LOCAL AND GLOBAL MINIMIZERS FOR A VARIATIONAL ENERGY INVOLVING A FRACTIONAL NORM

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Abstract. We study existence, unicity and other properties of the minimizers of the energy functional

$$\|u\|_{H^s(\Omega)}^2 + \int_\Omega W(u)\,dx,$$

where $\|u\|_{H^s(\Omega)}$ denotes the total contribution from Ω in the H^s norm of u and W is a double-well potential. We also deal with the solutions of the related fractional elliptic Allen-Cahn equation on the entire space $\mathbb{R}^n$.

The results collected here will also be useful for forthcoming papers, where the second and the third author will study the Γ-convergence and the density estimates for level sets of minimizers.

Contents

2010 *Mathematics Subject Classification.* Primary 82B26, 49J45; Secondary 49Q20.

Key words and phrases. Phase transitions, nonlocal energy, Gagliardo norm, fractional Laplacian.

The first author has been supported by Istituto Nazionale di Alta Matematica "F. Severi" (Indam). The second and the third authors have been partially supported by National Science Foundation (NSF) and FIRB "Project Analysis and Beyond", respectively.

1. Introduction

In this paper we study existence, unicity, some qualitative properties and related issues for the minimizers of a nonlocal energy functional involving a Gagliardo-type norm.

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain and denote by $\mathcal{C}\Omega$ its complement. We deal with the functional $\mathcal{F}$ defined by

$$\mathcal{F}(u,\Omega) = \mathcal{K}(u,\Omega) + \int_\Omega W(u)\,dx, \tag{1.1}$$

where $\mathcal{K}(u,\Omega)$ is given by

$$\mathcal{K}(u,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \tag{1.2}$$

with $s \in (0,1)$, and the function $W \in C^1(\mathbb{R})$ is a double-well potential with wells at $+1$ and -1; i.e., W is a non-negative function vanishing only at $\{-1,+1\}$.

The functional in (1.1) is a non-scaled Allen-Cahn-Ginzburg-Landau-type energy with its kinetic term $\mathcal{K}$ given by some nonlocal fractional integrals, in place of the classical Dirichlet integral. The energy $\mathcal{K}(u,\Omega)$ of a function u, with prescribed boundary data outside Ω, can be view as the contribution in Ω of the H^s norm of u

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

Nonlocal models involving the H^s norm are quite important in physics, since they naturally arise from many problems that exhibit long range interactions among particles.

In the specific case in (1.1) with the potential W given by a double-well function, an adequate scaling of the kinetic term $\mathcal{K}$ brings to the energy for a liquid-liquid two-phase transition model. A Γ-convergence theory for such energy has been recently developed by two of the authors in [15]. They show that suitably scalings of the functional $\mathcal{F}$ Γ-converge to the standard minimal surface functional when $s \in [1/2,1)$ and to the nonlocal one when $s \in (0,1/2)$. As in the classical case with the singular perturbation given by the Dirichlet energy, the functional in (1.1) is strictly related to the elliptic Allen-Cahn equation.

The nonlocal analogue of the Allen-Cahn equation is given by the following Euler-Lagrange equation for the energy $\mathcal{F}(u) := \mathcal{F}(u,\mathbb{R}^n)$

$$(-\Delta)^s u(x) + W'(u(x)) = 0 \quad \text{for any } x \in \mathbb{R}^n, \tag{1.3}$$

As usual, for any $s \in (0,1)$, $(-\Delta)^s$ denotes the s-power of the Laplacian operator and, omitting a multiplicative constant $c = c(n,s)$, we have

$$(-\Delta)^s u(x) \;=\; P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \;=\; \lim_{\varepsilon\to 0}\int_{\mathcal{C}B_\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

Here "*P.V.*" is a commonly used abbreviation for "in the principal value sense".

In the same spirit of a celebrate De Giorgi conjecture about the level sets of the solutions of the elliptic analogue of (1.3), it seems natural to study the solutions u of (1.3) that satisfy the following two conditions:

$$\partial_{x_n} u(x) > 0 \quad \text{for any } x \in \mathbb{R}^n \tag{1.4}$$

and, possibly,

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \quad \text{for any } x' \in \mathbb{R}^{n-1}. \tag{1.5}$$

We refer to [7, 18, 19, 6] for several results in this direction. Here, by means of a technical variation of the classical sliding method, we can prove that the solutions of the fractional elliptic Allen-Cahn equation (1.3) satisfy a minimizing property for the functional $\mathcal{F}$ defined in (1.1). Precisely, we will show that a solution u of (1.3) satisfying conditions (1.4)-(1.5) is such that, for any $r > 0$,

$$\mathcal{F}(u, B_r) \le \mathcal{F}(u+\phi, B_r) \tag{1.6}$$

for any measurable function ϕ supported in B_r (see Proposition 3.2).

In the case of Ω being an open one-dimensional set, we will carefully characterize such class of minimizers. For any $s \in (0,1)$, we will prove that the 1-D minimizers, with respect to the definition in (1.6), are monotone increasing and unique up to translations. Moreover, by further regularity assumptions on the potential W, we have that the 1-D minimizers satisfy certain regularity properties and then we will analyze their asymptotic behavior and the ones of their derivative (see Theorem 4.1).

As a further matter, we will be able to extend the 1-D results to construct a minimizer in higher dimension u^* and we will estimate the energy (1.1) of u^* on the ball B_R, proving that, as R gets larger and larger, the contribution in $\mathcal{K}(u^*, B_R)$ from $\mathcal{C}B_R$ becomes negligible if $s \ge 1/2$, however when $s < 1/2$ this does not happen (see Theorem 4.3).

Finally, it is worthing notice that, in order to prove all the above cited results, we need to perform careful computations on the strongly nonlocal form of the functional $\mathcal{F}$. Hence, it was important for us to understand some modifications of classical techniques to deal with the fractional energy term, in particular to manage the contributions coming from far. Therefore, we collect some general and independent results involving the Gagliardo-type norm in (1.2), to be applied here and in [14, 15], like compactness results, constructions of barriers and various estimates; as well as regularity properties for the solutions of equation (1.3) (see Section 2 below).

2. Preliminary results

In this section we state and prove some general results involving the Gagliardo norm $\|\cdot\|_{H^s}$. Here and in the sequel, we will assume that the fractional exponent s is a real number belonging to $(0,1)$.

2.1. A compactness remark.

We start by giving full details of a compactness result of classical flavor.

Lemma 2.1. *Let $n \geq 1$, Ω be a bounded open subset of $\mathbb{R}^n$ and $\mathcal{T}$ be a bounded subset of $L^2(\Omega)$. Suppose that*

$$\sup_{f\in\mathcal{T}} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \,<\, +\infty.$$

Then $\mathcal{T}$ is precompact in $L^2(\Omega)$.

Proof. The proof is a modification of the one of the classical Riesz-Frechet-Kolmogorov Theorem. We show that $\mathcal{T}$ is totally bounded in $L^2(\Omega)$, i.e., for any $\varepsilon \in (0,1)$ there exist $\beta_1,\dots,\beta_M \in L^2(\Omega)$ such that for any $f\in\mathcal{T}$ there exists $j\in\{1,\dots,M\}$ such that

$$\|f-\beta_j\|_{L^2(\Omega)} \leq \varepsilon. \tag{2.1}$$

For this, with a slight abuse of notation, any function $f\in\mathcal{T}$ will be implicitly assumed to be defined in the whole of $\mathbb{R}^n$, with $f := 0$ in $\mathcal{C}\Omega$.

We let

$$C := 1 + \sup_{f\in\mathcal{T}} \|f\|_{L^2(\Omega)} + \sup_{f\in\mathcal{T}} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

$$\rho = \rho_\varepsilon := \left(\frac{\varepsilon}{4\sqrt{C}\,n^{(n/2)+1}}\right)^{1/s} \quad \text{and} \quad \eta = \eta_\varepsilon := \frac{\varepsilon\,\rho^{n/2}}{2},$$

and we take a collection of nonoverlapping cubes $Q_1,\dots,Q_N$ of side ρ such that

$$\Omega \subseteq \bigcup_{j=1}^N Q_j.$$

For any $x\in\Omega$ we define

(2.2) $j(x)$ as the unique integer in $\{1,\dots,N\}$ for which $x\in Q_{j(x)}$.

Also, for any $f \in \mathcal{T}$, let

$$P(f)(x) := \frac{1}{|Q_{j(x)}|} \int_{Q_{j(x)}} f(y)\, dy.$$

Notice that

$$P(f+g) = P(f) + P(g) \text{ for any } f, g \in \mathcal{T}$$

and that $P(f)$ is constant, say equal to $q_j(f)$, in any Q_j, for $j \in \{1, \ldots, N\}$. Therefore, we can define

$$R(f) := \rho^{n/2}\big(q_1(f), \ldots, q_N(f)\big) \in \mathbb{R}^N.$$

We observe that $R(f+g) = R(f) + R(g)$. Moreover,

$$\begin{aligned} \|P(f)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{N} \int_{Q_j \cap \Omega} |P(f)|^2\, dx \\ &\leq \rho^n \sum_{j=1}^{N} |q_j(f)|^2 = |R(f)|^2 \leq \frac{|R(f)|^2}{\rho^n}. \end{aligned} \tag{2.3}$$

and, by Hölder inequality,

$$\begin{aligned} |R(f)|^2 &= \sum_{j=1}^{N} \rho^n |q_j(f)|^2 = \frac{1}{\rho^n} \sum_{j=1}^{N} \left| \int_{Q_j \cap \Omega} f(y)\, dy \right|^2 \\ &\leq \sum_{j=1}^{N} \int_{Q_j \cap \Omega} |f(y)|^2\, dy = \int_{\Omega} |f(y)|^2 = \|f\|^2_{L^2(\Omega)}. \end{aligned}$$

In particular,

$$\sup_{f \in \mathcal{T}} |R(f)|^2 \leq C,$$

that is, the set $R(\mathcal{T})$ is bounded in $\mathbb{R}^N$ and so, since it is finite dimensional, it is totally bounded. Therefore, there exist $b_1, \ldots, b_M \in \mathbb{R}^N$ such that

$$R(\mathcal{T}) \subseteq \bigcup_{i=1}^{M} B_\eta(b_i). \tag{2.4}$$

For any $i \in \{1, \ldots, M\}$, we write the coordinates of b_i as $b_i = (b_{i,1}, \ldots, b_{i,N}) \in \mathbb{R}^N$. For any $x \in \Omega$, we set

$$\beta_i(x) := \rho^{-n/2}\, b_{i,j(x)},$$

where $j(x)$ is as in (2.2).

Notice that β_i is constant on Q_j, i.e. if $x \in Q_j$ then

$$P(\beta_i)(x) = \rho^{-\frac{n}{2}} b_{i,j} = \beta_i(x) \tag{2.5}$$

and so $q_j(\beta_i) = \rho^{-\frac{n}{2}} b_{i,j}$; thus $R(\beta_i) = b_i$.

Furthermore, for any $f \in \mathcal{T}$, by Hölder inequality,

$$\begin{aligned}
\|f - P(f)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{N} \int_{Q_j\cap\Omega} |f(x) - P(f((x)|^2 \, dx \\
&= \sum_{j=1}^{N} \int_{Q_j\cap\Omega} \left| f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y)\,dy \right|^2 dx \\
&= \sum_{j=1}^{N} \int_{Q_j\cap\Omega} \frac{1}{|Q_j|^2} \left| \int_{Q_j} f(x) - f(y)\,dy \right|^2 dx \\
&\le \frac{1}{\rho^n} \sum_{j=1}^{N} \int_{Q_j\cap\Omega} \left[\int_{Q_j} \left| f(x) - f(y) \right|^2 dy \right] dx \\
&\le n^{(n/2)+1} \rho^{2s} \sum_{j=1}^{N} \int_{Q_j\cap\Omega} \left[\int_{Q_j} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\,dy \right] dx \\
&\le n^{(n/2)+1} \rho^{2s} \sum_{j=1}^{N} \int_{Q_j\cap\Omega} \left[\int_{\Omega} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\,dy \right] dx \\
&= n^{(n/2)+1} \rho^{2s} \int_{\Omega} \left[\int_{\Omega} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\,dy \right] dx \\
&\le C\, n^{(n/2)+1}\, \rho^{2s} = \frac{\varepsilon^2}{16}.
\end{aligned}$$

Consequently, for any $j \in \{1, ..., M\}$, recalling (2.3),

$$\begin{aligned}
\|f - \beta_j\|_{L^2(\Omega)} &\le \|f - P(f)\|_{L^2(\Omega)} + \|P(\beta_j) - \beta_j\|_{L^2(\Omega)} + \|P(f - \beta_j)\|_{L^2(\Omega)} \\
&\le \frac{\varepsilon}{2} + \frac{|R(f) - R(\beta_j)|}{\rho^{n/2}}. \qquad (2.6)
\end{aligned}$$

Now, given any $f \in \mathcal{T}$, we recall (2.4) and we take $j \in \{1, \dots, M\}$ such that $R(f) \in B_\eta(b_j)$. Then, (2.5) and (2.6) give that

$$\|f - \beta_j\|_{L^2(\Omega)} \le \frac{\varepsilon}{2} + \frac{|R(f) - b_j|}{\rho^{n/2}} \le \frac{\varepsilon}{2} + \frac{\eta}{\rho^{n/2}} = \varepsilon.$$

This proves (2.1), as desired. □

2.2. **Toolbox.** In this section, we collect some useful, general estimates to be applied here in the sequel as well as in [14] and [15].

Lemma 2.2 deals with the kernels of the Gagliardo norm in the case of n-dimensional balls B_R. We provide a lower bound estimate, with respect to the radius R of the contribution coming from far of the energy.

Lemma 2.3 and Lemma 2.4 estimate the fractional derivative of bounded functions on the whole space $\mathbb{R}^n$. We also provide some estimates of the energy with respect to the L^∞-norm of the functions and their derivatives. The case of radial symmetric functions is analyzed in Lemma 2.5.

In Lemma 2.7 we construct an useful barrier that will give us an estimate for certain subsolutions of the equation related to the eigenfunctions of the fractional operator $(-\Delta)^s$.

Lemma 2.2. *Let $n \geq 1$ and $R \geq 1$. Then,*

$$\text{if } s \in (0,\,1/2), \qquad \int_{B_R}\int_{B_{2R}\setminus B_R} \frac{dx\,dy}{|x-y|^{n+2s}} \leq \frac{3\,\omega_{n-1}^2\,R^{n-2s}}{2s\,(1-2s)}. \tag{2.7}$$

$$\text{If } s = 1/2, \qquad \int_{B_R}\int_{\mathcal{C}B_{R+1}} \frac{dx\,dy}{|x-y|^{n+2s}} \leq \omega_{n-1}^2\,R^{n-1}\,\left(2^n + \log(3R)\right). \tag{2.8}$$

$$\text{If } s \in (1/2,\,1), \qquad \int_{B_R}\int_{\mathcal{C}B_{R+1}} \frac{dx\,dy}{|x-y|^{n+2s}} \leq \frac{\omega_{n-1}^2\,R^{n-1}}{2s-1}. \tag{2.9}$$

Proof. For any fixed $y \in \mathbb{R}^n$,

$$\begin{aligned} 2s\int_{B_1} \frac{dx}{|x-y|^{n+2s}} &= -\int_{B_1} \operatorname{div}\left(\frac{x-y}{|x-y|^{n+2s}}\right)dx \\ &= -\int_{\partial B_1} \frac{x-y}{|x-y|^{n+2s}}\cdot x\,d\mathcal{H}^{n-1}(x) \\ &\leq \int_{\partial B_1} |x-y|^{1-n-2s}\,d\mathcal{H}^{n-1}(x). \end{aligned} \tag{2.10}$$

Accordingly, if $s \in (0,\,1/2)$,

$$\begin{aligned} 2s\int_{B_1}\int_{B_2\setminus B_1} \frac{dx\,dy}{|x-y|^{n+2s}} &\leq \int_{\partial B_1}\left[\int_{B_2\setminus B_1} |x-y|^{1-n-2s}\,dy\right]d\mathcal{H}^{n-1}(x) \\ &\leq \int_{\partial B_1}\left[\int_{B_3} |\zeta|^{1-n-2s}\,d\zeta\right]d\mathcal{H}^{n-1}(x) = \frac{3^{1-2s}\,\omega_{n-1}^2}{1-2s}, \end{aligned}$$

which is finite by our assumption on s, and so, by changing variable $\tilde{x} := x/R$ and $\tilde{y} := y/R$,

$$\begin{aligned} 2s\int_{B_R}\int_{B_{2R}\setminus B_R} \frac{dx\,dy}{|x-y|^{n+2s}} &= R^{n-2s}\int_{B_1}\int_{B_2\setminus B_1} \frac{d\tilde{x}\,d\tilde{y}}{|\tilde{x}-\tilde{y}|^{n+2s}} \\ &\leq \frac{3^{1-2s}\,\omega_{n-1}^2\,R^{n-2s}}{1-2s}, \end{aligned}$$

proving (2.7).

On the other hand, if $s \in (1/2, 1)$, we set $\varepsilon := 1/R$, and we use (2.10) to conclude that

$$\begin{aligned}\int_{B_1}\int_{\mathcal{C}B_{1+\varepsilon}} \frac{dx\,dy}{|x-y|^{n+2s}} &\le \int_{\partial B_1}\left[\int_{\mathcal{C}B_{1+\varepsilon}} |x-y|^{1-n-2s}\,dy\right] d\mathcal{H}^{n-1}(x)\\ &\le \int_{\partial B_1}\left[\int_{\mathcal{C}B_{\varepsilon}} |\zeta|^{1-n-2s}\,d\zeta\right] d\mathcal{H}^{n-1}(x) \le \frac{\omega_{n-1}^2\varepsilon^{1-2s}}{2s-1},\end{aligned}$$

hence (2.9) follows from scaling.

Finally, when $s = 1/2$, we use (2.10) in the following way:

$$\begin{aligned}\int_{B_1}\int_{\mathcal{C}B_{1+\varepsilon}} \frac{dx\,dy}{|x-y|^{n+2s}} &\le \int_{B_1}\int_{B_2\setminus B_{1+\varepsilon}} \frac{dx\,dy}{|x-y|^{n+1}} + \int_{B_1}\int_{\mathcal{C}B_2} \frac{dx\,dy}{(|y|/2)^{n+1}}\\ &\le \int_{\partial B_1}\left[\int_{B_2\setminus B_{1+\varepsilon}} |x-y|^{-n}\,dy\right] d\mathcal{H}^{n-1}(x) + 2^n\omega_{n-1}^2\\ &\le \int_{\partial B_1}\left[\int_{B_3\setminus B_{\varepsilon}} |\zeta|^{-n}\,d\zeta\right] d\mathcal{H}^{n-1}(x) + 2^n\omega_{n-1}^2\\ &= \omega_{n-1}^2\left(2^n + \log\frac{3}{\varepsilon}\right),\end{aligned}$$

hence (2.8) follows again from scaling. □

Similarly, one can estimate the kernel interaction of smooth functions as follows.

Lemma 2.3. *Let $n \ge 1$ and $x \in \mathbb{R}^n$, $\rho > 0$ and $\psi \in L^\infty(\mathbb{R}^n) \cap W^{1,\infty}(B_\rho(x))$. Then,*

(2.11)
$$\int_{\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy \le \frac{4\,\omega_{n-1}}{(1-s)\,s}\left[\|\nabla\psi\|_{L^\infty(B_\rho(x))}\rho^{2(1-s)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s}\right].$$

Proof. We bound the left hand side of (2.11) by

$$\begin{aligned}\int_{B_\rho(x)} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy+&\int_{\mathcal{C}B_\rho(x)} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dy\\
&\le \int_{B_\rho(x)} \frac{\|\nabla\psi\|_{L^\infty(B_\rho(x))}}{|x-y|^{n+2s-2}}\,dy+\int_{\mathcal{C}B_\rho(x)} \frac{4\,\|\psi\|^2_{L^\infty(\mathbb{R}^n)}}{|x-y|^{n+2s}}\,dy\\
&= \int_{B_\rho} \frac{\|\nabla\psi\|_{L^\infty(B_\rho(x))}}{|\zeta|^{n+2s-2}}\,d\zeta+\int_{\mathcal{C}B_\rho} \frac{4\,\|\psi\|^2_{L^\infty(\mathbb{R}^n)}}{|\zeta|^{n+2s}}\,d\zeta\\
&\le \omega_{n-1}\left(\frac{\|\nabla\psi\|_{L^\infty(B_\rho(x))}\,\rho^{2(1-s)}}{2(1-s)}+\frac{4\,\|\psi\|_{L^\infty(\mathbb{R}^n)^2}\rho^{-2s}}{s}\right)\end{aligned}$$

and this easily implies (2.11). □

Lemma 2.4. *Let $n\ge 1$. Let $x\in\mathbb{R}^n$, $\rho>0$ and $\psi\in L^\infty(\mathbb{R}^n)$.*

Suppose that there exists $\Xi\in\mathbb{R}^n$ and $K\in\mathbb{R}$

$$\psi(y)-\psi(x)-\Xi\cdot(y-x)\le K|x-y|^2, \tag{2.12}$$

for any $y\in B_\rho(x)$. Then,

$$\int_{\mathbb{R}^n} \frac{\psi(y)-\psi(x)}{|x-y|^{n+2s}}\,dy \le \omega_{n-1}\left(\frac{K\rho^{2(1-s)}}{2(1-s)}+\frac{\|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s}}{s}\right). \tag{2.13}$$

Analogously, if we replace (2.14) with the assumption that there exists $\tilde{\Xi}\in\mathbb{R}^n$ and $\tilde{K}\in\mathbb{R}$ such that

$$\psi(y)-\psi(x)-\tilde{\Xi}\cdot(y-x)\ge -\tilde{K}|x-y|^2, \tag{2.14}$$

for any $y\in B_\rho(x)$, we obtain that

$$\int_{\mathbb{R}^n} \frac{\psi(x)-\psi(y)}{|x-y|^{n+2s}}\,dy \le \omega_{n-1}\left(\frac{\tilde{K}\rho^{2(1-s)}}{2(1-s)}+\frac{\|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s}}{s}\right). \tag{2.15}$$

In particular, if $\psi\in L^\infty(\mathbb{R}^n)\cap W^{2,\infty}(B_\rho(x))$ we have that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \frac{\psi(y)-\psi(x)}{|x-y|^{n+2s}}\,dy\right|\\
&\qquad\le \frac{\omega_{n-1}}{(1-s)\,s}\Big(\|D^2\psi\|_{L^\infty(B_\rho(x))}\rho^{2(1-s)}+\|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s}\Big).\end{aligned} \tag{2.16}$$

Proof. We prove (2.13) under assumption (2.12), since the proof of (2.15) under assumption (2.14) is the same, and then (2.16) follows from (2.12) and (2.14) by choosing $\Xi=\tilde{\Xi}=\nabla\psi(x)$ and $K=\tilde{K}:=\|D^2\psi\|_{L^\infty(B_\rho(x))}$. The proof below is similar to the one of Lemma 2.3, but we give the details for the facility of the reader.

Notice that, by symmetry,

$$\int_{B_\rho(x)} \frac{\Xi\cdot(x-y)}{|x-y|^{n+2s}}\,dy = 0.$$

Consequently, we bound the left hand side of (2.13) by

$$\begin{aligned}
&\int_{B_\rho(x)} \frac{\psi(y)-\psi(x)+\Xi\cdot(x-y)}{|x-y|^{n+2s}}\,dy + \int_{\mathcal{C}B_\rho(x)} \frac{|\psi(x)-\psi(y)|}{|x-y|^{n+2s}}\,dy \\
&\qquad\qquad \le \int_{B_\rho(x)} \frac{K}{|x-y|^{n+2s-2}}\,dy + \int_{\mathcal{C}B_\rho(x)} \frac{2\|\psi\|_{L^\infty(\mathbb{R}^n)}}{|x-y|^{n+2s}}\,dy \\
&\qquad\qquad = \int_{B_\rho} \frac{K}{|\zeta|^{n+2s-2}}\,d\zeta + \int_{\mathcal{C}B_\rho} \frac{2\|\psi\|_{L^\infty(\mathbb{R}^n)}}{|\zeta|^{n+2s}}\,dy \\
&\qquad\qquad = \omega_{n-1}\Big[\frac{K\,\rho^{2(1-s)}}{2(1-s)} + \frac{\|\psi\|_{L^\infty(\mathbb{R}^n)}\rho^{-2s}}{s}\Big],
\end{aligned}$$

that is (2.13). □

Lemma 2.5. *Let $n \ge 1$ and $s \in (0,1)$. Let $x \in \mathbb{R}^n$. Let $\psi \in L^\infty(\mathbb{R}^n)$ be continuous, radial and radially non-decreasing, with*

$$\sup_{\mathbb{R}^n} \psi = \max_{\mathbb{R}^n} \psi = M.$$

Suppose that $\psi \in W^{2,\infty}(\{\psi < M\})$. Then,

$$\int_{\mathbb{R}^n} \frac{\psi(x)-\psi(y)}{|x-y|^{n+2s}}\,dy \le \frac{\omega_{n-1}}{(1-s)\,s}\Big(\|D^2\psi\|_{L^\infty(\{\psi<M\})} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\Big). \tag{2.17}$$

Proof. By the radial symmetry of ψ, we have that

$$\{\psi < M\} = B_\kappa$$

for some $\kappa > 0$. Accordingly,

$$\begin{aligned}
&\text{for any } z,\, y \text{ in the closure of } B_\kappa, \\
&\psi(y) \ge \psi(z) + \nabla\psi(z)(y-z) - \|D^2\psi\|_{L^\infty(\{\psi<M\})}(z-y)^2.
\end{aligned} \tag{2.18}$$

Also, fixed any $x \in \mathbb{R}^n$, we define

$$z := \begin{cases} x & \text{if } x \in B_\kappa, \\ x/\kappa & \text{otherwise.} \end{cases}$$

Notice that $|z| \le k$, that $\psi(x) = \psi(z)$, that $\psi(z) - \psi(y) \ge 0$ if and only if $|z| \ge |y|$. Also, if $|x| > k$ and α is the angle between the vector $x - z$ and $y - z$, the convexity of B_κ implies that

$$\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

and so $\cos\alpha \le 0$. Hence,

$$\begin{aligned} |x-y| &= \sqrt{|z-y|^2+|z-x|^2-2|z-y|\,|z-x|\cos\alpha} \\ &\ge \sqrt{|z-y|^2+|z-x|^2} \ \ge\ |z-y|, \end{aligned}$$

if $|x| > k$ (and, obviously, the estimate holds for $|x| \le k$ too, since is the case $z = x$).

Thus, we use the above observations to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\psi(x)-\psi(y)}{|x-y|^{n+2s}}\,dy &\le \int_{B_{|z|}} \frac{\psi(z)-\psi(y)}{|x-y|^{n+2s}}\,dy \ \le\ \int_{B_{|z|}} \frac{\psi(z)-\psi(y)}{|z-y|^{n+2s}}\,dy \\ &\le \int_{B_{|z|}\cap B_1(z)} \frac{\psi(z)-\psi(y)}{|z-y|^{n+2s}}\,dy + \int_{B_{|z|}\cap \mathcal{C}B_1(z)} \frac{\psi(z)-\psi(y)}{|z-y|^{n+2s}}\,dy \\ &\le \int_{B_1(z)} \|D^2\psi\|_{L^\infty(\{\psi<M\})}\,|z-y|^{2-n-2s}\,dy \\ &\qquad + \int_{\mathcal{C}B_1(z)} \frac{2\,\|\psi\|_{L^\infty(\mathbb{R}^n)}}{|z-y|^{n+2s}}\,dy \\ &\le \omega_{n-1}\left[\frac{\|D^2\psi\|_{L^\infty(\{\psi<M\})}}{1-s} + \frac{\|\psi\|_{L^\infty(\mathbb{R}^n)}}{s}\right], \end{aligned}$$

which implies the desired result. □

Now, we recall the construction of an useful barrier given by the following lemma, that is used in [14, 15] and also here in the asymptotic analysis of the one-dimensional minimizers of the energy (1.1) (see the forthcoming Theorem 4.1). The proof can be found in [14, Lemma 3.1]; it relies on a fine construction around the power function $t \mapsto |t|^{-2s}$ together with the estimates proved here in the previous lemmas.

Lemma 2.6. ([14]). *Let $n \ge 1$. Given any $\tau > 0$, there exists a constant $C \ge 1$, possibly depending on n, s and τ, such that the following holds: for any $R \ge C$, there exists a rotationally symmetric function*

$$w \in C\big(\mathbb{R}^n, [-1 + CR^{-2s}, 1]\big), \tag{2.19}$$

with

$$w = 1 \text{ in } \mathcal{C}B_R, \tag{2.20}$$

such that

$$\int_{\mathbb{R}^n} \frac{w(y)-w(x)}{|x-y|^{n+2s}}\,dy \le \tau\big(1+w(x)\big) \tag{2.21}$$

and

$$\frac{1}{C}(R+1-|x|)^{-2s} \le 1 + w(x) \le C\big(R+1-|x|\big)^{-2s} \tag{2.22}$$

for any $x \in B_R$.

We finish this section by considering the following equation related to the eigenfunctions of the operator $(-\Delta)^s$,

$$-(-\Delta)^s v(x) - \alpha v(x) = 0, \tag{2.23}$$

where α is a positive constant.

In Corollary 2.8 we show that the function v being a subsolution of equation (2.23) away from the origin is bounded (up to a multiplicative constant) by the function $x \mapsto |x|^{-(1+2s)}$. This estimate will be crucial in the analysis of the global minimizers of the functionals $\mathcal{F}$ (see Section 4).

First, we need to prove the following result.

Lemma 2.7. *Let $\eta \in C^2(\mathbb{R}, (0, +\infty))$, with $\|\eta\|_{C^2(\mathbb{R})} < +\infty$, and*

$$\eta(x) = \frac{1}{|x|^{1+2s}} \qquad \text{for any } x \in \mathbb{R} \setminus (-1, 1).$$

Then there exists $\kappa \in (0, +\infty)$, possibly depending on s and η, such that

$$\limsup_{x \to \pm\infty} \frac{-(-\Delta)^s \eta(x)}{\eta(x)} \le \kappa.$$

Proof. We will denote by C suitable positive quantities, possibly different from line to line, and possibly depending on s and η. For all $(x, y) \in \mathbb{R}^2$ with $|x| \ge 2$, we define

$$i(x, y) := \frac{\eta(y) - \eta(x) - \chi_{(-1/4,1/4)}(x - y)\, \eta'(x)(y - x)}{|x - y|^{1+2s}}.$$

For any fixed $y \in \mathbb{R}$, we have that

$$\lim_{x \to \pm\infty} |x|^{1+2s} i(x, y) = \lim_{x \to \pm\infty} \frac{|x|^{1+2s}}{|x - y|^{1+2s}} \big(\eta(y) - \eta(x)\big) = \eta(y). \tag{2.24}$$

Also, if $|y| \le 1$ and $|x| \ge 2$, we have that $|x - y| \ge |x| - |y| \ge |x|/2$ and so

$$|x|^{1+2s} |i(x, y)| = \frac{|x|^{1+2s} \big|\eta(y) - \eta(x)\big|}{|x - y|^{1+2s}} \le 16 \sup_{\mathbb{R}} |\eta|. \tag{2.25}$$

Using (2.25), (2.24) and the Bounded Convergence Theorem, we conclude that

$$\begin{aligned} &\lim_{x \to \pm\infty} |x|^{1+2s} \int_{-1}^{1} \frac{\eta(y) - \eta(x) - \chi_{(-1/4,1/4)}(x - y)\, \eta'(x)(y - x)}{|x - y|^{1+2s}}\, dy \\ &\quad = \int_{-1}^{1} \lim_{x \to \pm\infty} |x|^{1+2s} i(x, y)\, dy = \int_{-1}^{1} \eta(y)\, dy. \end{aligned} \tag{2.26}$$

Now, fixed $|x|\geq 2$, we estimate the contribution in $\mathbb{R}\setminus(-1,1)$. We write $\mathbb{R}\setminus(-1,1)=P\cup Q\cup R\cup S$, where

$$\begin{aligned} P &= \Big\{y\in\mathbb{R}\setminus(-1,1) \text{ s.t. } |x|/2<|y|\leq 2|x| \text{ and } |x-y|\geq 1/4\Big\},\\ Q &= \Big\{y\in\mathbb{R}\setminus(-1,1) \text{ s.t. } |x|/2<|y|\leq 2|x| \text{ and } |x-y|< 1/4\Big\},\\ R &= \Big\{y\in\mathbb{R}\setminus(-1,1) \text{ s.t. } |y|>2|x|\Big\},\\ S &= \Big\{y\in\mathbb{R}\setminus(-1,1) \text{ s.t. } |y|\leq |x|/2\Big\}. \end{aligned}$$

We observe that, if $y\in P$,

$$\begin{aligned} |i(x,y)| &= \frac{|\eta(y)-\eta(x)|}{|x-y|^{1+2s}} \leq \frac{|\eta(y)|+|\eta(x)|}{|x-y|^{1+2s}} = \frac{(1/|y|^{1+2s})+(1/|x|^{1+2s})}{|x-y|^{1+2s}}\\ &\leq \frac{C}{|x|^{1+2s}|x-y|^{1+2s}}. \qquad (2.27) \end{aligned}$$

As a consequence,

$$|x|^{1+2s}\int_P i(x,y)\,dy \leq C\int_P \frac{dy}{|x-y|^{1+2s}} \leq C\int_{\{|x-y|\geq 1/4\}}\frac{dy}{|x-y|^{1+2s}}\leq C.$$

Moreover, if $y\in Q$, we can use the Taylor expansion of the function $1/|t|^{1+2s}$ to obtain that

$$\begin{aligned} \eta(y)-\eta(x)-\chi_{(-1/4,1/4)}(x-y)\,\eta'(x)(y-x)& \\ &= \eta(y)-\eta(x)-\eta'(x)\cdot(y-x)\\ &= \frac{1}{|y|^{1+2s}}-\frac{1}{|x|^{1+2s}}+\frac{(1+2s)}{|x|^{3+2s}}x(y-x)\\ &= \frac{(1+2s)(2+2s)}{|\xi|^{3+2s}}|x-y|^2, \end{aligned}$$

for an appropriate ξ which lies on the segment joining x to y. Notice also that if $y\in Q$, then $y\geq 0$ if and only if $x\geq 0$, therefore both x and y lie either in $[|x|/2,+\infty)$ or in $(-\infty,-|x|/2]$. In any case, $|\xi|\geq |x|/2$ and so, for any $y\in Q$,

$$\begin{aligned} |i(x,y)| &= \frac{|\eta(y)-\eta(x)-\chi_{(-1/4,1/4)}(x-y)\,\eta'(x)(y-x)|}{|x-y|^{1+2s}}\\ &= \frac{C}{|\xi|^{3+2s}}|x-y|^{1-2s} \leq \frac{C}{|x|^{3+2s}}|x-y|^{1-2s}. \end{aligned}$$

As a consequence,

$$\begin{aligned} |x|^{1+2s}\int_Q i(x,y)\,dy &\leq \frac{C}{|x|^2}\int_Q |x-y|^{1-2s}\\ &\leq \frac{C}{|x|^2}\int_{|x-y|<1/4}|x-y|^{1-2s} \leq \frac{C}{|x|^2} \leq C. \end{aligned}$$

Furthermore, if $y \in R$, we have that $|x-y| \geq |y|-|x| \geq |x| > 1/4$, thus we can estimate the function $i(x,y)$ as in (2.27) and we obtain

$$|i(x,y)| \leq \frac{C}{|x|^{1+2s}|x-y|^{1+2s}}.$$

In particular,

$$\begin{aligned} |x|^{1+2s}\int_R i(x,y)\,dy &\leq C\int_{\{|y|\geq 2|x|\}} \frac{dy}{|x-y|^{1+2s}} \\ &\leq C\int_{\{|x-y|\geq |x|\}} \frac{dy}{|x-y|^{1+2s}} = \frac{C}{|x|^{2s}} \leq C. \end{aligned}$$

As for the last contribution, if $y \in S$ then $|x-y| \geq |x|-|y| \geq |x|/2 \geq 1$ and so

$$\begin{aligned} |i(x,y)| &\leq \frac{|\eta(y)|+|\eta(x)|}{|x-y|^{1+2s}} = \frac{(1/|y|^{1+2s})+(1/|x|^{1+2s})}{|x-y|^{1+2s}} \\ &\leq \frac{C}{|x|^{1+2s}|y|^{1+2s}}. \end{aligned}$$

Accordingly,

$$|x|^{1+2s}\int_S i(x,y)\,dy \leq C\int_{\{1\leq|y|\leq|x|/2\}} \frac{dy}{|y|^{1+2s}} \leq \frac{C}{|x|^2} \leq C.$$

All in all, we obtain that

$$\begin{aligned} &\limsup_{x\to\pm\infty} |x|^{1+2s}\int_{\mathbb{R}\setminus(-1,1)} \frac{\eta(y)-\eta(x)-\chi_{(-1/4,1/4)}(x-y)\,\nabla\eta(x)\cdot(y-x)}{|x-y|^{1+2s}}\,dy \\ &= \limsup_{x\to\pm\infty} |x|^{1+2s}\bigg(\int_P i(x,y)\,dy + \int_Q i(x,y)\,dy \\ &\quad + \int_R i(x,y)\,dy + \int_S i(x,y)\,dy\bigg) \leq C. \end{aligned}$$

From this and (2.26), the desired result plainly follows. □

Corollary 2.8. *Let α, $\beta > 0$ Let $v : \mathbb{R} \to \mathbb{R}$ be bounded and uniformly continuous, with $-(-\Delta)^s v(x) \geq \alpha v(x)$ for any $x \in \mathbb{R}\setminus(-\beta,\beta)$. Then, there exists a constant $\bar{C} > 0$, possibly depending on s, α and β, such that*

$$v(x) \leq \frac{\bar{C}}{|x|^{1+2s}} \qquad \text{for any } x \in \mathbb{R}.$$

Proof. Take η and κ as in Lemma 2.7. Define

$$a := \left(\frac{\alpha}{2\kappa}\right)^{1/(2s)}$$

and $\zeta(x) := \eta(ax)$.

Then,

$$\limsup_{x\to\pm\infty} \frac{-(-\Delta)^s\zeta(x)}{\zeta(x)} = a^{2s}\limsup_{x\to\pm\infty} \frac{-(-\Delta)^s\eta(ax)}{\eta(ax)} \leq a^{2s}\kappa = \frac{\alpha}{2}.$$

As a consequence, there exists $\beta' \geq \beta$ such that

$$-(-\Delta)^s \zeta(x) \leq \alpha \zeta(x) \text{ for any } x \in \mathbb{R} \setminus (-\beta', \beta'). \tag{2.28}$$

Now, we set

$$\bar{C} := \frac{4\|v\|_{L^\infty(\mathbb{R})}}{\min\limits_{[-a\beta', a\beta']} \eta} = \frac{4\|v\|_{L^\infty(\mathbb{R})}}{\min\limits_{[-\beta', \beta']} \zeta}.$$

We claim that

$$v(x) \leq \bar{C}\zeta(x) \text{ for any } x \in \mathbb{R}. \tag{2.29}$$

In order to prove the above inequality, we take b in $[0, +\infty)$ and we define $v_b(x) := \bar{C}\zeta(x) + b - v(x)$. When $b > \|v\|_{L^\infty(\mathbb{R})}$, we have that $v_b(x) > 0$ for any $x \in \mathbb{R}$. Now, let b_o the first b for which v_b touches 0 from above: we have that $v_{b_o}(x) \geq 0$ and that there exists a sequence $x_k \in \mathbb{R}$ such that $v_{b_o}(x_k) \leq 2^{-k}$, for $k \in \mathbb{N}$. We claim that

$$b_o = 0. \tag{2.30}$$

Indeed, we have, if k is sufficiently large,

$$\|v\|_{L^\infty(\mathbb{R})} \geq 2^{-k} \geq v_{b_o}(x_k) \geq \bar{C}\zeta(x_k) - v(x_k) \geq \bar{C}\zeta(x_k) - \|v\|_{L^\infty(\mathbb{R})}$$

and so

$$\zeta(x_k) \leq \frac{2\|v\|_{L^\infty(\mathbb{R})}}{\bar{C}} = \frac{\min\limits_{[-\beta', \beta']} \zeta}{2}.$$

Therefore, $|x_k| > \beta'$.

Hence, recalling (2.28),

$$\begin{aligned} \int_{\mathbb{R}} \frac{v_{b_o}(y) - v_{b_o}(x_k)}{|x_k - y|^{1+2s}} \, dy &= -(-\Delta)^s v_{b_o}(x_k) \\ &= -\bar{C}(-\Delta)^s \zeta(x_k) + (-\Delta)^s v(x_k) \leq \alpha(\bar{C}\zeta(x_k) - v(x_k)) \\ &= \alpha v_{b_o}(x_k) - \alpha b_o \leq 2^{-k}\alpha - \alpha b_o. \end{aligned} \tag{2.31}$$

Now, we define $v_k(x) := v_{b_o}(x + x_k)$. Notice that $v_k(x) \geq 0$ for any $x \in \mathbb{R}^n$ and $v_k(0) \leq 2^{-k}$. Also, by the Theorem of Ascoli, up to subsequence, we may suppose that v_k converges to some v_∞ locally uniformly as $k \to +\infty$.

It follows that $v_\infty(x) \geq 0 = v_\infty(0)$ for any $x \in \mathbb{R}$. Then, from (2.31),

$$\begin{aligned} -\alpha b_o &\geq \lim_{k \to +\infty} \int_{\mathbb{R}} \frac{v_{b_o}(y) - v_{b_o}(x_k)}{|x_k - y|^{1+2s}} \, dy \\ &= \lim_{k \to +\infty} \int_{\mathbb{R}} \frac{v_k(t) - v_k(0)}{|t|^{1+2s}} \, dt \\ &= \int_{\mathbb{R}} \frac{v_\infty(t)}{|t|^{1+2s}} \, dt \geq 0. \end{aligned}$$

This completes the proof of (2.30).

Now, from (2.30), we conclude that, for any $x \in \mathbb{R}$,

$$0 \le v_{b_o}(x) = \bar{C}\zeta(x) + b_o - v(x) = \bar{C}\zeta(x) - v(x)$$

and so $v(x) \le \bar{C}\zeta(x)$. □

2.3. **Regularity properties of the fractional Allen-Cahn equation.** The following propositions recall how the fractional Laplacian operators interact with the C^α-norms. Their proofs can be found in [16, Chapter 2], which presents some general properties of the $(-\Delta)^s$ operators and provides characterization of its supersolutions (see also [17]).

Proposition 2.9. ([16, Proposition 2.1.10]) *Let $n \ge 1$. Let $w \in C^{0,\alpha}(\mathbb{R}^n)$, for $\alpha \in (0,1]$. Let $u \in L^\infty(\mathbb{R}^n)$ be such that*

$$-(-\Delta)^s u(x) = w(x) \quad \text{for any } x \in \mathbb{R}^n. \tag{2.32}$$

Then,

(i) *If $\alpha + 2s \le 1$, then $u \in C^{0,\alpha+2s}(\mathbb{R}^n)$. Moreover*

$$\|u\|_{C^{0,\alpha+2s}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^N)} + \|w\|_{C^{0,\alpha}(\mathbb{R}^n)}\big)$$

for a constant C depending only on n, α and s.

(ii) *If $\alpha + 2s > 1$, then $u \in C^{1,\alpha+2s-1}(\mathbb{R}^n)$. Moreover*

$$\|u\|_{C^{1,\alpha+2s-1}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{C^{0,\alpha}(\mathbb{R}^n)}\big)$$

for a constant C depending only on n, α and s.

Proposition 2.10. ([16, Proposition 2.1.11]) *Let $n \ge 1$. Let u and $w \in L^\infty(\mathbb{R}^n)$ be such that*

$$-(-\Delta)^s u(x) = w(x) \quad \text{for any } x \in \mathbb{R}^n.$$

Then,

(i) *If $2s \le 1$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s$. Moreover*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big)$$

for a constant C depending only on n, α and s.

(ii) *If $2s > 1$, then $u \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s - 1$. Moreover*

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big)$$

for a constant C depending only on n, α and s.

Since we deal with the case of w in (2.32) being the derivative of a double-well potential W, we have to extrapolate the regularity informations for the solutions of equation (1.3); this can be obtained by iterating the results in Proposition 2.9 and Proposition 2.10. In the following two lemmas we arrange some regularity results in the form to be applied in the sequel of this paper (as well as in [14] and [15]).

Lemma 2.11. *Let $n \geq 1$. Let $u \in L^\infty(\mathbb{R}^n)$ be such that*

$$-(-\Delta)^s u(x) = W'(u(x)) \quad \text{for any } x \in \mathbb{R}^n, \tag{2.33}$$

with $W \in C^1(\mathbb{R})$. Then,

(i) *If $s \in (0, 1/2]$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s$. Moreover,*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|W'(u)\|_{L^\infty(\mathbb{R}^n)}\big).$$

(ii) *If $s \in (1/2, 1)$, then $u \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s - 1$. Moreover,*

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|W'(u)\|_{L^\infty(\mathbb{R}^n)}\big),$$

for a constant C depending only on n, α and s.

Proof. Let u in $L^\infty(\mathbb{R}^n)$ be a solution of equation (2.33). Since W belongs to $C^1(\mathbb{R})$, it suffices to apply Proposition 2.10(i)-(ii) by chosing $w(x) := W'(u(x))$. □

Lemma 2.12. *Let $n \geq 1$ and let $u \in L^\infty(\mathbb{R}^n)$ satisfy equation (2.33), with $W \in C^2(\mathbb{R})$. Then $u \in C^{2,\alpha}(\mathbb{R}^n)$, with α depending on s.*

Proof. Let $s \in (1/2, 1)$ and let u in $L^\infty(\mathbb{R}^n)$ be a solution of the equation (2.33). Then, $u \in C^{1,\alpha}(\mathbb{R}^n)$ with its $C^{1,\alpha}$ norm bounded as in Lemma 2.11(i). Moreover u' satisfies

$$-(-\Delta)^s u'(x) = W''(u(x))u'(x) \quad \text{for any } x \in \mathbb{R}^n. \tag{2.34}$$

By the hypothesis on W and u, we can apply Proposition 2.10(ii) to the solution u' of equation (2.34) with $w := W''(u(x))u'(x)$. It follows that u' belongs to $C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s - 1$ and thus the claim is proved.

Let $s = 1/2$. Then, by the fact that W is in C^2 together with the regularity of u provided by Lemma 2.11(i), Proposition 2.9(ii) with $w := W'(u)$ yields that the function u belongs to $C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 1$. Now, we can argue as for the case $s \in (1/2, 1)$ to obtain the desired regularity for u by Proposition 2.10(ii).

Finally, let $s \in (0, 1/2)$ and let $u \in L^\infty(\mathbb{R}^n)$ be a solution of (2.33). Lemma 2.11(i) yields $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s$. Then, for $s \in (1/4, 1/2)$ we can apply Proposition 2.9(ii) and we get $u \in C^{1,\alpha+2s-1}(\mathbb{R}^n)$. Hence, u' is well defined and it satisfies equation (2.34) with $w = W''(u)u'$ belonging to $C^{0,\alpha+2s-1}(\mathbb{R}^n)$ and again by Proposition 2.9(ii) we get $u' \in C^{1,\alpha+2s-1}$ for any $\alpha < 2s$.

For $s \in (0, 1/4]$, we can use Proposition 2.9(i) in order to obtain $u \in C^{0,\alpha+2s}(\mathbb{R}^n)$ for any $\alpha < 2s$. Thus, when $s \in (1/6, 1/4)]$, we can apply twice Proposition 2.9(ii) arguing as in the case $s \in (1/4, 1/2)$ and we get $u' \in C^{1,\alpha+4s-1}(\mathbb{R}^n)$, for any $\alpha < 2s$.

By iterating the above procedure on $k \in \mathbb{N}$, we obtain that, when $s \in (1/(2k+2),\, 1/2k]$, u belongs to $C^{2,\alpha+2k-1}$ for any $\alpha < 2s$. $\square$

We conclude this section observing that the equation we deal with behaves well under limits:

Lemma 2.13. *Let $s \in (0,1)$ and $W \in C^1(\mathbb{R})$. For any $k \in \mathbb{N}$, let $u_k \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that*

$$-(-\Delta)^s u_k(x) = W'(u_k(x)) \quad \text{for any } x \in B_k.$$

Suppose that $\sup_k \|u_k\|_{L^\infty(\mathbb{R}^n)} < \infty$ and that u_k converges a.e. to a function u. Then,

$$-(-\Delta)^s u(x) = W'(u(x)) \quad \text{for any } x \in \mathbb{R}^n.$$

Proof. Given any $\phi \in C_0^\infty(\mathbb{R})$ supported in B_k,

$$\begin{aligned} \int_{\mathbb{R}} W'(u_k(x))\,\phi(x)\,dx &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{u_k(y)-u_k(x)}{|x-y|^{n+2s}}\,dy \right] \phi(x)\,dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u_k(x)\big(\phi(y)-\phi(x)\big)}{|x-y|^{n+2s}}\,dx\,dy. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{\phi(x)-\phi(y)}{|x-y|^{n+2s}}\,dy \Big| dx &= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{|\phi(x)-\phi(x-y)|}{|y|^{n+2s}}\,dy \Big| dx \\ \leq \int_{\mathbb{R}} dx \left[\Big| \int_{B_1} \frac{\phi(x)-\phi(x+y)+\nabla\phi(x)y}{|y|^{n+2s}}\,dy \Big| + \Big| \int_{\mathcal{C}B_1} \frac{2\|\phi\|_{L^\infty}}{|y|^{n+2s}}\,dy \Big| \right] dx \\ \leq \int_{\mathbb{R}} dx \Big| \int_0^1 \frac{\|\nabla^2\phi\|_{L^\infty}}{r^{n+2s}} r^{n+1}\,dr + \int_1^\infty \frac{2\|\phi\|_{L^\infty}}{r^{1+2s}}\,dr \Big| \;&<\; +\infty. \end{aligned}$$

Thus, by Dominated Convergence Theorem,

$$\begin{aligned} \int_{\mathbb{R}} W'(u(x))\phi(x)\,dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u(x)\big(\phi(y)-\phi(x)\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy \right] \phi(x)\,dx \\ &= \int_{\mathbb{R}} -(-\Delta)^s u(x)\phi(x)\,dx, \end{aligned}$$

which gives the desired claim, since ϕ is arbitrary. $\square$

3. Minimization by sliding

In the forthcoming Proposition 3.2 we prove a minimization result which is a technical variation of a classical sliding method (see, e.g., Lemma 9.1 in [20], and also [2, 4] for a different variational approach for the classical local functional). First, we need the following lemma, in which we point out that the problem of minimizing the energy in a given ball has a solution.

Lemma 3.1. *Let $R > 0$ and $u_o : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Suppose that there exists a measurable function $\tilde{u}$ which coincides with u_o in $\mathcal{C}B_R$ and such that $\mathcal{F}(\tilde{u}, B_R) < +\infty$. Then, there exists a measurable function $u_\star$ such that $\mathcal{F}(u_\star, B_R) \le \mathcal{F}(v, B_R)$ for any measurable function v which coincides with u_o in $\mathcal{C}B_R$.*

Proof. We take a minimizing sequence, that is, let u_k be such that $u_k = u_o$ in $\mathcal{C}B_R$, $\mathcal{F}(u_k, B_R) \le \mathcal{F}(\tilde{u}, B_R)$ and

$$\lim_{k\to+\infty} \mathcal{F}(u_k, B_R) = \inf \mathcal{F}(v, B_R), \tag{3.1}$$

where the infimum is taken over any v that coincides with u_o in $\mathcal{C}B_r$. Then, (3.1) and Lemma 2.1 give that, up to subsequence, u_k converges almost everywhere to some $u_\star$. Thus, the desired result follows from (3.1) and Fatou Lemma. □

Now, we are in position to prove that every monotone solution of equation (1.3), with the limit condition (1.5), is a local minimizer for the corresponding energy functional $\mathcal{F}$.

Proposition 3.2. *Let $s \in (0, 1)$ and let $u \in C^1(\mathbb{R}^n)$ be a solution of*

$$-(-\Delta)^s u(x) = W'(u(x)), \quad \text{for any } x \in \mathbb{R}^n.$$

Suppose that

$$\partial_{x_n} u(x) > 0, \quad \text{for any } x \in \mathbb{R}^n$$

and

$$\lim_{x_n\to\pm\infty} u(x', x_n) = \pm 1, \quad \text{for any } x' \in \mathbb{R}^{n-1}. \tag{3.2}$$

Then, for any $r > 0$, we have that $\mathcal{F}(u, B_r) \le \mathcal{F}(u + \phi, B_r)$ for any measurable ϕ supported in B_r.

Proof. We argue by contradiction. Suppose that there exists r, $c_o > 0$ and ϕ supported in B_r such that $\mathcal{F}(u, B_r) - \mathcal{F}(u + \phi, B_r) \ge c_o$.

In view of Lemma 3.1, we can take $u_\star$ minimizing $\mathcal{F}(v; B_r)$ among all the measurable functions v such that $v = u$ in $\mathcal{C}B_r$. By construction,

$$\mathcal{F}(u_\star, B_r) \le \mathcal{F}(u + \phi, B_r) \le \mathcal{F}(u, B_r) - c_o.$$

In particular, $u_\star$ and u cannot be equal to each other, so we assume, without loss of generality, that there exists $P \in \mathbb{R}^n$ such that

$$u(P) < u_\star(P). \tag{3.3}$$

By cutting at the levels ± 1, which possibly makes $\mathcal{F}$ decrease, we see that $|u_\star| \le 1$.

Moreover, by the minimizing property of $u_\star$,

$$(-\Delta)^s u_\star(x) + W'(u_\star(x)) = 0 \text{ for any } x \in B_r \tag{3.4}$$

and then, by Lemma 2.11, $u_\star$ is continuous.

We claim that

$$|u_\star| < 1. \tag{3.5}$$

To check this, let us argue by contradiction and suppose that, say, $u_\star(\bar{x}) = +1$, for some $\bar{x} \in \mathbb{R}^n$.

Since $|u| < 1$ by our assumptions and $u_\star = u$ in $\mathcal{C}B_r$, we have that $\bar{x} \in B_r$. Then (3.4) and the fact that $W'(+1) = 0$ would give that

$$\int_{\mathbb{R}^n} \frac{1 - u_\star(y)}{|\bar{x} - y|^{1+2s}}\, dy \;=\; 0.$$

Since the integrand is always nonnegative, $u_\star$ must be identically equal to $+1$. But this is in contradiction with the fact that $u_\star = u$ in $\mathcal{C}B_r$, hence it proves (3.5).

Now, we claim that there exists $\bar{k} \in \mathbb{R}$ such that,

$$\text{if } k \ge \bar{k}, \text{ then } u(x', x_n + k) \ge u_\star(x) \text{ for any } x = (x', x_n) \in \mathbb{R}^n. \tag{3.6}$$

To prove (3.6), we argue by contradiction and we suppose that, for any $k \in \mathbb{N}$, there exists $x^{(k)} = (x^{(k)\prime}, x_n^{(k)}) \in \mathbb{R}^n$ for which $u(x^{(k)\prime}, x_n^{(k)} + k) < u_\star(x^{(k)})$. Since u is monotone and $k \ge 0$, it follows that $u(x^{(k)}) < u_\star(x^{(k)})$ and therefore $x^{(k)} \in B_r$.

Thus, up to subsequence, we suppose that

$$\lim_{k \to +\infty} x^{(k)} = x_\star,$$

for some $x_\star$ in the closure of B_r. Consequently, by (3.2),

$$+1 \;=\; \lim_{k \to +\infty} u(x^{(k)\prime}, x_n^{(k)} + k) \;\le\; \lim_{k \to +\infty} u_\star(x^{(k)}) \;=\; u_\star(x_\star) \;\le\; \sup_{B_r} u_\star.$$

Since this is in contradiction with (3.5), we have proved (3.6).

Then, by (3.6) and the monotonicity of u, we have that, if $k > \bar{k}$, then $u(x', x_n + k) > u_\star(x)$ for any $x = (x', x_n) \in \mathbb{R}^n$. We take $\bar{k}$ as small as possible with this property, i.e., $u(x', x_n + k) \ge u_\star(x)$ for any $k \ge \bar{k}$ and any $x \in \mathbb{R}^n$, and there exists an infinitesimal sequence $\eta_j > 0$ and points $p^{(j)} \in \mathbb{R}^n$ for which $u(p^{(j)\prime}, p_n^{(j)} + \bar{k} - \eta_j) \le u_\star(p^{(j)})$.

So, recalling (3.3), we have that $u(P) < u_\star(P) \le u(P', P_n + \bar{k})$ and then the monotonicity of u implies that

$$\bar{k} > 0. \tag{3.7}$$

We claim that

$$p^{(j)} \in B_r. \tag{3.8}$$

Indeed, if $p^{(j)}$ belonged to $\mathcal{C}B_r$ we would have that

$$u(p^{(j)\prime}, p_n^{(j)} + \bar{k} - \eta_j) \;\le\; u_\star(p^{(j)}) \;=\; u(p^{(j)}).$$

Hence, by the monotonicity of u, we would have that $\bar{k} - \eta_j \le 0$ and so, by taking the limit in j, that $\bar{k} \le 0$. This is in contradiction with (3.7) and so (3.8) is proved.

Then, by (3.8), we may suppose that $\lim_{j\to+\infty} p^{(j)} = \zeta$, for some ζ in the closure of B_r. As a consequence, the function $w(x) := u(x', x_n + \bar{k}) - u_\star(x)$ satisfies $w(x) \ge 0$ for any $x \in \mathbb{R}^n$ and $w(\zeta) = 0$.

Thus, recalling (3.4), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{w(y)}{|\zeta - y|^{n+2s}}\,dy &= -(-\Delta)^s w(\zeta)\\ &= -(-\Delta)^s u(\zeta', \zeta_n + \bar{k}) + (-\Delta)^s u_\star(\zeta)\\ &= W'(u(\zeta', \zeta_n + \bar{k})) - W'(u_\star(\zeta)) = 0.\end{aligned}$$

Since the integrand is nonnegative, this implies that w vanishes identically, and so

$$u(x', x_n + \bar{k}) = u_\star(x).$$

Taking into account the above equality, (3.7) and the strict monotonicity of u it yields that

$$u(x) < u_\star(x) \quad \text{for any } x \in \mathbb{R}^n.$$

This is in contradiction with the fact that u and $u_\star$ coincide in $\mathcal{C}B_r$ and so Proposition 3.2 is proved. □

4. The 1D minimizer

We are ready to deal with 1-D minimizers (for related observations when $s \in (1/2, 1)$ see [10] and [12]).

First, for the convenience of the reader, we introduce the setting in which we work and we recall some previous definitions.

We denote by

$$\mathcal{X} = \big\{ f \in L^1_{\rm loc}(\mathbb{R}) \ \text{ s.t. } \ \lim_{x\to\pm\infty} f(x) = \pm 1 \big\} \tag{4.1}$$

the space of admissible functions and we define the functional $\mathcal{G} : \mathcal{X} \to \mathbb{R} \cap \{+\infty\}$ as follows

$$\mathcal{G}(u) = \begin{cases} \displaystyle\liminf_{R\to+\infty} \frac{1}{R^{1-2s}} \mathcal{F}(u,[-R,R]) & \text{if } s \in (0,\,1/2), \\[2ex] \displaystyle\liminf_{R\to+\infty} \frac{1}{\log R} \mathcal{F}(u,[-R,R]) & \text{if } s = 1/2, \\[2ex] \mathcal{F}(u,\mathbb{R}) & \text{if } s \in (1/2,\,1), \end{cases} \tag{4.2}$$

where, for every $I \in \mathbb{R}$, $\mathcal{F}(\cdot, I)$ is defined by (1.1); that is,

$$\mathcal{F}(u,I) = \frac{1}{2}\int_I\int_I \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dx\,dy + \int_I\int_{\mathcal{C}I} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dx\,dy + \int_I W(u)\,dx,$$

with $W \in C^{2,1}(\mathbb{R})$ a double-well potential with wells at $\{-1,+1\}$ satisfying

$$W''(\pm 1) > 0. \tag{4.3}$$

In forthcoming Theorem 4.1 we state and prove some basic features of the minimizers of the functional $\mathcal{G}$. To this aim we define the following set of functions $\mathcal{M}$ in $\mathcal{X}$.

$$\begin{aligned} \mathcal{M} = \Big\{ & u \in \mathcal{X} \ \text{ s.t. } \ \mathcal{G}(u) < +\infty \ \text{ and } \ \mathcal{F}(u,[-a,a]) \le \mathcal{F}(u+\phi,[-a,a]) \\ & \text{for any } a > 0 \text{ and any } \phi \text{ measurable and supported in } [-a,a] \ \Big\}. \end{aligned} \tag{4.4}$$

Theorem 4.1. *Let $\mathcal{M}$ be the set of function defined by (4.4). Then*

(i) *$\mathcal{M}$ is non-empty;*

(ii) *if $u_o \in \mathcal{M}$, then $(-\Delta)^s u_o(x) + W'(u_o(x)) = 0$ for any $x \in \mathbb{R}$;*

(iii) *for any $x_o \in \mathbb{R}$, the set*

$$\mathcal{M}^{(x_o)} := \big\{ u \in \mathcal{M} \ \ s.t. \ \ x_o = \sup\{t \in \mathbb{R} \ \ s.t. \ \ u(t) < 0\} \big\}$$

consists of only one element, which will be denoted by $u^{(x_o)}$, and $u^{(x_o)}(x) = u^{(0)}(x - x_o)$;

(iv) *$u^{(0)} \in C^2(\mathbb{R})$ is such that*

$$\big(u^{(0)}\big)'(x) > 0 \ \text{ for any } x \in \mathbb{R}, \tag{4.5}$$

and

$$\mathcal{M}^{(x_o)} \equiv \big\{ u \in \mathcal{M} \ \ s.t. \ u(x_o) = 0 \big\}; \tag{4.6}$$

(v) *there exists* $C \geq 1$ *such that*

$$|u^{(0)}(x) - \operatorname{sign}(x)| \leq C\,|x|^{-2s}, \tag{4.7}$$

$$\left|\big(u^{(0)}\big)'(x)\right| \leq C\,|x|^{-(1+2s)} \tag{4.8}$$

for any large $x \in \mathbb{R}$.

Proof.
Proof of (i).

We will prove assertion (i) by taking the limit of a suitably sequence of functions in X, by means of Lemma 3.1.

First, we set

$$H(x) := \begin{cases} -1 & \text{if } x \leq -1, \\ x & \text{if } x \in (-1,+1), \\ +1 & \text{if } x \geq +1. \end{cases}$$

By a direct computation, for any $R > 2$,

$$\begin{aligned}
&\text{if } s = (0,\,1/2), \quad \int_{-R}^{-1}\int_{+1}^{R} \frac{dx\,dy}{|x-y|^{1+2s}} \leq C_1 R^{2s-1};\\
&\text{if } s = 1/2, \quad \int_{-R}^{-1}\int_{+1}^{R} \frac{dx\,dy}{|x-y|^{1+2s}} \leq C_1 \log R;\\
&\text{if } s \in (1/2,\,1), \quad \int_{-R}^{-1}\int_{+1}^{R} \frac{dx\,dy}{|x-y|^{1+2s}} \leq C_1;\\
&\text{if } s \in (0,1), \quad \int_{-1}^{+1}\int_{-1}^{+1} |x-y|^{1-2s}\,dx\,dy \leq C_1\\
&\text{and} \quad \int_{-1}^{+1}\left[\int_{+1}^{R} \frac{|1-x|^2\,dy}{|x-y|^{1+2s}}\right] dx \;\leq\; \int_{-1}^{+1}\left[\int_{1}^{2} |x-y|^{1-2s}\,dy\right] dx\\
&\qquad\qquad + \int_{-1}^{+1}\left[\int_{2}^{+\infty} \frac{4\,dy}{|x-y|^{1+2s}}\right] dx \;\leq\; C_1,
\end{aligned}$$

for a suitable $C_1 > 0$, possibly depending on s.

This entails that

$$\mathcal{F}(H,[-R,R]) \leq \begin{cases} C_2\Big(1 + R^{2s-1}\Big) & \text{if } s = (0,\,1/2), \\ C_2\Big(1 + \log R\Big) & \text{if } s = 1/2, \\ C_2 & \text{if } s \in (1/2,\,1), \end{cases} \tag{4.9}$$

for a suitable $C_2 > 0$.

Consequently, we use Lemma 3.1 to obtain that, for any $K \in \mathbb{N}$, $K \geq 2$, there exists v_K such that $v_K(x) = H(x)$ if $|x| \geq K$ and $\mathcal{F}(v_K,\,[-K,+K]) \leq \mathcal{F}(v,\,[-K,+K])$ for any measurable v such that $v(x) = H(x)$ if $|x| \geq K$.

In fact, without loss of generality, we take the monotone non-decreasing rearrangement of v_K (that we still denote by v_K): such rearrangement is equidistributed with respect to the original functional (see (1.4) in [11]) and therefore it lowers the energy (see (1.6) in [11] and [21, Theorem 5.47]).

The minimization property of v_K yields that

$$\int_{\mathbb{R}} \frac{v_K(y) - v_K(x)}{|x-y|^{1+2s}}\, dy = -(-\Delta)^s v_K(x) = W'(v_K(x)) \quad \text{for any } x \in [-K, K], \tag{4.10}$$

and so, by Lemma 2.11, we have that v_K is continuous, with modulus of continuity bounded independently of K. Also, we fix a point $c_o \in (-1, 1)$ such that

$$W'(c_o) \neq 0. \tag{4.11}$$

By continuity, there must be a point $p_K \in [-K, +K]$ such that $v_K(p_K) = c_o$

In fact, we claim that

$$\lim_{K \to +\infty} K - |p_K| = +\infty. \tag{4.12}$$

To prove (4.12), ????????????????

This proves (4.12).

We set

$$u_K(x) := v_K(x + p_K),$$

so $u_K(0) = c_o$. As a consequence, we may suppose that u_K converges locally uniformly to some $u_* \in C(\mathbb{R}; [-1, +1])$, with

$$u_*(0) = c_o \tag{4.13}$$

and

$$u_* \text{ is non-decreasing.} \tag{4.14}$$

By (4.10) and Lemma 2.13,

$$(-\Delta)^s u_*(x) + W'(u_*(x)) = 0 \text{ for any } x \in \mathbb{R}. \tag{4.15}$$

This and Lemma 2.12 imply that $u_* \in C^2(\mathbb{R})$. From (4.14), we already know that $u_*' \geq 0$. Now, we will prove that

$$u_*'(x) > 0 \text{ for any } x \in \mathbb{R}\,. \tag{4.16}$$

For this, suppose, by contradiction, that $u_*'(\bar{x}) = 0$, for some $\bar{x} \in \mathbb{R}$. Then, by differentiating the equation in (4.15), we have that

$$\begin{aligned} -\int_{\mathbb{R}} \frac{u_*'(y)}{|\bar{x}-y|^{1+2s}}\, dy &= \int_{\mathbb{R}} \frac{u_*'(\bar{x})-u_*'(y)}{|\bar{x}-y|^{1+2s}}\, dy \\ &= (-\Delta)^s u_*'(\bar{x}) \\ &= (-\Delta)^s u_*'(\bar{x}) + W''\big(u_*(\bar{x})\big)\, u_*'(\bar{x}) = 0. \end{aligned}$$

Since the integrand is non-negative, we would obtain that u_*' vanishes identically. This would give that u_* is constantly equal to c_o, due to (4.13). But then (4.15) gives that $W'(c_o) = 0$, which is in contradiction with (4.11). This proves (4.16).

Now, we prove that

$$\mathcal{G}(u_*) < +\infty. \tag{4.17}$$

Indeed, by (4.9), we get

$$\begin{aligned} \mathcal{F}(u_K, [p_K - K, p_K + K]) &= \mathcal{F}(v_K, [-K, +K]) \\ &\leq \mathcal{F}(H, [-K, +K]) \\ &\leq C_2 \cdot \begin{cases} (1+K^{2s-1}) & \text{if } s = (0,\, 1/2), \\ (1+\log K) & \text{if } s = 1/2, \\ 1 & \text{if } s \in (1/2,\, 1), \end{cases} \end{aligned}$$

This, (4.12) and Fatou Lemma imply (4.17).

Moreover, u_* is such that

$$\lim_{x\to\pm\infty} u_*(x) = \pm 1. \tag{4.18}$$

We can prove (4.18) arguing by contradiction. By (4.16), we know that there exists a_-, a_+ such that

$$-1 \leq a_- < a_+ \leq +1$$

and

$$\lim_{x\to\infty} u_*(x) = a_\pm.$$

Let us show that $a_- = -1$. Suppose, by contradiction, that

$$a_- > -1. \tag{4.19}$$

Then, we set $a_* := (a_- + a_+)/2 \in (-1, a_+)$ and we infer from (4.19) that

$$i := \inf_{[a_*, a_+]} W > 0.$$

Recalling (4.14), we have that there exists $\kappa \in \mathbb{R}$ such that, if $x \geq \kappa$, then $u_*(x) \in [a_*, a_+]$. So, from (4.17),

$$\begin{aligned} +\infty &> \mathcal{G}(u_*) \\ &\geq \lim_{R\to+\infty} (\log R)^{-1} \int_\kappa^R W(u_*)\, dx \\ &\geq \lim_{R\to+\infty} i\,(R-\kappa)\,(\log R)^{-1} \;=\; +\infty, \end{aligned}$$

and this contradiction proves that $a_- = -1$. Analogously, one proves that $a_+ = +1$. This finishes the proof of (4.18).

By (4.18) and Proposition 3.2, we obtain that

$$\mathcal{F}(u_*, [-a,a]) \leq \mathcal{F}(u_* + \phi, [-a,a]) \quad \text{for any } a>0 \text{ and any } \phi \text{ measurable and supported in } [-a,a]\ . \tag{4.20}$$

By collecting the results in (4.17), (4.18) and (4.20), we obtain (i).

Proof of (ii).

Assertion (ii) follows from the minimizing property in (i).

Proof of (iii).

Now, we prove that there exists $x_* \in \mathbb{R}$ such that

$$\mathcal{M}^{(x_*)} \text{ has only one element.} \tag{4.21}$$

For this, we consider the previously constructed minimizer u_* and we take $x_* \in \mathbb{R}$ such that $u_* \in \mathcal{M}^{(x_*)}$. Let us take $u \in \mathcal{M}^{(x_*)}$. By cutting at the levels ± 1, we see that $|u| \leq 1$. Thus, for any fixed $\varepsilon > 0$, there exists $k(\varepsilon) \in \mathbb{R}$ such that, for $k \in (-\infty, k(\varepsilon)]$, we have

$$u(x-k) + \varepsilon > u_*(x) \quad \text{for any } x \in \mathbb{R}.$$

Now we take k as large as possible with the above property; that is, we take k_ε such that

$$u(x - k_\varepsilon) + \varepsilon \geq u_*(x) \tag{4.22}$$

for any $x \in \mathbb{R}^n$ and, for any $j \geq 1$ there exist a sequence $\eta_{j,\varepsilon} \geq 0$ and points $x_{j,\varepsilon} \in \mathbb{R}$ such that

$$\lim_{j\to+\infty} \eta_{j,\varepsilon} \;=\; 0$$

and $u(x_{j,\varepsilon} - (k_\varepsilon + \eta_{j,\varepsilon})) + \varepsilon \leq u_*(x_{j,\varepsilon})$.

We observe that $x_{j,\varepsilon}$ must be a bounded sequence in j. Otherwise, if

$$\lim_{j\to+\infty} x_{j,\varepsilon} = \pm\infty,$$

then

$$\pm 1 + \varepsilon \;=\; \lim_{j\to+\infty} u(x_{j,\varepsilon} - (k_\varepsilon + \eta_{j,\varepsilon})) + \varepsilon \;\leq\; \lim_{j\to+\infty} u_*(x_{j,\varepsilon}) \;=\; \pm 1,$$

which is a contradiction.

Therefore, we may suppose that

$$\lim_{j\to+\infty} x_{j,\varepsilon} = x_\varepsilon,$$

for some $x_\varepsilon \in \mathbb{R}$. By (ii) and by Lemma 2.11, we know that u and u_* are continuous (recall 3.4), therefore

$$u(x_\varepsilon - k_\varepsilon) + \varepsilon = u_*(x_\varepsilon). \tag{4.23}$$

Thus, if we set

$$u_\varepsilon(x) := u(x - k_\varepsilon) + \varepsilon,$$

we have that $u_\varepsilon \geq u_*$, $u_\varepsilon(x_\varepsilon) = u_*(x_\varepsilon)$ and, by (ii),

$$-(-\Delta)^s u_\varepsilon(x) \;=\; -(-\Delta)^s u(x-k_\varepsilon) \;=\; W'(u(x-k_\varepsilon)) \;=\; W'(u_\varepsilon(x)-\varepsilon).$$

Consequently,

$$\begin{aligned} 0 \;&\leq\; \int_{\mathbb{R}} \frac{(u_\varepsilon - u_*)(y)}{|x_\varepsilon - y|^{1+2s}}\, dy \;=\; -(-\Delta)^s (u_\varepsilon - u_*)(x_\varepsilon) \\ &=\; W'(u_*(x_\varepsilon)-\varepsilon) - W'(u_*(x_\varepsilon)). \end{aligned} \tag{4.24}$$

Now, we claim that

$$|x_\varepsilon| \text{ is bounded.} \tag{4.25}$$

Indeed, suppose that, for some subsequence,

$$\lim_{\varepsilon\to0^+} |x_\varepsilon| = +\infty.$$

Then,

$$\lim_{\varepsilon\to0^+} u_*(x_\varepsilon) = \pm 1. \tag{4.26}$$

By taking into account hypothesis (4.3) on the potential W, we have that

$$W'(t) \geq W'(r) + c(t-r) \text{ when } r \leq t,\ r,t \in [-1,\,-1+c] \cup [+1-c,\,+1], \tag{4.27}$$

for some $c > 0$. Then, by (4.26) there exists $\varepsilon_o > 0$ such that both $u_*(x_\varepsilon)$ and $u_*(x_\varepsilon)-\varepsilon$ belong, for $\varepsilon \in (0,\varepsilon_o)$, to $[-1,\,-1+c]\cup[+1-c,\,+1]$, where $c > 0$ is the one given by (4.27). It follows

$$W'(u_*(x_\varepsilon)) \;\geq\; W'(u_*(x_\varepsilon)-\varepsilon) + c\varepsilon \;>\; W'(u_*(x_\varepsilon)-\varepsilon),$$

and this is in contradiction with (4.24). Thus (4.25) is proved.

As a consequence, we may suppose, up to subsequences, that

$$\lim_{\varepsilon\to0^+} x_\varepsilon = x_o, \tag{4.28}$$

for some $x_o \in \mathbb{R}$.

We also have that

$$|k_\varepsilon| \text{ is bounded.} \tag{4.29}$$

Indeed, if

$$\lim_{\varepsilon\to 0^+} k_\varepsilon = \pm\infty,$$

we would obtain from (4.23) and (4.28) that

$$\mp 1 \;=\; \lim_{\varepsilon\to 0^+} u(x_\varepsilon - k_\varepsilon) + \varepsilon \;=\; \lim_{\varepsilon\to 0^+} u_*(x_\varepsilon) \;=\; u_*(x_o),$$

and so, from (ii),

$$0 \;=\; W'(u_*(x_o)) \;=\; -(-\Delta)^s u_*(x_o) \;=\; \int_{\mathbb{R}} \frac{u(y) \pm 1}{|x_o - y|^{1+2s}}\, dy.$$

Since the integrand is either non-negative or non-positive, it follows that u_* is identically equal to ± 1, which is a contradiction. This proves (4.29).

Accordingly, we may suppose that

$$\lim_{\varepsilon\to 0^+} k_\varepsilon = k_o,$$

for some $k_o \in \mathbb{R}$. Hence,

$$\lim_{\varepsilon\to 0^+} (u_\varepsilon - u_*)(y) = \lim_{\varepsilon\to 0^+} u(y - k_\varepsilon) + \varepsilon - u_*(y) = u(y - k_o) - u_*(y), \quad \forall y \in \mathbb{R},$$

and so, passing to the limit in (4.24), we conclude that

$$\int_{\mathbb{R}} \frac{u(y - k_o) - u_*(y)}{|x_\varepsilon - y|^{1+2s}}\, dy \;=\; 0. \tag{4.30}$$

On the other hand, by passing to the limit in (4.22), we see that $u(x - k_o) \geq u_*(x)$ for any $x \in \mathbb{R}$, that is, the integrand in (4.30) is non-negative. Consequently,

$$u_*(x) = u(x - k_o) \text{ for any } x \in \mathbb{R}. \tag{4.31}$$

We claim that

$$k_o = 0. \tag{4.32}$$

To check this, we argue as follows. Since u belongs to $\mathcal{M}^{(x_*)}$, we have that

$$\text{if } u(x) < 0 \text{ then } x \leq x_*,$$

and that

$$\text{there exists an infinitesimal sequence } \varepsilon_j > 0 \text{ such that } u(x_* - \varepsilon_j) < 0.$$

Hence, by (4.31),

$$\text{if } u_*(x) < 0 \text{ then } x \leq x_* + k_o \tag{4.33}$$

and

(4.34)

there exists an infinitesimal sequence $\varepsilon_j > 0$ such that $u_*(x_* + k_o - \varepsilon_j) < 0$.

On the other hand, since $u_* \in \mathcal{M}^{(x_*)}$, we have that

$$\text{if } u_*(x) < 0 \text{ then } x \le x_* \tag{4.35}$$

and

(4.36)

there exists an infinitesimal sequence $\delta_j > 0$ such that $u_*(x_* - \delta_j) < 0$.

By (4.34) and (4.35), we have that $x_*+k_o-\varepsilon_j \le x_*$ and so, by passing to the limit, $k_o \le 0$. But, from (4.33) and (4.36), we have that $x_* - \delta_j \le x_* + k_o$, that is, passing to the limit $x_* \le k_o$.

The observations above prove (4.32), that is $k_o = 0$. Then, from (4.31) and (4.32), we have that $u = u_*$, and this proves (4.21).

From (4.21) we can easily deduce that the set $\mathcal{M}^{(x_o)}$ consists of only one element, for any $x_o \in \mathbb{R}$.

Take any $u \in \mathcal{M}^{(x_o)}$ and set $\tilde{u}(x) = u(x + (x_* - x_o))$ for every $x \in \mathbb{R}$. Since such translate function $\tilde{u}$ belongs to $\mathcal{M}^{(x_*)}$, it follows that $\tilde{u} \equiv u_*$. Accordingly, $u \in \mathcal{M}^{(x_o)}$ is such that $u(x) = u_*(x - (x_* - x_o))$; i.e., $\mathcal{M}^{(x_o)}$ consists of only one element. By the arbitrariness of $x_o \in \mathbb{R}$, (iii) is proved.

Proof of (iv).

First, in view of (ii) and the regularity assumptions on the function W, by Lemma 2.12 we can deduce that $u^{(0)}$ belongs to $C^2(\mathbb{R})$.

Moreover, we know from (iii) that $\mathcal{M}^{(0)}$ only consists of one element and, in the proof of (i), we built one with positive derivative (recall (4.16)). In particular such $u^{(0)}$ is continuous and strictly monotone increasing. This completes the proof of (iv).

Proof of (v).

For this, we take c as in (4.27) and we choose $\tau = c$ in Lemma 2.6. Then for any $R \ge C$, we consider the barrier w constructed in Lemma 2.6.

From (2.19), we know that there exists $K \in \mathbb{R}$ such that, if $k \in (-\infty, K]$, then $w(x - k) > u^{(0)}(x)$ for any $x \in \mathbb{R}$. We take $\bar{k}$ as large as possible with this property, i.e.

$$w(x - k) > u^{(0)}(x) \text{ for any } k < \bar{k} \text{ and any } x \in \mathbb{R} \tag{4.37}$$

and there exists an infinitesimal sequence $\eta_j \in [0, 1)$ and points $x_j \in \mathbb{R}$ for which

$$w(x_j - (\bar{k} + \eta_j)) \le u^{(0)}(x_j). \tag{4.38}$$

From (iv) and the asymptotic behavior at ∞ and the strict monotonicity of $u^{(0)}$, we know that $|u^{(0)}(x)| < 1$ for any $x \in \mathbb{R}$. Hence, by (4.38),

$$w(x_j - (\bar{k} + \eta_j)) < 1.$$

This and (2.20) gives that

$$\big|x_j - (\bar{k} + \eta_j)\big| \le R, \tag{4.39}$$

therefore

$$|x_j| \le R + |\bar{k}| + 1.$$

Thus, up to subsequence, we may suppose that

$$\lim_{j\to+\infty} x_j = \bar{x},$$

for some $\bar{x} \in \mathbb{R}$. Moreover, (4.39) implies that

$$\bar{x} - \bar{k} \in [-R, R], \tag{4.40}$$

while (4.38) and (4.37) give that $w(\bar{x} - \bar{k}) = u^{(0)}(\bar{x})$.

Thus, we set $v(x) := w(x - \bar{k}) - u^{(0)}(x)$ and we see that $v(x) \ge 0$ for any $x \in \mathbb{R}$ and $v(\bar{x}) = 0$.

Note that if $x - \bar{k} \in [-R, R]$ and $u^{(0)}(x) \in [-1, -1 + c]$, then

$$\begin{aligned} \int_{\mathbb{R}} \frac{v(y) - v(x)}{|x - y|^{1+2s}}\, dy &= \int_{\mathbb{R}} \frac{w(y - \bar{k}) - w(x - \bar{k})}{|x - y|^{1+2s}}\, dy + (-\Delta)^s u^{(0)}(x) \\ &\le \tau(1 + w(x - \bar{k})) - W'(u^{(0)}(x)) \\ &\le \tau(1 + w(x - \bar{k})) - c(u^{(0)}(x) + 1) \\ &= cv(x), \end{aligned} \tag{4.41}$$

thanks to (ii), (2.21) and (4.27).

We claim that

$$u^{(0)}(\bar{x}) > -1 + c. \tag{4.42}$$

The proof of (4.42) is by contradiction: if $u^{(0)}(\bar{x}) \in [-1, -1 + c]$ we deduce from (4.40) and (4.41) that

$$\int_{\mathbb{R}} \frac{v(y)}{|\bar{x} - y|^{1+2s}}\, dy = \int_{\mathbb{R}} \frac{v(y) - v(\bar{x})}{|\bar{x} - y|^{1+2s}}\, dy \le cv(\bar{x}) = 0.$$

Since the first integrand is non-negative, we would have that v vanishes identically, i.e. $w(x - \bar{k}) = u^{(0)}(x)$ for any $x \in \mathbb{R}^n$. But then

$$+1 = \lim_{x\to-\infty} w(x - \bar{k}) = \lim_{x\to-\infty} u^{(0)}(x) = -1$$

and this contradiction proves (4.42).

From (2.22), (4.40) and (4.42), we obtain

$$C\big(R + 1 - |\bar{x} - \bar{k}|\big)^{-2s} \ge 1 + w(\bar{x} - \bar{k}) = 1 + u^{(0)}(\bar{x}) > c,$$

hence

$$|\bar{x}-\bar{k}| \geq R - C' \tag{4.43}$$

for a suitable $C' > 0$.

We now observe that

$$\bar{x}-\bar{k} \geq 0. \tag{4.44}$$

Indeed, if, by contradiction, $\bar{x}-\bar{k} < 0$, we define $\hat{k} := 2\bar{x}-\bar{k} < \bar{k}$ and we use (4.37) to obtain

$$w(\bar{k}-\bar{x}) = w(\bar{x}-\hat{k}) > u^{(0)}(\bar{x}) = w(\bar{x}-\bar{k}).$$

Since w is even, this is a contradiction, and (4.44) is proved.

We deduce from (4.40), (4.43) and (4.44) that

$$\bar{x}-\bar{k} \in [R-C',\, R]. \tag{4.45}$$

We fix $\kappa \in \mathbb{R}$ such that $u^{(0)}(-\kappa) = -1 + c$. We remark that $-\kappa \leq \bar{x}$ and so

$$u^{(0)}(x-\kappa) \leq u^{(0)}(x+\bar{x}), \tag{4.46}$$

for any $x \in \mathbb{R}$, thanks to (4.42) and the monotonicity of $u^{(0)}$.

Now, we take any

$$y \in \left[\frac{R}{2},\, R\right]. \tag{4.47}$$

Then, by (4.45), we have that

$$\bar{x}-y-\bar{k} \in \left[-\frac{R}{2},\, \frac{R}{2}\right],$$

and so, by (2.22),

$$\begin{aligned} 1 + w(\bar{x}-y-\bar{k}) &\leq C\big(R+1-|\bar{x}-y-\bar{k}|\big)^{-2s} \\ &\leq C(R/2)^{-2s} \leq 4C\, y^{-2s}. \end{aligned}$$

By the above inequality, (4.37) and (4.46) we obtain that

$$u^{(0)}(-\kappa-y) \leq u^{(0)}(\bar{x}-y) \leq w(\bar{x}-y-\bar{k}) \leq -1+4C\, y^{-2s}$$

for any y as in (4.47).

Since κ is a constant and R may be taken arbitrarily large, this says that, when x is negative and very large,

$$u^{(0)}(x) \leq -1 + C\,|x|^{-2s},$$

with a suitably renamed $C > 0$. Analogously, one can prove that

$$u^{(0)}(x) \geq +1 - C\,|x|^{-2s}$$

when x is positive and very large, and these estimates prove the formula in (4.7).

Finally, in order to prove the estimate in (4.8), we observe that the function $(u^{(0)})'$ satisfies the following equation

$$-(-\Delta)^s (u^{(0)})'(x) = W''(u^{(0)}(x))(u^{(0)})' \quad \text{for any } x \in \mathbb{R}.$$

Then, since $\lim_{x\to\pm\infty} u^{(0)} = \pm 1$ and the C^2 potential W attains its minimum on ± 1, there exist $\alpha, \beta > 0$ such that $(u^{(0)})'$ satisfies

$$-(-\Delta)^s (u^{(0)})' \geq \alpha (u^{(0)})'(x) \quad \text{for any } x \in \mathbb{R} \setminus (-\beta, \beta).$$

Hence, if we choose $v = (u^{(0)})'$, Corollary 2.8 yields the desired estimate in (4.8).

The proof of Theorem 4.1 is complete. □

Remark 4.2. *Alternative proof of Theorem 4.1*(i)

We note that when $s \in (1/2, 1)$ the functional $\mathcal{G}$ coincides with $\mathcal{F}$ on $\mathcal{X}$. Hence, in view of Proposition 3.2 and the fact that global minimizers of $\mathcal{F}$ are solutions of the equation (1.3), we can provide an alternative proof of assertion (i) in Theorem 4.1, by showing the existence of a monotone global minimizer which satisfies the limit condition (3.2). We will prove that the following infimum

$$\gamma_1 := \inf \left\{ \mathcal{G}(v) : v : \mathbb{R} \to \mathbb{R}, \lim_{x\to\pm\infty} v(x) = \pm 1 \right\} \tag{4.48}$$

is achieved by an increasing function.

The key of the proof is given by the fact that the energy functional $\mathcal{G}$ is decreasing with respect to monotone rearrangements. The proof is adapted from [3, Theorem 2.4], in which the authors deals with a nonlocal functional deriving from Ising spin systems.

First, we recall that the energy $\mathcal{G}$ is also decreasing under truncations by -1 and $+1$ and then it is not restrictive to minimize the problem (4.48) with the additional condition $|u| \leq 1$.

We denote by X the class of all $v : \mathbb{R} \to [1, 1]$ such that $\lim\limits_{x\to\pm\infty} v(x) = \pm 1$; we denote by $X^\star$ the class of $v \in X$ such that v is increasing and $v(0) = 0$.

We claim that the infimum of $\mathcal{G}$ on X is equal to the infimum of $\mathcal{G}$ on $X^\star$.

In fact, since $X^\star \subset X$ we have $\inf\limits_{v\in X^\star} \mathcal{G}(v) \geq \inf\limits_{v\in X} \mathcal{G}(v)$, while the reverse inequality follows mainly by the fact that the singular perturbation term in the energy $\mathcal{G}$ is decreasing under monotone rearrangements (see for instance [1, Theorem 2.11] and [11, Theorem I.1] for monotonicity on the real line and on bounded intervals, respectively).

Now, we are in position to show that the infimum of $\mathcal{G}$ on $X^\star$ is achieved, by the direct method.

Take a minimizing sequence $(u_n) \subset X^\star$. Since u_n is increasing and converging to -1 and $+1$ at $\pm\infty$, its distributional derivative u_n' is a positive measure on $\mathbb{R}$ with $\|u_n'\| = |Du_n(\mathbb{R})| = 2 < \infty$, $\forall n \in \mathbb{N}$. Then there exist $u_* \in BV_{\mathrm{loc}}(\mathbb{R})$ and a subsequence (u_{n_k}) such that u_{n_k} converges to u_* almost everywhere as k goes to $+\infty$ (see for instance, [9, Helly's First Theorem]). By construction, u is increasing and satisfies $u_*(x) = 0$.

Let us show that $\lim_{x\to\pm\infty} u_*(x) = \pm 1$.

Since u_* is increasing in $[-1, 1]$, there exist $a < 0$ and $b > 0$ such that

$$\lim_{x\to-\infty} u_*(x) = a \quad \text{and} \quad \lim_{x\to+\infty} u_*(x) = b.$$

By contradiction, we assume that either $a \neq -1$ or $b \neq 1$. Then, since W is continuous and strictly positive in $(-1, 1)$, we obtain

$$\int_{\mathbb{R}} W(u_*)\, dx = +\infty.$$

This is impossible, because, by Fatou's Lemma, we have

$$\int_{\mathbb{R}} W(u)\, dx \le \liminf_{n\to+\infty} \int_{\mathbb{R}} W(u_n)\, dx \le \liminf_{n\to+\infty} \mathcal{G}(u_n) < +\infty. \tag{4.49}$$

Hence, u_* belongs to $X^\star$.

Finally, since $\mathcal{G}$ is lower semicontinuous on sequences such that $u_n \to u_*$ pointwise, the minimum problem γ_1 has a solution and this concludes the proof of Theorem 4.1(i).

It is worth mentioning that an ulterior proof of the existence of minimizers for (4.48) can be found in [10], where it was studied the 1-D functional $\tilde{\mathcal{F}}$ given by

$$\tilde{\mathcal{F}}(u) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^p} dx\, dy + \int_{\mathbb{R}} W(u)\, dx, \qquad (p > 2).$$

Our case is analogous if we take $p = 1 + 2s$, since the exponent of the term $|u(x) - u(y)|$ does not play any special role in the proof (see [10, Proposition 3.3]).

4.1. **Extending the 1D minimizer to any dimension.** When $n \ge 2$, we define

$$\varpi := \frac{1}{\left(\int_{\mathbb{R}^{n-1}} \frac{d\zeta}{(1 + |\zeta|^2)^{(n+2s)/2}}\right)^{\frac{1}{2s}}}. \tag{4.50}$$

This constant is needed just to keep track of the dependence of $(-\Delta)^s$ on the dimension, as we will see in Theorem 4.3 below. It also appears in [8] and [5].

Here, we let $u^{(0)}$ be as in Theorem 4.1 and we extend it to all the dimensions by setting, for any $x \in \mathbb{R}^n$,

$$u^*(x) = u^*(x_1, \dots, x_n) := u^{(0)}(\varpi x_n). \tag{4.51}$$

In the following theorem, we estimate the energy $\mathcal{F}(u^*)$ on the ball B_R.

Theorem 4.3. *Let u^* be defined by (4.51). Then, for any $r > 0$, we have that*

$$\mathcal{F}(u^*; B_r) \leq \mathcal{F}(u^* + \phi, B_r) \tag{4.52}$$

for any measurable ϕ supported in B_r.

Moreover, let $\mathcal{G}$ be the 1-D functional defined by (4.2) and let $u^{(0)}$ be as in Theorem 4.1, then the following results hold as $R \to +\infty$.

(i) *If $s \in (0, 1/2)$, then*

$$c_1 \leq \frac{1}{R^{n-2s}} \int_{B_R} \int_{\mathcal{C}B_R} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}} dx\, dy \leq c_2.$$

(ii) *If $s = 1/2$, then*

$$\frac{\mathcal{F}(u^*; B_R)}{R^{n-1} \log R} \to b^* \quad \text{and} \quad \frac{1}{R^{n-1} \log R} \int_{B_R} \int_{\mathcal{C}B_R} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+1}} dx\, dy \to 0.$$

(iii) *If $s \in (1/2, 1)$, then*

$$\frac{\mathcal{F}(u^*; B_R)}{R^{n-1}} \to b^* \quad \text{and} \quad \frac{1}{R^{n-1}} \int_{B_R} \int_{\mathcal{C}B_R} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}} dx\, dy \to 0,$$

where c_1 and c_2 are positive constants and $b^* = \dfrac{\omega_{n-1}}{\varpi} \mathcal{G}(u^{(0)})$.

Finally, there exists $C > 0$ such that for any $R \geq 2$ and $\delta \in (0, 1/2)$ we have

$$\mathcal{F}(u^*; B_R \setminus B_{(1-\delta)R}) \leq C \delta R^{n-1}. \tag{4.53}$$

Proof. First, we recall that, by construction, the function u^* coincides with $u^{(0)}$ along the n-th coordinate x_n. Then, Theorem 4.1 yields

$$\partial_{x_n} u^*(x) \; = \; (u^{(0)})'(x_n) \; > \; 0 \quad \forall x \in \mathbb{R}^n \tag{4.54}$$

and

$$\lim_{x_n \to \pm\infty} u^*(x', x_n) = \lim_{x_n \to \pm\infty} u^{(0)}(x_n) \; = \; \pm 1 \quad \forall x' \in \mathbb{R}^{n-1}. \tag{4.55}$$

In view of (4.54) and (4.55), it remains to show that u^* satisfies $-(-\Delta)^s u^*(x) = W'(u^*(x))$, for any $x \in \mathbb{R}^n$, and (4.52) will follow by Proposition 3.2. This is straightforward, since, by setting

$$z' := (y' - x')/|y_n - x_n| \quad \text{and} \quad z_n := \varpi y_n \tag{4.56}$$

the change of variable formula yields

$$\begin{aligned} -(-\Delta)^s u^*(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} \frac{u^{(0)}(\varpi y_n) - u^{(0)}(\varpi x_n)}{|x_n - y_n|^{n+2s} \left(1 + \frac{|x'-y'|^2}{|x_n-y_n|^2}\right)^{(n+2s)/2}} \, dy' \right] dy_n \\ &= \varpi^{2s} \int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} \frac{u^{(0)}(z_n) - u^{(0)}(\varpi x_n)}{|\varpi x_n - z_n|^{1+2s} \, (1+|z'|^2)^{(n+2s)/2}} \, dz' \right] dz_n \\ &= \int_{\mathbb{R}} \frac{u^{(0)}(z_n) - u^{(0)}(\varpi x_n)}{|\varpi x_n - z_n|^{1+2s}} \, dz_n \;=\; W'(u^{(0)}(\varpi x_n)) \\ &= W'(u^*(x)). \end{aligned}$$

Now, we will prove the claims in (i), (ii) and (iii).

We need to carefully estimate the contribution on B_R and on $\mathcal{C}B_R$ of the H^s_0 norm of the function u^*.

Let $s \in (0,1)$, we observe that by the estimate in (4.8) it follows that there exists a constant $C_1 > 0$ such that

$$\|(u^{(0)})'(x_n)\|_{L^\infty\left([x_n-(|x_n|/2), x_n+|x_n|/2]\right)} \le C_1 |x_n|^{-2(1+s)}$$

for any x_n large enough.

Accordingly, Lemma 2.3 (used here with $\rho := |x_n|/2$) gives

$$\int_{\mathbb{R}} \frac{|u^{(0)}(x_n) - u^{(0)}(y)|^2}{|x_n - y|^{1+2s}} \, dy \le C_2 |x_n|^{-2s}, \tag{4.57}$$

for any $x_n \in \mathbb{R}^n$ with $|x_n|$ large enough, for a suitable constant $C_2 > 0$.

From (4.57), we obtain that, for any $x \in \mathbb{R}^n$ with $|x_n|$ large enough,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}} \, dy &\le C_3 \int_{\mathbb{R}} \frac{|u^{(0)}(\varpi x_n) - u^{(0)}(\varpi y_n)|^2}{|x_n - y_n|^{1+2s}} \, dy_n \\ &\le C_4 |x_n|^{-2s}, \end{aligned} \tag{4.58}$$

for suitable C_3, $C_4 > 0$.

Also, if $x \in \mathbb{R}^n$ with $|x_n| \le R/2$, we have that

$$\int_{\mathcal{C}B_R} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}} \, dy \le \int_{\mathcal{C}B_R} \frac{4}{(|y|/2)^{n+2s}} \, dy \le C_5 R^{-2s} \tag{4.59}$$

for a suitable $C_5 > 0$.

Hence, for any $R \geq 4$, by (4.58) and (4.59), we get

$$\begin{aligned}
&\int_{B_R}\int_{\mathcal{C}B_R} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad \leq \int_{B_R\cap\{|x_n|\leq R/2\}}\int_{\mathbb{R}^n} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad\quad + \int_{B_R\cap\{|x_n|> R/2\}}\int_{\mathcal{C}B_R} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad \leq C_5 \int_{B_R\cap\{|x_n|\leq R/2\}} R^{-2s}\,dx + C_4\int_{B_R\cap\{|x_n|> R/2\}} |x_n|^{-2s}\,dx \\
&\qquad \leq C_6\, R^{n-2s}, \qquad (4.60)
\end{aligned}$$

for a suitable $C_6 > 0$.

Note that by (4.60) it follows

$$\text{if } s=1/2, \quad \frac{1}{R^{n-1}\log R}\int_{B_R}\int_{\mathcal{C}B_R} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}dx\,dy \;\leq\; C_6\frac{1}{\log R} \overset{R\to+\infty}{\longrightarrow} 0,$$

$$\forall s\in(1/2,\,1), \quad \frac{1}{R^{n-1}}\int_{B_R}\int_{\mathcal{C}B_R} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}dx\,dy \;\leq\; C_6\frac{1}{R^{2s-1}} \overset{R\to+\infty}{\longrightarrow} 0,$$

which shows the asymptotic behavior as R goes to infinity of the contribution in the H^s_0 norm of u^* on $\mathcal{C}B_R$, as stated in claim (ii) and (iii).

For the case $s\in(0,\,1/2)$, the estimate in (4.60) yields

$$\frac{1}{R^{n-2s}}\int_{B_R}\int_{\mathcal{C}B_R} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}dx\,dy \;\leq\; C_6, \qquad (4.61)$$

which provides an upper bound for any R large enough. Moreover, by construction of u^*, we can obtain a lower bound as follows.

$$\begin{aligned}
\int_{B_R}\int_{\mathcal{C}B_R} \frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}dx\,dy \;&\geq\; C_7\int_{B_{R/2}}\int_{\mathcal{C}B_{2R}} \frac{dx\,dy}{|x-y|^{n+2s}} \\
&\geq\; C_7\int_{B_{R/2}}dx\int_{\mathcal{C}B_{2R}}\frac{dy}{|y|^{n+2s}} \;=\; C_8R^{n-2s}, \qquad (4.62)
\end{aligned}$$

for suitable positive constants C_7 and C_8, provided that R is large enough. Hence, (4.61) together with (4.62) gives the estimates of the contribution in the H^s_0 norm of u^* on $\mathcal{C}B_R$ for the case $s\in(0,\,1/2)$ as in claim (i).

Now, notice that for any $s \in (0,1)$ using the change of variable in (4.56), $t := \varpi x_n$, $\rho = x'/R$, we have

$$
\begin{aligned}
&\frac{1}{R^{n-1}}\int_{B_R}\int_{B_R}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&=\frac{1}{R^{n-1}}\int_{B_R}\int_{B_R}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\frac{1}{R^{n-1}}\int_{B_R}\int_{\mathcal{C}B_R}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&=\frac{1}{\varpi}\int_{-\varpi R}^{\varpi R}\left[\int_{B_{\sqrt{1-(t^2/\varpi^2)}}}\Big(\int_R\frac{|u^0(t)-u^{(0)}(z_n)|^2}{|t-z_n|^{1+2s}}\,dz_n\Big)\,dx'\right]\,dt\\
&\quad-\frac{1}{R^{n-1}}\int_{B_R}\int_{\mathcal{C}B_R}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&=\frac{1}{\varpi}\int_{-\varpi R}^{\varpi R}\left[\int_{\mathbb{R}}\frac{|u^{(0)}(t)-u^{(0)}(z_n)|^2}{|t-z_n|^{1+2s}}\left(\int_{B_{\sqrt{1-(|x_n|/R)^2}}}\frac{1}{(1+|z'|^2)^{(n+2s)/2}}\right)\,dz_n\right]\,dt\\
&\quad-\frac{1}{R^{n-1}}\int_{B_R}\int_{\mathcal{C}B_R}\frac{|u^*(x)-u^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy
\end{aligned}
\tag{4.63}
$$

and

$$
\frac{1}{R^{n-1}}\int_{B_R}W(u^*(x))\,dx \;=\; \frac{\omega_{n-1}}{\varpi}\int_{-\varpi R}^{\varpi R}W'(u^{(0)}(t))\left(1-\frac{t^2}{\varpi^2R^2}\right)^{n-1}dt. \tag{4.64}
$$

We define the scaling constant λ_R depending of s as follows

$$
\lambda_R=\begin{cases}\dfrac{1}{R^{1-2s}} & \text{if } s\in(0,\,1/2),\\[2ex] \dfrac{1}{\log R} & \text{if } s=1/2,\\[2ex] 1 & \text{if } s\in(1/2,\,1).\end{cases}
$$

Thus, recalling that $\mathcal{G}(u^{(0)})$ is finite, due to Theorem 4.1(i), we make use of (4.60), (4.63), (4.64) and the Dominated Convergence Theorem to obtain that

$$\liminf_{R\to+\infty} \lambda_R R^{1-n} \left(\frac{1}{2} \int_{B_R} \int_{B_R} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}} \, dx \, dy + \int_{B_R} W(u^*(x)) \, dx \right)$$

$$= \liminf_{R\to+\infty} \lambda_R \cdot \left\{ \frac{1}{\varpi} \int_{-\varpi R}^{\varpi R} \Big[\int_{\mathbb{R}} \frac{|u^{(0)}(t) - u^{(0)}(z_n)|^2}{|t - z_n|^{1+2s}} \left(\int_{B_{\sqrt{1-(|x_n|/R)^2}}} \frac{1}{(1+|z'|^2)^{(n+2s)/2}} \right) dz_n \Big] \, dt + \frac{\omega_{n-1}}{\varpi} \int_{-\varpi R}^{\varpi R} W'(u^{(0)}(t)) \left(1 - \frac{t^2}{\varpi^2 R^2}\right)^{n-1} dt \right\}$$

$$= \frac{\omega_{n-1}}{\varpi} \mathcal{G}(u^{(0)}).$$

This completes the proof of claim (ii) and (iii).

Finally, using Lemma 2.3 with $\rho := 1$, we obtain

$$\int_{\mathbb{R}^n} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{n+2s}} \, dy \le C_9,$$

for any $x \in \mathbb{R}^n$, for a suitable $C_9 > 0$, and so

$$\mathcal{F}(u^*; B_R \setminus B_{(1-\delta)R}) \le \Big(C_9 + \sup_{r \in [-1,1]} W(r) \Big) \big| B_R \setminus B_{(1-\delta)R} \big|,$$

that is (4.53). The proof of the theorem is complete. □

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