations 33 (2008), no. 1, 1–35.

[12] E. Gagliardo, *Un criterio di compattezza rispetto alla convergenza in media*, Ricerche Mat. 6 (1957), 34–48.

[13] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova 27 (1957), 284–305.

[14] A. Garroni and G. Palatucci, *A singular perturbation result with a fractional norm*, Variational problems in material science, Progress in Nonlinear Differential Equations and Their Applications 68 (2006), Birkhäuser, Basel, 111–126.

[15] M. d. M. González, *Gamma convergence of an energy functional related to the fractional Laplacian*, Calc. Var. Partial Differential Equations 36 (2009), no. 2, 173–210.

[16] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. 98 (1987), no. 2, 123–142.

[17] L. Modica and S. Mortola, *Un esempio di $\Gamma^{-}$-convergenza*, Boll. Un. Mat. Ital. B (5) 14 (1977), no. 1, 285–299.

[18] N. C. Owen and P. Sternberg, *Nonconvex variational problems with anisotropic perturbations*, Nonlinear Anal. 16 (1991), no. 7-8, 705–719.

[19] G. Palatucci, O. Savin and E. Valdinoci, *Local and global minimizers for a variational energy involving a fractional norm*, Preprint, 2010.

[20] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math. (2) 169 (2009), no. 1, 41–78.

[21] O. Savin and E. Valdinoci, *Γ-convergence for nonlocal phase transitions*, Preprint, 2010.

[22] O. Savin and E. Valdinoci, *Density estimates for a nonlocal variational model via the Sobolev inequality*, Forthcoming.

[23] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. 256 (2009), no. 6, 1842–1864.

OVIDIU SAVIN

Mathematics Department, Columbia University,

2990 Broadway, New York , NY 10027, USA.

Email: `savin@math.columbia.edu`

Enrico Valdinoci
Dipartimento di Matematica, Università di Roma Tor Vergata,
Via della Ricerca Scientifica 1, 00133 Roma, Italy.
Email: `enrico@mat.uniroma3.it`