$C^{1,\alpha}$ REGULARITY OF SOLUTIONS TO PARABOLIC MONGE-AMPÉRE EQUATIONS

PANAGIOTA DASKALOPOULOS* AND OVIDIU SAVIN**

Abstract. We study interior $C^{1,\alpha}$ regularity of viscosity solutions of the parabolic Monge-Ampére equation

$$u_t = b(x,t)\,(\det D^2 u)^p,$$

with exponent $p > 0$ and with coefficients b which are bounded and measurable. We show that when p is less than the critical power $\frac{1}{n-2}$ then solutions become instantly $C^{1,\alpha}$ in the interior. Also, we prove the same result for any power $p > 0$ at those points where either the solution separates from the initial data, or where the initial data is $C^{1,\beta}$.

1. Introduction

In this paper we investigate interior regularity of viscosity solutions of the parabolic Monge-Ampére equation

$$u_t = b(x,t)\,(\det D^2 u)^p, \tag{1.1}$$

with exponent $p > 0$ and with coefficients b which are bounded measurable and satisfy

$$\lambda \le b(x,t) \le \Lambda \tag{1.2}$$

for some fixed constants $\lambda > 0$ and $\Lambda < \infty$. We assume that the function u is convex in x and increasing in t.

Equations of the form of (1.1) appear in geometric evolution problems and in particular in the motion of a convex n-dimensional hyper-surface Σ_t^n embedded in $\mathbb{R}^{n+1}$ under Gauss curvature flow with exponent p, namely the equation

$$\frac{\partial P}{\partial t} = K^p\,\mathbf{N} \tag{1.3}$$

where each point P moves in the inward direction $\mathbf{N}$ to the surface with velocity equal to the p-power of its Gaussian curvature K. If we express the surface $\Sigma^n(t)$

$**$: Partially supported by NSF grant 0701037 and a Sloan Fellowship.

locally as a graph $x_{n+1} = u(x,t)$, with $x \in \Omega \subset \mathbb{R}^n$, then the function u satisfies the parabolic Monge-Ampére equation

$$u_t = \frac{(\det D^2 u)^p}{(1+|\nabla u|^2)^{\frac{(n+2)p-1}{2}}}. \tag{1.4}$$

Since any convex solution satisfies locally the bound $|\nabla u| \leq C$, equation (1.4) becomes of the form (1.1).

The case $p = 1$ corresponds to the well studied Gauss curvature flow which was first introduced by W. Firey in [9] as a model for the wearing process of stones. It follows from the work of Tso [15] that uniformly strictly convex hyper-surfaces will become instantly C^∞ smooth and they remain smooth up to their vanishing time T. However, convex surfaces which are not necessarily uniformly strictly convex, may not become instantly strictly convex and smooth (c.f. [12], [5]) and their regularity poses an interesting problem that we will investigate in this paper.

Equations of the form (1.3) for different powers of $p > 0$ were studied by B. Andrews in [1] (see also in [6]). He showed that when $p \leq 1/n$ any convex hyper-surface will become instantly strictly uniformly convex and smooth.

It can be seen from radially symmetric examples that, when $p > 1/n$, surfaces evolving by (1.3) or (1.1) may have a flat side that persists for some time before it disappears. These surfaces are of class C^{1,γ_p} with $\gamma_p := \frac{p}{np-1}$. Since $\gamma_p < 1$ if $p > \frac{1}{n-1}$, solutions which are not strictly convex fail, in general, to be of class $C^{1,1}$ in this range of exponents. In particular, solutions to the Gauss curvature flow ($p = 1$) with flat sides are no better than $C^{1,\frac{1}{n-1}}$ while the flat sides persist. The $C^{1,\alpha}$ regularity of solutions of (1.3) for any $p > 0$ will be addressed in this work.

In dimension $n = 2$, the regularity for the Gauss curvature flow ($p = 1$) is well understood. It follows from the work of B. Andrews in [2] that, in this case, all surfaces become instantly of class $C^{1,1}$ and remain so up to a time when they become strictly convex and therefore smooth, before they contract to a point. Also, it follows from the works of the first author with Hamilton [7] and Lee [8] that $C^{1,1}$ is the optimal regularity here, as can be seen from evolving surfaces Σ_t^2 in $\mathbb{R}^3$ with flat sides. The optimal regularity of surfaces with flat sides and interfaces was further discussed in [7, 8].

We mention that $C^{1,\alpha}$ and $W^{2,p}$ interior estimates were established by Gutiérrez and Huang in [11] for equations similar to (1.1) for $p = -1$ and by Huang and Lu for $p = \frac{1}{n}$. However, their work requires uniform convexity of the initial data and strict monotonicity of the function on the lateral boundary.

If w is a solution to the Monge-Ampére equation

$$\det D^2 w = 1, \quad x \in \Omega \subset \mathbb{R}^n,$$

then $u(x,t) = w(x) + t$ solves equation (1.1) with $b \equiv 1$ for any p. The question of regularity for the Monge-Ampére equation is closely related to the strict convexity of w. Strict convexity does not always hold in the interior as it can be seen from a classical example due to Pogorelov [14]. However, Caffarelli [3] showed that if the convex set D where w coincides with a tangent plane contains at least a line segment then all extremal points of D must lie on $\partial\Omega$. We prove the parabolic version of this result for solution of (1.1). Our result says that, if at a time t the convex set D where u equals a tangent plane contains at least a line segment then, either the extremal points of D lie on $\partial\Omega$ or $u(\cdot,t)$ coincides with the initial data on D (see Theorem 5.3). The second behavior occurs for example in those solutions with flat sides. In other words, a line segment in the graph of u at time t either originates from the boundary data at time t or from the initial data.

We prove a similar result for angles instead of line segments, which is crucial for our estimates. We show that if at a time t the solution u admits a tangent angle from below then either the set where u coincides with the edge of the angle has all extremal points on $\partial\Omega$ or the initial data has the same tangent angle from below (see Theorem 6.1).

The $C^{1,\alpha}$ regularity is closely related to understanding whether or not solutions separate instantly away from the edges of a tangent angle of the initial data. It turns out that when $p > \frac{1}{n-2}$ the set where u coincides with the edge of the angle may persist for some time (see Proposition 4.8), hence C^1 regularity does not hold in this case without further hypotheses. If $p < \frac{1}{n-2}$ we prove that, at any time t after the initial time, solutions are $C^{1,\alpha}$ in the interior of any section of $u(\cdot,t)$ which is included in Ω (see Theorem 8.1). For the critical exponent $p = \frac{1}{n-2}$ we show that solutions are C^1 with a logarithmic modulus of continuity for the gradient (see Theorem 8.2).

In the case of any power $p > 0$ we prove $C^{1,\alpha}$ estimates at all points (x,t) where u separates from the initial data (see Theorem 8.4). Also, if we assume that the initial data is $C^{1,\beta}$ in some direction e then we show that the solution is $C^{1,\alpha}$ in the same direction e for all later times (see Theorem 8.3).

In particular, our methods can be applied for solutions with flat sides. If the initial data has a flat side $D \subset \mathbb{R}^n$, then solutions are $C^{1,\alpha}$ for all later times in the interior of D. A similar statement holds for solutions that contain edges of tangent angles: they are $C^{1,\alpha}$ along the direction of the edge for all later times. To be more precise we state these results below.

Theorem 1.1. *Let u be a viscosity solution of (1.1) in $\Omega \times [0,T]$ with $u(x,0) \geq 0$ in Ω, $u(x,0) \geq 1$ on $\partial\Omega$. There exists $\alpha > 0$ depending on n, λ, Λ, p such that*

a) $u(x,t)$ is $C^{1,\alpha}$ in x at all points (x,t) with x an interior point of the set $\{u(x,0) = 0\}$ and $u(x,t) < 1$.

b) If $u(x,0) \geq |x_n|$ then $u(x,t)$ is $C^{1,\alpha}$ in the x' variables at all points $((x',0),t)$ with x' an interior point of the set $\{x' \, : \, u((x',0),0) = 0\}$ and $u(x,t) < 1$.

We finally remark that the equations (1.1) for negative and positive powers are in some sense dual to each other. Indeed, if u is a solution of (1.1) and $u^*(\xi,t)$ is the Legendre transform of $u(\cdot,t)$ then

$$u^*_t = -\tilde{b}(\xi,t)(\det D^2 u^*)^{-p}, \qquad \lambda \leq \tilde{b}(\xi,t) \leq \Lambda.$$

The paper is organized as follows. In section 2 we introduce the notation and some geometric properties of sections of convex functions. In sections 3 we derive estimates for subsolutions and supersolutions. In section 4 we discuss the separation of solutions away from constant solutions such as planes and angles. In sections 5 and 6 we discuss the geometry of lines and angles. In section 7 and 8 we quantify the results of section 6 and prove the main theorems concerning $C^{1,\alpha}$ regularity.

2. Preliminaries

We use the standard notation $B_r(x_0) := \{x \in \mathbb{R}^n \, : \, |x - x_0| < r\}$ to denote the open ball of radius r and center x_0, and we write shortly B_r for $B_r(0)$. Also, given a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote by x' the point $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Throughout the paper we refer to positive constants depending on n, λ, Λ and p as universal constants. We denote them by abuse of notation as c for small constants and C for large constants, although their values change from line to line. If a constant depends on universal constants and other parameters d, δ etc. then we denote them by $c(d,\delta)$, $C(d,\delta)$.

We use the following definition to say that a function is $C^{1,\alpha}$ in a pointwise sense.

Definition 2.1. *A function w is $C^{1,\alpha}$ at a point x_0 if there exists a linear function $l(x)$ and a constant C such that*

$$|w(x) - l(x)| \leq C|x - x_0|^{1+\alpha}.$$

A function is $C^{1,\alpha}$ in a set D if it is $C^{1,\alpha}$ at each $x \in D$.

A function is $C^{1,\alpha}$ at a point x_0 in the direction $\mathbf{e}$ if it is $C^{1,\alpha}$ at x_0 when restricted to the line $x_0 + s\mathbf{e}$, $s \in \mathbb{R}$.

Next we introduce the notion of a section. We denote by $S_h(x,t) \subset \mathbb{R}^n$ a section at height h of the function u at the point (x,t) defined by

$$S_h(x,t) := \{ \, y \in \Omega : \; u(y,t) \le u(x,t) + p_h \cdot (y-x) + h \, \},$$

for some $p_h \in \mathbb{R}^n$. Sometimes, in order to simplify the notation, we denote such sections as S_h, $S_h(t)$ whenever there is no possibility of confusion.

We define the notion of d-balanced convex set with respect to a point.

Definition 2.2 (d-balanced convex set)**.** *A convex set S with $0 \in S$ is called d-balanced with respect to the origin, if there exists a linear transformation A (which maps the origin into the origin) such that*

$$B_1 \subset A\,S \subset B_d.$$

Clearly, the notion of d-balanced set around 0 is invariant under linear transformations. Next we recall

John's lemma *Every convex set in $\mathbb{R}^n$ is C_n-balanced with respect to its center of mass, with C_n depending only on n.*

It is often convenient to consider sections at a point x that have x as center of mass. We denote such sections by $T_h(x,t)$ instead of $S_h(x,t)$. The existence of centered sections is due to Caffarelli [4].

Theorem. [Centered sections] *Let w be a convex function defined on a bounded convex domain Ω. For each $x_0 \in \Omega$, and $h > 0$ there exists a centered h-section $T_h(x_0)$ at x_0*

$$T_h(x_0) := \{w(x) < w(x_0) + h + p_h \cdot (x - x_0)\,\}$$

(for some $p_h \in \mathbb{R}^n$) which has x_0 as its center of mass.

The following simple observations follow from the definition of d-balanced sets and will be used throughout the paper.

Remark 2.3. Assume that the h-section of w,

$$S_h(x_0) = \{w(x) < w(x_0) + h + p_h \cdot (x - x_0)\},$$

is d-balanced around x_0. Then,

$$-d\,h < w(x) - (w(x_0) + h + p_h \cdot (x - x_0)) < 0, \qquad \text{in } \; S_h(x_0).$$

Also, if we assume $w \geq 0$ and $w(x_0) = 0$, then $w(x) \leq d\,h$ for all $x \in S_h(x_0)$.

Next lemma proves the existence of certain balanced sections which are compactly included in the domain of definition. It says that if we have a d-balanced section S_h which is compactly included in Ω, then we can find $C_n d$-balanced sections for all smaller heights than h that are included in S_h.

Lemma 2.4. (a) *Assume that w is a convex function defined on a set $\Omega \subset \mathbb{R}^n$ with $w(0) = 0$ such that $S_1 := \{x : w(x) < 1\} \subset\subset \Omega$ is d-balanced around 0. Then, there exists a constant $C_n > 0$ depending only on n, such that for every $h < 1$ we can find a section S_h at height h with $S_h \subset S_1$ and S_h is $C_n d$-balanced around 0.*

(b) *Let us denote by $r(x)$ the volume of the maximal ellipsoid centered at x that is included in S_1. Then, there exists a number $C_n > 0$ such that the section S_h in part* (a) *is either C_n-balanced around 0 or $r(x^*) \geq 2\,r(0)$ where x^* is the center of mass of S_h.*

Proof. a) For $h < 1$ fixed, consider the section S_h at height h that has 0 as its center of mass. If $S_h \subset S_1$ we have nothing to prove. Assume not and let's say

$$S_h = \{w(x) < h + \alpha\, e_n \cdot x\,\}, \qquad \text{for some } \alpha > 0.$$

We decrease the slope α continuously till we obtain the section $S_{h,t} := \{w < h + t\, e_n \cdot x\,\}$ for which the set

$$\{\,(x, x_{n+1}) : \; x \in \overline{S}_{h,t}, \; x_{n+1} = h + t\, e_n \cdot x\,\}$$

becomes tangent to the hyper-plane $x_{n+1} = 1$ at a point $(x_0, 1)$. We will show that $S_{h,t}$ satisfies (a) and (b).

Clearly $S_{h,t} \subset S_1$. At the point x_0 we have $x_0 \in \partial S_1$ and

$$S_1 \subset \{(x - x_0) \cdot e_n \leq 0\,\}.$$

Since S_1 is d-balanced, we may assume that $B_1 \subset S_1 \subset B_d$ hence $1 \leq x_0 \cdot e_n$. Also, $S_h \cap \{x_n = 0\} = S_{h,t} \cap \{x_n = 0\}$, hence the section $S_{h,t}$ is already C_n-balanced in $x' := (x_1, \cdots, x_{n-1})$ around 0.

Since $t \leq \alpha$, the center of mass x^* of $S_{h,t}$ satisfies $x^* \cdot e_n \leq 0$. This together with $x_0 \cdot e_n \geq 1$ and $x_0 \in \partial S_{h,t} \subset \bar{B}_d$, implies that $S_{h,t}$ is $C_n d$-balanced around 0 in all the directions.

b) If we assume that

$$-x^* \cdot e_n \le C_0(n)\, x_0 \cdot e_n \tag{2.1}$$

then we obtain that $S_{h,t}$ is $C(n, C_0(n))$ balanced with respect to 0. Assume now that (2.1) doesn't hold and denote by E the maximum volume ellipsoid centered at 0 which is included in S_1. After an affine transformation we have the following:

$$E = B_1 \subset S_1, \quad S_{h,t} \subset \{\, (x - x_0) \cdot e_n \le 0 \,\}, \quad x_0 \in \partial S_{h,t}$$

and

$$-x^* \cdot e_n > C_0(n)\, x_0 \cdot e_n \ge C_0(n)$$

which implies that $|x^*| \ge C_0(n)$. Since x^* is the center of mass of $S_{h,t}$ and $0 \in S_{h,t}$ we see from John's lemma that $(1+c_n)\, x^* \in S_{h,t} \subset S_1$. Hence if $C_0(n)$ is sufficiently large we can find an ellipsoid of volume 2 centered at x^* and included in the convex set generated by $(1 + c_n)\, x^*$ and B_1. This convex set is contained in S_1, and this concludes the proof of part (b). □

3. Estimates for subsolutions and supersolutions

In this section we use the scaling of the equation to derive estimates for viscosity subsolutions and supersolutions of

$$\lambda\, (\det D^2 u)^p \le u_t \le \Lambda\, (\det D^2 u)^p, \qquad x \in \Omega. \tag{3.1}$$

Throughout the paper we assume that u is convex in x, increasing in t and the domain Ω is convex and bounded.

Let us now introduce the scaling of equation (3.1). Given an affine transformation $A := \mathbb{R}^n \to \mathbb{R}^n$ and $h > 0, m > 0$ positive constants, the function

$$v(x,t) := \frac{1}{h}\, u(A\, x, m\, t)$$

is a solution of equation (3.1) provided

$$m = \frac{(\det A)^{2p}}{h^{np-1}}.$$

The equation is not affected by adding or subtracting a linear function in x. For this reason we write our comparison results using constant functions instead of linear functions.

Lemma 3.1. *Let u be a viscosity subsolution in B_1 i.e.*

$$u_t \le \Lambda(\det D^2 u)^p$$

with

$$u(0,0) \ge -1, \quad u(x,0) \le 0 \ \ on\ \partial B_1.$$

Then

$$u(0,t) \ge -2 \quad for\ t \ge -c,$$

with $c > 0$ universal.

Proof. If $u(0,-c) \le -2$ then, by convexity, u at time $-c$ is below the cone generated by $(0,-2)$ and ∂B_1 i.e

$$u(x,-c) \le -2 + 2|x| \quad \text{ in } B_1.$$

This implies that $u \le w$ on the boundary of the parabolic cylinder $B_1 \times [-c, 0]$ for

$$w(x,t) := m(t+c) + 2|x|^2 - \frac{3}{2} \quad \text{with } m = \Lambda 4^{np} \ .$$

Since $w_t = \Lambda(\det D^2 w)^p$ we obtain by the maximum principle that $u(0,0) \le w(0,0)$ and we reach a contradiction by choosing $c = 1/(4m)$.

□

Remark 3.2. The conclusion can be replaced by $u(0,t) \ge -(1+\delta)$ for $t \ge -c(\delta)$.

The scaling of the equation and the previous lemma give the following:

Proposition 3.3. *Assume that u is a viscosity subsolution in a convex set S with center of mass 0. If*

$$u(0,0) \ge -h, \quad u(x,0) \le 0 \ \ on\ \partial S,$$

then

$$u(0,t) \ge -2h \quad for \quad t \ge -c\, \frac{|S|^{2p}}{h^{np-1}},$$

with c universal.

Proof. From John's lemma there exists a linear transformation A such that

$$B_1 \subset A^{-1} S \quad \text{with} \quad \det A \ge c(n)|S|.$$

The proposition follows by applying Lemma 3.1 to the rescaled solution

$$v(x,t) := \frac{1}{h}\, u(Ax, m\,t), \qquad m = \frac{(\det A)^{2p}}{h^{np-1}}.$$

□

Remark 3.4. We obtain a slightly different version of Proposition 3.3 by requiring S to be only d-balanced around the origin and by replacing the conclusion by $u(0,t) \geq -(1+\delta)h$. In this case we need to take the constant $c = c(d,\delta)$ depending also on d and δ as can be seen from the proofs of Lemma 3.1 and Proposition 3.3.

Remark 3.5. In general we apply Proposition 3.3 at a point (x_0, t_0) in an h-section $S_h = S_h(x_0, t_0)$ which is d-balanced around x_0 to conclude that

$$u(x_0, t) \geq u(x_0, t_0) - h \quad \text{for} \quad t \geq t_0 - c\frac{|S_h|^{2p}}{h^{np-1}}.$$

Remark 3.6. At a given point we can apply the Proposition directly in the sections given by its tangent plane. Indeed, taking S to be the set

$$S_h = S_h(0,0) := \{u < h + P(x)\}, \quad P(x) := u(0,0) + \nabla u(0,0) \cdot x$$

we conclude that $u(x^*, t) \geq P(x^*) - 2h$ with x^* the center of mass of S_h. This, by John's lemma, implies a bound in whole S_h

$$u(x,t) \geq P(x) - C(n)\, h, \qquad \text{for all } \ x \in S_h, \ \ t \geq -c\,\frac{|S_h|^{2p}}{h^{np-1}},$$

with $C(n)$ depending only on the dimension.

Corollary 3.7. *Assume that u is a bounded subsolution of equation* (3.1) *in the cylinder $Q_1 := B_1 \times [-1, 0]$. Then, u is uniformly Hölder continuous in time t on the cylinder $Q_{1/2} := B_{1/2} \times [-1/2, 0]$, namely $u \in C^{1,\beta}(Q_{1/2})$, with $\beta = 1/(np+1)$.*

Proof. Since u is bounded on Q_1, the convexity of $u(\cdot, t)$ implies that $|\nabla u|$ is bounded by a constant M in $Q_{3/4}$. Then, by Proposition 3.3 applied in $B_h(x)$, with $x \in B_{1/2}$ and $h < 1/4$, we have

$$-2M\, h \leq u(x,t) - u(x,0) \leq 0 \qquad \text{if} \ \ - c\,\frac{|B_h(x)|^{2p}}{h^{np-1}} \leq t \leq 0. \tag{3.2}$$

Taking $t = -c_1\, h^{np+1}$, we find that for all t small enough

$$|u(x,t) - u(x,0)| \leq C(M)\, t^{1/(np+1)}$$

from which the desired result readily follows. □

As a consequence we obtain compactness of viscosity solutions.

Corollary 3.8. *A sequence of bounded solutions of* (3.1) *in $\Omega \times [-T, 0]$ has a subsequence that converges uniformly on compact sets to a solution of the same equation.*

Next we discuss the case of supersolutions.

Lemma 3.9. *Let u be a viscosity supersolution in $S \subset B_1$ i.e.*

$$u_t \geq \lambda\, (\det D^2 u)^p$$

with

$$u(x,0) \geq -1 \text{ in } S, \qquad u(x,0) \geq 0 \text{ on } \partial S.$$

Then

$$u(x,t) \geq -\frac{1}{2} \quad \text{for } t \geq C,$$

with $C > 0$ universal.

Proof. The lemma follows by comparison of our solution u with the function

$$w(x,t) = \frac{1}{2}\,(|x|^2 - 1) + \lambda\,(t - C)$$

on the cylinder $S \times [0, C]$. The function w is a solution of the equation $w_t = \lambda(\det D^2 w)^p$ and, since $S \subset B_1$, satisfies $w \leq 0$ on $\partial S(0) \times [0, C]$. In addition, by choosing $C = 1/\lambda$, we have $w(x,0) \leq -1 \leq u(x,0)$ for $x \in S$. The comparison principle implies $u(x,C) \geq w(x,C) \geq -1/2$ in S.

□

Remark 3.10. We can replace $-1/2$ by $-\delta$ in the lemma above by taking $C = C(\delta)$ depending also on δ.

Remark 3.11. If we assume that S is d-balanced around 0 and $u(0,0) = -1$, $u(x,0) = 0$ on ∂S, then the same conclusion holds by taking $C = C(d)$ depending also on d. Indeed, in this case we obtain $u(x,0) \geq -C(d)$ for all $x \in S$ and the desired conclusion follows as before.

The scaling of the equation and the previous lemma give the following:

Proposition 3.12. *Let u be a supersolution in Ω and assume*

$$u(x,0) \geq 0, \qquad \text{and } S_h := \{u(x,0) < h\} \subset\subset \Omega.$$

Then

$$u(\cdot,t) \geq \frac{h}{2}, \qquad \text{for} \quad t \geq C\, \frac{|S_h|^{2p}}{h^{np-1}},$$

with C universal.

Proof. Let A be a linear transformation such that $A^{-1}S_h \subset B_1$ so that $\det A \leq C(n)|S_h|$. We then apply the previous lemma to the re-scaled solution

$$v_h = \frac{1}{h}u(Ax, mt) - 1, \qquad m = \frac{(\det A)^{2p}}{h^{np-1}}.$$

□

Remark 3.13. In view of Remark 3.11 we obtain a version of Proposition 3.12 for sections $S_h = S_h(x_0, t_0)$ which are d-balanced around x_0 and are compactly included in Ω, and conclude that

$$u(x_0, t) \geq u(x_0, t_0) + (1-\delta)h \quad \text{for} \quad t \geq t_0 + C(\delta, d)\frac{|S_h|^{2p}}{h^{np-1}}.$$

4. Separation from constant solutions

In this section we consider the case when the solution u at the initial time $t = 0$ is above a given function w depending only on $n-1$ variables, u and w coincide at the origin, and $u > w$ on $\partial\Omega$. We investigate whether u separates from w instantaneously for positive times, i.e $u(0,t) > w(0)$ for all $t > 0$. Of particular interest is the case of angles given by $w = |x_n|$.

Throughout this section we assume that $u(x, 0) \geq 0$. For $h > 0$ we will consider the sub-level set $S_h(t)$ of our solution $u(\cdot, t)$ in Ω which is defined as

$$S_h(t) := \{x \in \Omega : \; u(x,t) < h\}.$$

We will also consider balls $B'_\rho \subset \mathbb{R}^{n-1}$, namely

$$B'_\rho := \{x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} : \; |x'| < \rho\}.$$

Proposition 4.1. *Let u be a supersolution in $\Omega \times [0, T]$ with $u \geq 0$ at $t = 0$. Assume that $S_h(0) \cap \{x_n < 2\,\beta\}$ is compactly included in Ω and is included as well in the cylinder $\{0 < x_n < 2\beta\} \times S'$ for a bounded domain $S' \subset \mathbb{R}^{n-1}$ and two positive constants $h > 0, \beta > 0$. Then,*

$$S_{h/4}(t_0) \subset \{x_n > \beta\}, \qquad \text{for } t_0 = C\,\frac{(\beta\,|S'|)^{2p}}{h^{np-1}},$$

with C universal.

Proof. We apply Proposition 3.3 for

$$\tilde{u} = u + \frac{h}{2\beta}x_n$$

and see that $\tilde{u} \geq u \geq 0$. Also $\{\tilde{u}(x, 0) < h\}$ is compactly included in Ω and is included in $\{0 < x_n < 2\beta\} \times S'$. We conclude that $\tilde{u}(x, t_0) \geq \frac{3}{4}h$ with t_0

given above. This implies that if $x_n \leq \beta$ then $u(x,t_0) \geq \tilde{u}(x,t_0) - \frac{h}{2} \geq \frac{h}{4}$ hence $S_{h/4}(t_0) \subset \{x_n > \beta\}$. □

From Proposition 4.1 we obtain the following corollary.

Corollary 4.2. *Let u be a supersolution in $\Omega \times [0,T]$ and assume that*

$$u(x,0) \geq w(x') \geq 0, \qquad u(0,0) = w(0) = 0, \quad u(x,0) > 0 \quad \text{on } \partial\Omega,$$

for a function w defined on $\mathbb{R}^{n-1}$. Suppose that w satisfies

$$\frac{a_{h_j}^{2p}}{h_j^{np-1}} \to 0, \qquad \text{for a sequence } h_j \to 0. \tag{4.1}$$

with

$$a_h := |\{w(x') < h\} \cap \pi_n(\Omega)|, \qquad \text{where } \pi_n(x) := x'.$$

Then,

$$u(0,t) > 0 \qquad \text{for any } t > 0.$$

Proof. Let $h > 0$ be small such that the sub-level sets $S_h(0)$ of u is compactly supported in Ω. Since $u \geq w$ we obtain that

$$S_h(0) \subset (\{w(x') < h\} \cap \pi_n(\Omega)) \times [b, \infty),$$

for some $b < 0$ (since $0 \in S_h(0)$). We apply Proposition 4.1 for $h_j \leq h$ (hence $S_{h_j}(0) \subset S_h(0)$), with $\beta = -b$. We conclude that

$$S_{h_j/4}(t_j) \subset \{x_n > 0\}, \qquad t_j = C\beta^{2p} \frac{a_{h_j}^{2p}}{h_j^{np-1}},$$

and obtain $u(0,t_j) \geq h_j/4 > 0$ for a sequence $t_j \to 0$. □

Remark 4.3. If $p > 1/2$ and the sequence above is bounded, then the conclusion of Corollary 4.2 still holds true.

Next we investigate the case when w is identically 0.

Proposition 4.4. *Let u be a supersolution in $\Omega \times [0,T]$ with $p \leq 1/n$. Assume that $u \geq 0$ at $t = 0$ and $u(x,0) > 0$ on $\partial\Omega$. Then, $u > 0$ in Ω for any $t > 0$.*

Proof. For $p < 1/n$ the proposition follows from Corollary 4.2.

Let $p = 1/n$. Assume that for $h > 0$ small we have $S_h(0) \subset B_\rho$ for some ρ in $0 < \rho \leq 1$, and $S_h(0)$ is compactly supported in Ω. We first show that for $\beta \in (0,\rho]$ small, we have

$$S_{h/4}(t_0) \subset B_{\rho-\beta}(0), \qquad \text{for } t_0 = C\,\beta^{1+\frac{1}{n}}. \tag{4.2}$$

To this end, we will apply Proposition 4.1 for each $x_0 \in \partial B_\rho$ in the direction $(-x_0)$. Let us assume for simplicity that $x_0 = (0, \cdots, 0, -\rho)$. Then, since $S_h(0) \subset B_\rho$, we have

$$S_h(0) \cap \{-\rho < x_n < -\rho + 2\beta\} \subset B'_{2\sqrt{\beta}} \times (-\rho, -\rho + 2\beta).$$

Applying Proposition 4.1, we obtain that

$$S_{h/4}(t_0) \subset \{x_n > -\rho + \beta\}, \qquad \text{for } t_0 = C\,(\beta\,\beta^{\frac{n-1}{2}})^{2/n}.$$

and (4.2) readily follows.

We will now use (4.2) to show that $u > 0$ for $t > 0$. Let $t > 0$ and fixed. Choose $\beta := 1/k > 0$ with k the smallest integer so that $C\,\beta^{\frac{1}{n}} \leq t$, with C the constant from (4.2). Using this β we repeat the argument above k times, starting at $\rho = 1$, to conclude that

$$S_{h/4^k}(t_0) \subset B_{1-k\,\beta}, \qquad \text{for } t_0 = C\,k\,\beta^{1+\frac{1}{n}}.$$

This shows that $S_{h/4^k}(t_0) = \emptyset$, for $t_0 = C\beta^{\frac{1}{n}} \leq t$ hence $u(\cdot, t) \geq h/4^k > 0$. □

Remark 4.5. For $p > 1/n$ there exist radial solutions with a flat side that persists for some time.

Remark 4.6. In the proof we showed in fact that if $u \geq 0$, $u(x, 0) \geq h$ on ∂B_1 then

$$u(\cdot, t) \geq h e^{-Ct^{-n}}$$

for some C universal.

In the next results we investigate the case of angles i.e when $w(x) = |x_n|$. First proposition shows that u separates instantly from the edge of the angle when the exponent $p \leq \frac{1}{n-2}$. The second proposition shows that this is not the case when $p > \frac{1}{n-2}$.

Proposition 4.7. *Assume u is a supersolution, and $p \leq \frac{1}{n-2}$. If $u(x, 0) \geq |x_n|$ and $u(x, 0) > 0$ on $\partial\Omega$, then $u > 0$ for any $t > 0$.*

Proof. If $p > \frac{1}{n-2}$ then the proposition follows from Corollary 4.2 since $a_h \leq Ch$.

Let $p = \frac{1}{n-2}$. Since $u \geq |x_n|$ we may assume without loss of generality that $S_h(0) \subset B'_1 \times [-h, h]$. For each $x'_0 \in B'_1$ we apply Proposition 4.1 in the direction $(-x'_0)$, in a manner similar to that used in Proposition 4.4, to show that

$$S_{\frac{h}{4}}(t_0) \subset B'_{1-\beta} \times [-h/4, h/4], \qquad \text{for } t_0 = C\,\frac{(\beta\,|S'|)^{2p}}{h^{np-1}}.$$

Notice that this time $|S'| = h\,|B''_{2\sqrt{\beta}}|$, where B''_r is an $n-2$ dimensional ball, hence (since $p = \frac{1}{n-2}$)

$$t_0 \geq C\,\frac{(h\,\beta^{\frac{n}{2}})^{2p}}{h^{np-1}} = C\beta^{\frac{n}{n-2}}.$$

Now the proof continues as in the proof of Proposition 4.4 and we obtain

$$u(\cdot,t) \geq he^{-Ct^{-\frac{n-2}{2}}}.$$

□

Proposition 4.8. *There exists a non-trivial solution u of equation*

$$u_t = (\det D^2 u)^p, \qquad \text{on } \mathbb{R}^n \times [0,\infty) \tag{4.3}$$

for which $u(x,0) \geq |x_n|$ and $u(0,t) = 0$, for all $t \in [0,\delta]$, for some $\delta > 0$.

Proof. We will seek for a solution u of the form

$$u(x,t) = f(t)\,v(\frac{x}{g(t)}) \tag{4.4}$$

for some functions $f = f(t)$ and $v = v(y)$. The function u satisfies (4.3) if and only if

$$(-f')\left(\frac{x}{f}\,\nabla v(\frac{x}{f}) - w\right) = f^{-n\,p}\,(\det D^2 v)^p.$$

We pick a function f which satisfies

$$-f' = f^{-n\,p}. \tag{4.5}$$

Solving (4.5) gives us

$$f(t) = [(1+n\,p)\,(T-t)]^{\frac{1}{1+np}} \tag{4.6}$$

for any constant $T > 0$. We will next show that there exists a function $v = v(y)$ such that

$$y \cdot \nabla v - v = (\det D^2 v)^p, \qquad v(y) \geq |y_n|, \qquad v(0) = 0. \tag{4.7}$$

The existence of such a function v implies the claim of our proposition. To this end, we seek for v of the form

$$v(y', y_n) = \tilde{v}(|y'|, y_n) = \varphi(|y'|)\,g(\frac{y_n}{\varphi(|y'|)}), \tag{4.8}$$

with $g(s) \geq |s|$. A direct computation shows that,

$$\tilde{v}_1 = \varphi'\,g - \varphi'\,\frac{y_n}{\varphi}\,g' = \varphi'\,(g - s\,g'), \qquad \tilde{v}_2 = g'(\frac{y_n}{\varphi}) = g'(s)$$

with $s = y_n/\varphi$. Also,

$$\tilde{v}_{11} = \varphi''\,(g - s\,g') + \varphi'\,s\,g''\,\frac{y_n}{\varphi^2}\,\varphi', \quad \tilde{v}_{12} = -\frac{\varphi'}{\varphi}\,s\,g'', \quad \tilde{v}_{22} = \frac{1}{\varphi}\,g''.$$

Using that $y_n/\varphi = s$, we get

$$y \cdot \nabla v - v = |y'|\,\varphi'\,(g - s\,g') + y_n\,g' - \varphi\,g = (|y'|\,\varphi' - \varphi)\,(g - s\,g'),$$

and also,

$$\det D^2 v = \frac{\varphi''}{\varphi}\,g''\,(g - s\,g')^{n-1}\left(\frac{\varphi'}{|y'|}\right)^{n-2}.$$

Separating the functions g and ϕ, we conclude that v satisfies (4.5), if

$$g''\,(g - s\,g')^{n-1-\frac{1}{p}} = 1 \qquad \text{and} \qquad \varphi''\left(\frac{\varphi'}{|y'|}\right)^{n-2} = (|y'|\,\varphi' - \varphi)^{\frac{1}{p}}\,\varphi.$$

For the second equation we seek for a solution in the form $\varphi(r) = C_{n,p}\,r^{\beta}$ with $\beta > 1$. We find that φ satisfies the above equation if

$$(\beta - 2)\,(n - 1) = \frac{\beta}{p} + \beta$$

which after we solve for β yields to

$$\beta = \frac{2\,(n-1)}{(n - 2 - 1/p)}.$$

Since we need $\beta > 1$, we have to restrict ourselves to the exponents $p > \frac{1}{n-2}$.

Next we find an even function g, convex of class $C^{1,\alpha}$, that solves the ODE for g in the viscosity sense and for which $g(s) = |s|$ for large values of s. Rewriting the ODE and the conditions above in terms of the Legendre transform g^* of g we find

$$(g^*)'' = |g^*|^{n-1-1/p} \quad \text{in } [-1, 1], \quad g^*(1) = g^*(-1) = 0.$$

The existence of g^* follows by scaling the negative part of any even solution $\tilde{g}$ to the ODE above, i.e $g^*(t) = a\tilde{g}(t/b)$ for appropriate constants a and b. We obtain the function g by taking the Legendre transform of g^*.

□

Remark 4.9. Proposition 4.8 shows that in the Gauss curvature flow (1.3) with exponent p, if the initial data is a cube, then the edges ($n-1$-dimensional) move instantaneously if and only if $p \leq \frac{1}{n-2}$. In the particular case of the classical Gauss curvature flow with $p = 1$, the edges of the cube move instantaneously if and only $n \leq 3$.

5. The geometry of lines

Our goal in this section is to prove Theorem 5.3, which constitutes the parabolic version of the result of Caffarelli for Monge-Ampere equation. Theorem 5.3 deals with extremal points of the set $\{ u = 0 \}$ for a nonnegative solution u of (3.1). We begin by giving the definition of an *extremal point* of a convex set (cf. in [10], Chapter 5).

Definition 5.1. *Let D be a convex subset of $\mathbb{R}^n$. The point $x_0 \in \partial D$ is an extremal point of D if x_0 is not a convex combination of other points in $\overline{D}$.*

We now give the main results of this section. The first Theorem states that a constant segment in time can be extended backward all the way to the initial data.

Theorem 5.2. *Let u be a solution of equation (3.1) on $\Omega \times [-T, 0]$. Assume $u(0,t) = 0$ for $t \in [-\delta, 0]$ and there exists a section $S_{h_0}(0) := \{u(x,0) < h_0 + p_{h_0} \cdot x\}$ at $(0,0)$ that is compactly supported in Ω. Then $u(0,t) = 0$ for all $t \in [-T, 0]$.*

The second Theorem states that if the graph of u at a given time coincides with a tangent plane in a set D that has an extremal point in Ω, and D contains at least a line segment, then u agrees with the initial data on D.

In other words, a line segment at a given time either originates from the boundary data at that particular time or from the data at the initial time.

Theorem 5.3. *Let u be a solution of equation (3.1) on $\Omega \times [-T, 0]$, for some convex domain $\Omega \subset \mathbb{R}^n$. Suppose that at time $t = 0$ we have $u \geq 0$, and the set*

$$D := \{ u(x, 0) = 0\}$$

contains a line segment and D has an extremal point in Ω. Then,

$$u(x, -T) = 0, \qquad \text{for all } x \in D.$$

As a consequence of the theorems above we obtain the following:

Corollary 5.4. *Assume u is defined in $\Omega \times [-T, 0]$ and $u(x, -T) \geq 0$ on $\partial\Omega$. Then u is strictly convex in x and strictly increasing in t at all points (x,t) that satisfy $u(x, -T) < u(x,t) < 0$.*

We first prove Theorem 5.2.

Proof of Theorem 5.2. By continuity of u the section

$$S_{h_0}(-\sigma) := \{u(x,-\sigma) < h_0 + p_{h_0} \cdot x\}$$

at $(0,-\sigma)$ is also compactly included in Ω for a small $\sigma \in [0,\delta]$. Let d be sufficiently large so that $S_{h_0}(-\sigma)$ is d-balanced around 0. By Lemma 2.4, for each $h \leq h_0$ we can find a section $S_h(-\sigma)$ which is $C_n d$-balanced around 0. We apply Proposition 3.12 (see Remark 3.13) and use that $u(0,0) - u(-\sigma,0) = 0 < h/2$ to conclude

$$\sigma \leq C(d)\frac{|S_h(-\sigma)|^{2p}}{h^{np-1}}.$$

Assume next that $u(0,-t_0) = 0$, for some $t_0 > \sigma$. We apply Proposition 3.3 (see Remark 3.4) at $(0,-t_0)$ in the set $S := S_h(-\sigma)$ and conclude

$$u(0,t) \geq -h, \quad \text{for} \quad t \geq -t_0 - c(d)\frac{|S_h(-\sigma)|^{2p}}{h^{np-1}}.$$

Using the bound on σ we find that $u(0,t) = 0$ for $t \geq -t_0 - c(d)\sigma$ and the conclusion follows. □

Next lemma is the key step in the proof of Theorem 5.3.

Lemma 5.5. *Assume $u(se_n, 0) = 0$ for $s \in [0,2]$, and for some $t_0 > 0$*

$$u(e_n,-t_0) \geq -h, \qquad T_{6h}(0,-t_0) \subset B_\delta \subset\subset \Omega,$$

where $T_{6h}(0,-t_0)$ is the centered section at 0 at time $-t_0$. Then

$$u(e_n,-Mt_0) \geq -2h, \qquad \text{with } M = 1 + c\delta^{-2p}, \qquad (c \text{ universal}).$$

Proof. Since $u(2e_n,-t_0) \leq u(2e_n,0) = 0$, the convexity of $u(\cdot,-t_0)$ implies that $u(0,-t_0) \geq -2h$. We apply Proposition 3.12 (see Remark 3.13) in the section

$$T_{6h} := T_{6h}(0,-t_0) = \{u(x,-t_0) < u(0,-t_0) + 6h + p_{6h} \cdot x\}$$

and conclude that

$$t_0 \leq C\frac{|T_{6h}|^{2p}}{h^{np-1}}.$$

Indeed, otherwise we obtain $u(0,0) \geq h$ which contradicts the hypothesis. Since $T_{6h} \subset B_\delta$ and has 0 as center of mass, we find

$$|T_{6h}| \leq C\delta|T'_{6h}|, \qquad \text{where } T'_{6h} := \{x' \in \mathbb{R}^{n-1} | \ (x',0) \in T_{6h}\},$$

for some C depending only on n. Using the inequality for t_0 we conclude

$$\frac{|T'_{6h}|^{2p}}{h^{np-1}} \geq c\delta^{-2p} t_0. \tag{5.1}$$

Now we apply Proposition 3.3 (see Remark 3.4) for the function

$$\tilde{u} = u - p'_{6h} \cdot x' - 6h$$

in the convex set S which is the convex hull generated by the $n-1$ dimensional set $T'_{6h} \times \{0\}$ and the segment $[0, 2e_n]$. Notice that $\tilde{u}$ is negative at time $-t_0$ in S and $\tilde{u}(e_n, -t_0) \geq -7h$. Since S is d-balanced with respect to e_n with d depending only on n we conclude that

$$\tilde{u}(e_n, -t) \geq -8h \quad \text{for} \quad t \geq -t_0 - c\frac{(2|T'_{6h}|)^{2p}}{h^{np-1}},$$

with c universal. Using (5.1) we find $u(e_n, t) \geq -2h$ if $t \geq -t_0(1 + c\delta^{-2p})$.

□

Proof of Theorem 5.3. Assume for simplicity that $0 \in \Omega$ is an extremal point for D and $2e_n \in D$. We want to prove that $u(2e_n, -T) = 0$.

Fix $\delta > 0$ small, smaller than a universal constant to be specified later. There exists $\sigma > 0$ depending on u and δ such that

$$T_{6h}(0, -t) \subset B_\delta \subset\subset \Omega \quad \text{for all } h, t \in [0, \sigma].$$

Indeed, otherwise we can find a sequence of h_n, t_n tending to 0 for which the inclusion above fails. In the limit we obtain that 0 can be written as a linear combination of two other points in D (one of them outside B_δ) and contradict that 0 is an extremal point.

First we show that $u(x, -\sigma) = 0$ on the line segment $[0, 2e_n]$. Using the Holder continuity of u in t at the point $(e_n, 0)$ we find that for small $t_0 > 0$,

$$u(e_n, -t_0) \geq -h := -C(u)t_0^{\frac{1}{np+1}}.$$

We can apply Lemma 5.5 inductively and conclude that as long as $M^{k-1}t_0 \leq \sigma$, $2^{k-1}h \leq \sigma$ then

$$u(e_n, -M^k t_0) \geq -2^k h.$$

We choose δ small enough so that $M = 1 + c\delta^{-2p} > 4^{np+1}$. Then

$$2^k h \leq C(u)2^{-k}(M^k t_0)^{\frac{1}{np+1}} \leq C(u)2^{-k}(M\sigma)^{\frac{1}{np+1}}.$$

This shows that if we start with t_0 small enough then $M^{k-1}t_0 \leq \sigma$ implies $2^{k-1}h \leq \sigma$ and moreover, as $t_0 \to 0$ then $2^k h \to 0$ as well. We conclude that $u(e_n, -\sigma) = 0$ hence $u(x, -\sigma) = 0$ on the line segment $[0, 2e_n]$.

Now we can use Theorem 5.2 for the points $(se_n, 0)$ for small $s \geq 0$ which are included in a compact section at the origin at time $t = 0$. Since $u(se_n, t) = 0$ for

$t \in [-\sigma, 0]$, we conclude that $u(se_n, -T) = 0$ for small s. Then convexity in x and monotonicity in t imply $u(x, -T) = 0$ on the segment $[0, 2e_n]$.

□

6. The geometry of angles

Our goal in this section is to prove the analogue of Theorem 5.3 for angles. That is, if $u : \Omega \times [-T, 0] \to \mathbb{R}$ is a solution to (3.1) for which the graph of u at time $t = 0$ has a tangent angle from below, then this angle originates either from the initial data $u(\cdot, -T)$ or from the boundary data on $\partial\Omega$ at time $t = 0$.

Throughout this section we will denote by x' points $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and by $x = (x', x_n)$ points in $\mathbb{R}^n$. Our result states as follows.

Theorem 6.1. *Let $u : \Omega \times [-T, 0] \to \mathbb{R}$ be a solution of equation (3.1) with $\Omega \subset \mathbb{R}^n$. Assume that at time $t = 0$, we have $u(0, 0) = 0$, $u(x, 0) \geq |x_n|$ and 0 is an extremal point for the set $D := \{x \in \Omega : \ u(x, 0) = 0\}$. Then, $u(x, -T) \geq |x_n|$.*

The proof of Theorem 6.1 is more involved than that of Theorem 5.3. We introduce the following convenient notation.

Definition 6.2. *For negative times $t \leq 0$ we say that*

$$(h, \alpha) \in A_t(u) \subset \mathbb{R}^2_+$$

if there exist vectors $q_1, q_2 \in \mathbb{R}^n$ such that

$$u(x, t) \geq u(0, 0) - h + \max\{q_1 \cdot x, q_2 \cdot x\}$$

in Ω and $(q_1 - q_2) \cdot e_n \geq \alpha$. Whenever there is no possibility of confusion we write A_t instead of $A_t(u)$.

Remark 6.3. The statement $(h, \alpha) \in A_t$ is in fact a one-dimensional condition on $u(x, t)$. It says that, when restricted to the line se_n, we can find a certain angle below the graph of $u(\cdot, t)$. The vertex of the angle is at distance h below $u(0, 0)$ at the origin and the difference in the slopes of the lines that form the angle is α.

Clearly, if $(h, \alpha) \in A_{t_1}$ then $(h, \alpha) \in A_t$ for all $t \geq t_1$. The statement $(h, \alpha) \in A_t$ remains true if we add to u a linear function in x or if we perform an affine transformation in the x variable that leaves e_n invariant.

Next proposition is the key step in proving Theorem 6.1 and later for obtaining interior $C^{1,\alpha}$ estimates.

Proposition 6.4. *Let u be a solution of equation* (3.1) *with* $u(0,0)=0$. *Assume that at time* $-t_0$, ($t_0>0$) *the solution u satisfies for a fixed constant C_0 and a parameter* $\delta \le 1$:

i. $(h,\alpha)\in A_{-t_0}$ *and* $(C_0\, h, (1+\delta)\,\alpha)\notin A_{-t_0}$, *and*

ii. there exists a section (at distance h from the origin)

$$S_h := \{u(x,-t_0) < h + q\cdot x\,\}$$

of $u(\cdot,-t_0)$ which is d-balanced with respect to the origin and is compactly supported in Ω.

Then,

$$\left(C_0^2\, h, \frac{\alpha}{1+\frac{\delta}{2}}\right)\in A_{-t}, \qquad for \quad t_0\le t\le t_0 + c(d)\,\delta^{-2p}\,t_0$$

for some $c(d)>0$.

Remark 6.5. From the proof we will see that we can take the constant $C_0=100$.

Proof. Since $(h,\alpha)\in A_{-t_0}$, we have $u(x,t)\ge -h+\max\{q_1\cdot x, q_2\cdot x\}$, for some vectors q_1,q_2. Without loss of generality, we may assume that q_1,q_2 have only components in the e_n direction. This reduction is possible by first subtracting the linear map $\frac{q_1+q_2}{2}\cdot x$ and then performing a linear transformation that leaves e_n invariant. Thus, assume that

$$u(x,-t_0)\ge -h+\frac{\alpha}{2}\,|x_n|.$$

Since S_h is d-balanced, the inequality above and Remark 2.3 imply that

$$S_h\subset\{\,|x_n|<4d\,\frac{h}{\alpha}\,\}.$$

Thus, if $S_h' := S_h\cap\{\,x_n=0\,\}$, we have

$$|S_h|\le Cd\,\frac{h}{\alpha}\,|S_h'|.$$

Since $u(0,0)=0$ and $u(0,-t_0)\ge -h$, Proposition 3.12 implies that

$$t_0\le C(d)\,\frac{|S_h|^{2p}}{h^{np-1}},$$

and from the previous estimate we have

$$t_0\le C(d)\,\frac{(|S_h'|\,\frac{h}{\alpha})^{2p}}{h^{np-1}}. \tag{6.1}$$

On the other hand, since $(C_0\, h, (1+\delta)\, \alpha) \notin A_{-t_0}$ there exists $s_1 e_n \in \Omega$ with $s_1 > 0$, such that

$$u(s_1\, e_n, -t_0) < -C_0\, h + \frac{\alpha}{2}\,(1+2\,\delta)\, s_1.$$

Otherwise the angle with vertex at $-C_0$ and lines of slopes $-\alpha/2$, $\alpha/2 + \delta\alpha$ would be below the graph of $u(x, -t_0)$ on the line $x = se_n$ and we reach a contradiction.

Since $u(s_1\, e_n, -t_0) \geq -h + \frac{\alpha}{2}\, s_1$, the above yields the bound

$$s_1 \geq \frac{(C_0 - 1)\, h}{\alpha\, \delta} := s_0.$$

Moreover, since $u(0, -t_0) \leq u(0,0) \leq 0$ and

$$u(s_1\, e_n, -t_0) < -C_0\, h + \frac{\alpha}{2}\,(1+2\,\delta)\, s_1 < \frac{\alpha}{2}\,(1+2\,\delta)\, s_1$$

the convexity of $u(\cdot, -t_0)$ implies that

$$u(s\, e_n, -t_0) < \frac{\alpha}{2}\,(1+2\,\delta)\, s, \qquad \forall s \in [0, s_0] \subset [0, s_1].$$

Hence, if $s \in [0, s_0]$, then

$$u(s\, e_n, -t_0) < \frac{\alpha}{2}\, s + \alpha\, \delta\, s_0 = (C_0 - 1)\, h + \frac{\alpha}{2}\, s.$$

Recalling that $S_h := \{u(x, -t_0) < h + q' \cdot x' + q_n\, x_n\}$, it follows from the above discussion that the set

$$\{u(x, -t_0) < (C_0 - 1)\, h + q' \cdot x' + \frac{\alpha}{2}\, x_n\}$$

contains the convex set $\tilde{S}$ which is generated by $S'_h := S_h \cap \{x_n = 0\}$ and the segment $[0, s_0 e_n]$. It follows from the convexity of $\tilde{S}$ that

$$|\tilde{S}| \geq c_n\, |S'_h|\, s_0 = c_n\, |S'_h|\, \frac{(C_0 - 1)\, h}{\alpha\, \delta} \tag{6.2}$$

for some universal $c_n > 0$.

We apply Proposition 3.3 (see Remark 3.4) for $\tilde{S}$ which is Cd-balanced around $s_0 e_n/2$ and with $\tilde{h} = C_0 h$, $\tilde{\delta} = 1/30$, and find that (since $C_0 \geq 100$)

$$u(\frac{s_0\, e_n}{2}, -t) \geq -h + \frac{\alpha}{2}\,\frac{s_0}{2} - \frac{C_0\, h}{30} \geq \frac{\alpha}{2}\,(1 - \frac{\delta}{5})\,\frac{s_0}{2}$$

for

$$-t_0 - c(d)\,\frac{|\tilde{S}|^{2p}}{h^{np-1}} \leq -t \leq -t_0.$$

Observing that a similar consideration holds for negative x_n and using (6.2) we conclude

$$u(\pm\frac{s_0\, e_n}{2}, -t) \geq \frac{\alpha}{2}\,(1 - \frac{\delta}{5})\,\frac{s_0}{2}$$

for

$$-t_0 - c(d)\,\frac{\left(|S_h'|\,\frac{(C_0-1)\,h}{\alpha\,\delta}\right)^{2p}}{h^{np-1}} \le -t \le -t_0,$$

or, from (6.1), for

$$-t_0 - c(d)\delta^{-2p} t_0 \le -t \le -t_0.$$

It follows that for such t we have (since $u(0,-t) \le 0$)

$$\nabla u(\pm\frac{s_0\,e_n}{2}, -t)\cdot(\pm e_n) \ge \frac{\alpha}{2}\,(1-\frac{\delta}{5}).$$

Setting

$$\tilde q_1 = \nabla u(\frac{s_0\,e_n}{2}, -t) \quad \text{and} \qquad \tilde q_2 = \nabla u(-\frac{s_0\,e_n}{2}, -t)$$

we obtain

$$(\tilde q_1 - \tilde q_2)\cdot e_n \ge \alpha\,(1-\frac{\delta}{5}) \ge \frac{\alpha}{1+\frac{\delta}{2}}$$

since $\delta \le 1$. From the convexity of $u(\cdot,-t)$ and the inequalities

$$u(s_0\,e_n, -t) \le u(s_0\,e_n, -t_0) \le \frac{\alpha}{2}\,s_0 + (C_0-1)\,h$$

$$u(\frac{s_0 e_n}{2}, -t) \ge \frac{\alpha}{2}\,\frac{s_0}{2} - \frac{C_0-1}{20}h$$

we conclude that the tangent planes at $\pm\frac{s_0\,e_n}{2}$ for $u(\cdot,-t)$ are above $-2C_0 h$ (and therefore $-C_0^2 h$) at the origin. This implies that

$$\left(C_0^2\,h, \frac{\alpha}{1+\frac{\delta}{2}}\right) \in A_{-t}, \qquad \text{if} \quad t_0 \le t \le t_0 + c(d)\,\delta^{-2p}\,t_0$$

which finishes the proof of the proposition.

□

Remark 6.6. If hypothesis ii) is satisfied only for a time $-\tilde t$ with $\tilde t \le t_0$ i.e

$$S_h := \{u(x,-\tilde t) \le h + q\cdot x\} \subset\subset \Omega \qquad \text{and } S_h \text{ is } d\text{-balanced around } 0,$$

then the same conclusion holds in the smaller time interval

$$\left(h, \frac{\alpha}{1+\frac{\delta}{2}}\right) \in A_{-t}, \qquad \text{for } \; t_0 \le t \le t_0 + c(d)\,\delta^{-2p}\,\tilde t.$$

Indeed, the only difference appears when estimating $|S_h'|$ from below: in (6.1) we have to replace the left hand side t_0 by $\tilde t$.

Remark 6.7. If x^* denotes the center of mass of the d-balanced section S_h at time $-t_0$, then it follows from the proof of Proposition 6.4 and Remark 2.3 that

$$u(x,-t) \geq u(x^*,-t_0) - C(d)\,h + \max_{i=1,2}\{\tilde{q}_i \cdot (x - x^*)\}$$

for $t_0 \leq t \leq t_0 + c(d)\,\delta^{-2p}\,t_0$, with

$$(\tilde{q}_1 - \tilde{q}_2)\cdot e_n \geq \frac{\alpha}{1+\frac{\delta}{2}}, \qquad \tilde{q}_i = \nabla u\,(\frac{s_0 e_n}{2}, -t).$$

In other words, if $\tilde{u}$ is the translation of u defined by

$$\tilde{u}(x,t) = u(x + x^*, t - t_0) - u(x^*, -t_0)$$

then

$$\left(C(d)\,h, \frac{\alpha}{1+\frac{\delta}{2}} \right) \in A_{-t}(\tilde{u}), \qquad \text{for } 0 \leq t \leq c(d)\,\delta^{-2p}\,t_0.$$

We will now proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. We will denote throughout the proof by $u_0 := u(\cdot, 0)$. Since $u_0 \geq 0$, and 0 is an extremal point for the set $D = \{u_0 = 0\}$ we can find (as in the proof of Theorem 5.3) $\sigma_0 := \sigma_0(u) > 0$ small, depending on u, such that if $0 \leq h, t \leq \sigma_0$ then the section

$$T_{h,-t} := \{u(x,-t) \leq h + q \cdot x\}$$

of $u(\cdot, -t)$ that has $x = 0$ as center of mass is compactly supported in Ω. Thus, by John's lemma $T_{h,-t}$ is C_n-balanced with respect to the origin.

Let $0 < \delta < \delta_0$ with δ_0 small universal constant to be made precise later. Without loss of generality we may assume that u_0 is tangent to $|x_n|$ on the line $x' = 0$ at the origin, i.e. we have

$$\lim_{x_n \to 0^+} \frac{u_0(0,x_n)}{x_n} = 1 \qquad \text{and} \qquad \lim_{x_n \to 0^-} \frac{u_0(0,x_n)}{x_n} = -1. \tag{6.3}$$

Hence, by taking $\sigma_1 = \sigma_1(\delta, u)$ smaller than σ_0, depending also on δ, we can assume that

$$\left(\tilde{h}, 2\,(1+\frac{\delta}{2})\right) \notin A_0, \qquad \text{for } \tilde{h} \leq \sigma_1.$$

Choose $h << \sigma_1$. Since u_0 is Lipschitz in say $B_a \subset \Omega$ with $|\nabla u_0| < 1/a$, for some small a we find (using Proposition (3.3)) that at time $-t_0$, given by

$$t_0 := c(a)\,\frac{h^{n2p}}{h^{np-1}} = c(a)\,h^{np+1}$$

the we have $u(x,-t_0) \geq u_0(x) - h$ for $x \in B_a$. This easily implies

$$(h,\alpha) \in A_{-t_0}, \qquad \alpha := 2\,(1 - \frac{1}{a}\,h). \tag{6.4}$$

Also notice that

$$\left(\tilde{h}, \alpha\,(1+\delta)\right) \notin A_0, \qquad \text{if } \ h, \tilde{h} \leq \sigma_2 = \sigma_2(a,\sigma_1).$$

We choose δ_0 such that

$$M^2 := c(C_n)\,\delta^{-2p} \geq c(C_n)\,\delta_0^{-2p} := C_0^{10(np+1)}$$

where $c(d)$ is the constant that appears in Proposition 6.4.

Lemma 6.8. *As long as $M^k\,t_0 \leq \sigma_0$ and $C_0^{3k+1}\,h \leq \sigma_2$, there exists $0 \leq m \leq k$ such that*

$$\left(C_0^{3k-m}\,h, \alpha\,\frac{1}{1+\frac{\delta}{2}} \cdots \frac{1}{1+\frac{\delta}{2^m}}\right) \in A_{-M^k\,t_0}. \tag{6.5}$$

Proof. We will use induction in k. When $k=0$ we take $m=0$ and we use (6.4). Assume now that the statement holds for k and let m be the smallest so that (6.5) holds. If $m>0$, then

$$\left(C_0^{3k-(m-1)}\,h, \alpha\,\frac{1}{1+\frac{\delta}{2}} \cdots \frac{1}{1+\frac{\delta}{2^{m-1}}}\right) \notin A_{-M^k\,t_0}.$$

Combining this with (6.5), and applying Proposition 6.4 we find that

$$\left(C_0^{3k-m+2}\,h, \alpha\,\frac{1}{1+\frac{\delta}{2}} \cdots \frac{1}{1+\frac{\delta}{2^{m+1}}}\right) \in A_{-t}, \qquad \text{if } \ t \leq M^{k+2}\,t_0$$

which proves (6.5) for the pair $(k+1, m+1)$.

If $m=0$, then $(C_0^{3k}\,h, \alpha) \in A_{-M^k\,t_0}$. On the other hand, since $C_0^{3k+1}\,h \leq \sigma_2$ we have $(C_0^{3k+1}\,h, \alpha\,(1+\delta)) \notin A_0$, thus

$$(C_0^{3k+1}\,h, \alpha\,(1+\delta)) \notin A_{-M^k\,t_0}.$$

Hence, by Proposition 6.4

$$(C_0^{3k+2}\,h, \alpha\,\frac{1}{1+\frac{\delta}{2}}) \in A_{-t}$$

for $t \leq M^{k+2}\,t_0$ which again proves (6.5) for the pair $(k+1, 1)$. This concludes the proof of the lemma. □

We will now finish the proof of the theorem. Since $M \geq C_0^{5(np+1)}$ and $t_0 = c\, h^{np+1}$ we see that for the last k for which $M^k t_0 \leq \sigma_0$ we satisfy

$$C_0^{3k+1} h \leq C_0 M^{\frac{3}{5}\frac{k}{np+1}} h \leq C(\sigma_0)\, h^{\frac{2}{5}} < \sigma_2$$

if $h << \sigma_2$ is sufficiently small. Also, if δ is chosen small, depending on σ_0 and T, for the last k we also have $M^{k+2}\, t_0 \geq T$. We conclude from the lemma above that

$$(C(\sigma_0)\, h^{\frac{2}{5}}, \alpha\, e^{-\delta}) \in A_{-T}$$

and by letting $h \to 0$ we obtain

$$(0, 2\, e^{-\delta}) \in A_{-T}.$$

Finally, letting $\delta \to 0$ we conclude that $(0,2) \in A_{-T}$ which proves the theorem. □

7. $C^{1,\alpha}$ REGULARITY - I

In the next two sections we establish $C^{1,\alpha}$ interior regularity of solutions to (3.1). They are based on quantifying the result of Theorem 6.1. In the elliptic case $C^{1,\alpha}$ regularity is obtained by a compactness argument. However, in our setting compactness methods would only give C^1 continuity for exponents $p \leq \frac{1}{n-2}$. The reason for this is that in the parabolic setting it is more delicate to normalize a solution in space and time.

The main result of this section is the following Theorem (see Definition 2.1).

Theorem 7.1. *Let u be a solution to* (3.1) *in* $\Omega \times [-T, 0]$ *and assume there exists a section of $u(x,0)$ which is d-balanced around* 0 *and is compactly supported in* Ω.

a)If the initial data $u(x,-T)$ is $C^{1,\beta}$ at 0 *in the e direction then $u(x,0)$ is $C^{1,\alpha}$ at the origin in the e direction with $\alpha = \alpha(\beta, d)$ depending on β, d and the universal constants.*

b) If $u(0,0) > u(0,-T)$ then $u(x,0)$ is $C^{1,\alpha}$ at the origin with $\alpha = \alpha(d)$ depending on d and the universal constants.

Part b) will be improved in Theorem 8.4 in which we show that α can be taken to be a universal constant. As a consequence we obtain Theorem 1.1.

Proof of Theorem 1.1. In view of Remark 2.3, at a point (x,t) for which $u(x,t) \leq c_n$, with c_n small depending only on n the centered section $T_h(x,t)$ at x, for small h, is compactly supported in Ω. Clearly $u(x,0)$ is $C^{1,1}$ at an interior point of the set $\{u(x,0) = 0\}$. Thus we can apply Theorem 7.1 with d depending only on n and

$\beta = 1$ and obtain the desired result. If $c_n < u(x,t) < 1$ then we can apply directly Theorem 8.4 and obtain the same conclusion. The second part of the theorem follows similarly.

□

The following simple lemma gives the relation between the sets A_t defined in Definition 6.2 and $C^{1,\alpha}$ regularity. Its proof is straightforward and is left to the reader.

Lemma 7.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function with $f(0) = 0$ and let q be a sub-gradient of f at $x = 0$. If, for some x, we have $f(x) - q \cdot x \geq a\,|x|^{1+\alpha}$, then*

$$(h, a^{\frac{1}{1+\alpha}}\, h^{\frac{\alpha}{1+\alpha}}) \in A(f) \tag{7.1}$$

with $h = a\,|x|^{1+\alpha}$. Conversely, if for some number h, (7.1) holds, then

$$f(x) - q \cdot x \geq \frac{a}{4^{\alpha+1}}\,|x|^{1+\alpha}$$

for some x with $|x| = 4\,(\frac{h}{a})^{\frac{1}{\alpha+1}}$.

As a consequence we obtain the following useful corollary.

Corollary 7.3. *The function $u(x,0)$ is $C^{1,\alpha}$ at 0 in the e_n direction if and only if*

$$(h, Ch^{\frac{\alpha}{\alpha+1}}) \notin A_0$$

for some large C and for all small h.

Theorem 7.1 will follow from the following lemma.

Lemma 7.4. *Assume that $u : \Omega \times [-T,0] \to \mathbb{R}$ is a solution of (3.1) such that $u(0,0) = 0$, $u(x,-T) > 1$ on $\partial\Omega$, and*

$$B_{\frac{1}{d}} \subset \{\, u(x,0) < 1 \,\} \subset \{\, u(x,-T) < 1 \,\} \subset B_1. \tag{7.2}$$

Choose $\delta_0(d)$ sufficiently small, so that

$$c(C_n d)\, \delta_0^{-2p} = C_0^{12\,(np+1)} := M, \tag{7.3}$$

where $c(C_n d)$ and C_0 are the constants from Proposition 6.4 and C_n the constant from Lemma 2.4. Assume also that $(C_0^{-k}, (1+\delta_0)^{-l}) \in A_{-t_0}$, for some $k \geq 0$ and some $0 < t_0 \leq T$.

There exists a constant $C_1(d) > 0$ such that if m_0 is an integer satisfying

$$3\,m_0 \leq k - l - C_1(d) \qquad \text{and} \quad M^{m_0}\, t_0 \geq T,$$

then

$$\left(C_0^{C_1(d)+l+3\,m_0-k}, (1+\delta_0)^{-l-C_1(d)}\right) \in A_{-T}.$$

Proof. Define $\eta : \mathbb{N} \to \mathbb{Z}$ as

$$(C_0^{-k}, (1+\delta_0)^{-\eta(k)-1}) \in A_{-t_0} \quad \text{but} \quad (C_0^{-k}, (1+\delta_0)^{-\eta(k)}) \notin A_{-t_0}.$$

Clearly,

i) η is nondecreasing i.e $\eta(k+1) \geq \eta(k)$,
ii) $\eta(0) \geq -C_1(d)$, and
iii) $\eta(k) < l$ (by assumption).

For each integer m with $0 \leq m \leq \frac{k-l-C_1(d)}{3}$, we define s_m as the largest s, $0 \leq s \leq k$ that satisfies

$$\eta(k-s) \leq l + 3m - s.$$

Notice that we satisfy the inequality above when $s = 0$ and the opposite inequality when $s = k$. We obtain:

$$\eta(k - s_m) = l + 3m - s_m, \quad \text{thus} \quad s_m - 3m \leq l + C_1(d). \tag{7.4}$$

Also, from the definition of s_m we find that $s_{m+1} \geq s_m + 3$.

Claim: *There exists* $(r_1, r_2, r_3) \in \mathbb{Z}^3$, $r_i \geq 0$, *such that*

$$\left(C_0^{r_1-k}, (1+\delta_0)^{r_2-l} \frac{1}{(1+\frac{\delta_0}{2})\cdots(1+\frac{\delta_0}{2^{r_3}})}\right) \in A_{-t_m}, \qquad t_m = M^m t_0 \tag{7.5}$$

with

$$r_1 - r_2 + r_3 = 3m, \quad r_3 \leq m, \quad r_1 + r_3 \leq s_m \, (\Leftrightarrow 0 \leq r_2 \leq s_m - 3m). \tag{7.6}$$

Proof of Claim: In order to simplify the notation, instead of (7.5) we write

$$(r_1, r_2, r_3) \in \mathcal{A}_{-t_m}$$

We will use induction on m. For $m = 0$ the claim holds from our assumption $(C_0^{-k}, (1+\delta_0)^{-l}) \in A_{-t_0}$, if $(r_1, r_2, r_3) = (0, 0, 0)$.

Assume now that the claim holds for m. Consider the pairs

$$(r_1 + s, r_2, r_3 - s), \qquad \text{if } 0 \leq s \leq r_3$$

$$(r_1 + s, r_2 + s - r_3, 0), \qquad \text{if } s \geq r_3$$

where (r_1, r_2, r_3) comes from the induction step m. When $s = 0$ the first pair belongs to $\mathcal{A}_{-t_m}$, by the induction hypothesis, and when $s = s_m - r_1$ the second pair doesn't belong to $\mathcal{A}_{-t_m}$, since for that choice of s the second pair is

$$\left(C_0^{s_m-k}, (1+\delta_0)^{-(l+3m-s_m)}\right) = \left(C_0^{-(k-s_m)}, (1+\delta_0)^{-\eta(k-s_m)}\right) \notin A_{-t_0}$$

from the definition of the function η given above. Note that for $s = r_3$ the two pairs are the same.

It follows that either there exists an $s < r_3$ such that

$$(r_1 + s, r_2, r_3 - s) \in \mathcal{A}_{-t_m} \quad \text{and} \quad (r_1 + s + 1, r_2, r_3 - s - 1) \notin \mathcal{A}_{-t_m}$$

or, there exists an $r_3 \le s < s_m - r_1$ such that

$$(r_1 + s, r_2 + s - r_3, 0) \in \mathcal{A}_{-t_m} \quad \text{and} \quad (r_1 + s + 1, r_2 + s + 1 - r_3, 0) \notin \mathcal{A}_{-t_m}.$$

In either case we can apply Proposition 6.4. Indeed, the hypothesis (7.2) and Lemma 2.4 imply the existence of a section S_h of $u(\cdot, t)$ that satisfies ii) in Proposition 6.4 for any $h \le 1$ and any $t \in [-T, 0]$. More precisely, S_h is $C_n d$-balanced section around 0 and it is compactly supported in Ω. We conclude that either $(r_1 + s + 2, r_2, r_3 - s + 1)$ for some $0 \le s < r_3$ or $(r_1 + s + 2, r_2 + s - r_3, 1)$ for some $s \ge r_3$ belongs to $\mathcal{A}_{-Mt_m}$. Notice that in both cases the sum of the first and third component is less than $s_m + 3 \le s_{m+1}$. This concludes the proof of the claim.

The lemma follows now from the claim above. Since $M^{m_0} t_0 \ge T$ and

$$r_1 \le s_{m_0} \le l + 3m_0 + C_1(d), \qquad r_2 \ge 0,$$

we conclude that

$$\left(C_0^{C_1(d)+l+3m_0-k}, (1+\delta_0)^{-l}\, e^{-\delta_0}\right) \in A_{-T}.$$

□

Remark 7.5. If we assume that hypothesis (7.2) holds only on a smaller interval $t \in [-T_1, 0]$ instead of the full interval $[-T, 0]$ then the same conclusion holds by replacing $C_1(d)$ with a constant $C_1(d, T/T_1)$.

The only difference occurs in the inductive step that shows $(r_1, r_2, r_3) \in \mathcal{A}_{-t_m}$, and we have to distinguish whether $t_m \le T_1$ or $t_m > T_1$. The case when $t_m \le T_1$ is the same and we obtain $t_{m+1} = Mt_m$ as before. In the case when $t_m > T_1$ we apply Remark 6.6 of Proposition 6.4 and obtain $t_{m+1} = t_m + MT_1$. This second case occurs at most $T/(MT_1) = C(d, T/T_1)$ times and therefore we need to replace m_0 by $m_0 + C(d, T/T_1)$.

Remark 7.6. If in the assumption (7.2) we have a constant a instead of 1 i.e

$$B_{\frac{1}{d}} \subset \{\, x : \; u(x,0) < a \,\} \subset \{\, x : \; u(x,-T) < a \,\} \subset B_1$$

then the conclusion is the same, except that $k \geq 0$ is replaced by $k \geq C(a)$ and $C_1(d)$ is replaced by $C_1(d,a)$.

Indeed, $\tilde{u}(x,t) := \frac{1}{a}\, u(x, a^{1-np}\, t)$ satisfies the assumptions of the lemma with $\tilde{t}_0 = a^{np-1}\, t_0$ and $\tilde{T} = a^{np-1}\, T$ and $(C_0^{-k+C(a)}, (1+\delta_0)^{-l-C(a)}) \in A_{-\tilde{t}_0}(\tilde{u})$, hence the conclusion of the lemma follows.

Next we prove Theorem 7.1.

Proof of Theorem 7.1. From the continuity of u we can assume that, after a linear transformation, we have the following situation: $u(0,0) = 0$, $u(x,-T_1) > 1$ on $\partial\Omega$ and

$$B_{\frac{1}{2d}} \subset \{\, u(x,0) < 1 \,\} \subset \{\, u(x,-T_1) < 1 \,\} \subset B_1$$

for some small $T_1 \in (0,T]$.

Let $k \geq 0$, l be integers such that

$$(C_0^{-k}, (1+\delta_0)^{-l}) \in A_0.$$

In view of Corollary 7.3 it suffices to show that there exists $\varepsilon := \varepsilon(d,\beta)$ small (or $\varepsilon = \varepsilon(d)$ for the second part) such that $l \geq \varepsilon k$ for all large k. Assume by contradiction that

$$l < \varepsilon k \quad \text{for a sequence of } k \to \infty.$$

Then, from the Lipschitz continuity of $u(x,0)$ in $B_{1/4d}$ and Proposition 3.3 we find (as in the proof of Theorem 6.1) that $\big(2C_0^{-k}, (1+\delta_0)^{-l} - C(d)C_0^{-k}\big) \in A_{-t_0}$ or, for k large enough

$$(C_0^{1-k}, (1+\delta_0)^{-l-1}) \in A_{-t_0} \quad \text{with} \quad t_0 := c(d)C_0^{-k(np+1)}. \tag{7.7}$$

Now we can apply Remark 7.5 and conclude that if

$$3m_0 \leq k - l - C_1 \quad \text{and} \quad M^{m_0} t_0 \geq T$$

then

$$(C_0^{C_1+3m_0+l-k}, (1+\delta_0)^{-l-C_1}) \in A_{-T} \quad \text{with } C_1 = C_1(d, T/T_1).$$

We choose $m_0 = [\frac{k}{6}]$ to be the smallest integer greater than $k/6$. Clearly both inequalities for m_0 are satisfied for k large (we assume $\varepsilon \leq 1/6$) since $M = C_0^{12(np+1)}$ and

$$M^{m_0} t_0 \geq C_0^{2k(np+1)} t_0 \to \infty \quad \text{as } k \to \infty.$$

Thus

$$(C_0^{-k/6}, (1+\delta_0)^{-2\varepsilon k}) \in A_{-T} \quad \text{for a sequence of } k \to \infty.$$

We reached a contradiction if $u(0,-T) < 0$ (we choose $\varepsilon = 1/6$).

If we assume that $u(0,-T) = 0$ and $u(x,-T)$ is $C^{1,\beta}$ at 0 in the e_n direction then it follows from Corollary 7.3,

$$\frac{\log C_0}{6} \frac{\beta}{\beta+1} \leq 2\varepsilon \log(1+\delta_0)$$

and we reach a contradiction again by choosing $\varepsilon(d,\beta)$ small. □

8. $C^{1,\alpha}$ regularity - II

In this section we prove the main estimates. Let u be a solution defined in $\Omega \times [-T,0]$ and assume that $u > l(x)$ on $\partial\Omega \times [-T,0]$ for some linear function $l(x)$. We are interested in obtaining $C^{1,\alpha}$ estimates in x at time $t=0$ in any compact set K included in the section $\{u(x,0) < l(x)\}$. Theorem 7.1 gives such estimates but with the exponent α depending also on the distance from K to $\partial\{u(x,0) < l(x)\}$ which is not desirable.

We can assume that after rescaling we are in the following situation:

$$\lambda\,(\det D^2 u)^p \leq u_t \leq \Lambda\,(\det D^2 u)^p, \qquad \text{in } \Omega \times [-T,0], \tag{8.1}$$

$$u > 1 \text{ on } \partial\Omega \times [-T,0], \qquad \Omega \subset B_1(y) \text{ for some } y \in \mathbb{R}^n, \tag{8.2}$$

$$u_0(x) := u(x,0) \text{ satisfies } u_0(0) = 0. \tag{8.3}$$

First two theorems deal with the case $p < \frac{1}{n-2}$ and $p = \frac{1}{n-2}$. In view of the results of Section 3, $C^{1,\alpha}$ (or C^1) continuity is expected for these exponents regardless of the behavior of the initial data at time $-T$.

Theorem 8.1. *Let u be a solution of* (8.1)-(8.3) *with $0 < p < \frac{1}{n-2}$ and $T \leq 1$. Then,*

$$\|u_0\|_{C^{1,\alpha}(K)} \leq C(K)\,T^{-\gamma} \quad \textit{for any set} \quad K \subset\subset \{\,u_0(x) < 1\}.$$

The constants $\alpha, \gamma > 0$ are universal (depend only on n, p, λ and Λ), and $C(K)$ depends on the universal constants and the distance between K and $\partial\{u_0(x) < 1\,\}$.

The example in Proposition 4.8 shows that the Theorem 8.1 fails when $p > \frac{1}{n-2}$. For the critical exponent $p = \frac{1}{n-2}$ we obtain a logarithmic modulus of continuity of the gradient.

Theorem 8.2. *Under the same assumptions and notation as in Theorem 8.1, if* $p = \frac{1}{n-2}$, *then*

$$|\nabla u_0(x) - \nabla u_0(y)| \le C(K)\ |\log|x-y||^{-\alpha}\, T^{-\gamma}, \qquad \forall x, y \in K.$$

Next two theorems deal with the case of general exponents $p > 0$. First theorem states that if the initial data $u(x,-T)$ is $C^{1,\beta}$ in the e direction then $u(x,0)$ is $C^{1,\alpha}$ in the e direction with $\alpha = \alpha(\beta)$.

Theorem 8.3. *Let u be a solution of* (8.1)-(8.3) *with $p > 0$. If*

$$\partial_e u(\cdot,-T) \in C^{\beta}(\bar S), \qquad S := \{\, u(x,-T) < 1\},$$

for some $\beta > 0$ small, then for any set $K \subset\subset \{\, u_0(x) < 1\}$

$$\|\partial_e u_0\|_{C^{\alpha}(K)} \le C(K) \|\partial_e u(\cdot,-T)\|_{C^{\beta}(\bar S)}.$$

The constant $\alpha = \alpha(\beta) > 0$ depends on β and the universal constants.

The second Theorem is a pointwise $C^{1,\alpha}$ estimate at points that separated from the initial data at time $-T$.

Theorem 8.4. *Let u be a solution of* (8.1)-(8.3) *with $p > 0$. If*

$$u(0,0) - u(0,-T) := a > 0$$

then, there exists $q \in \mathbb{R}^n$ for which

$$|u_0(x) - q\cdot x| \le C(a)\,|x|^{1+\alpha}$$

with α universal and $C(a)$ depends on a, the distance from 0 to $\partial\{u_0 < 1\}$) and the universal constants.

The theorems above will follow from a refinement of Lemma 7.4. We show that we may choose δ_0 universal in Lemma 7.4 and satisfy the conclusion at a point $\tilde x$ possibly different from the origin. The key step is to use the part $b)$ of Lemma 2.4.

Lemma 8.5. *Let $u : \Omega \times [-T,0] \to \mathbb{R}$ be a solution of* (3.1) *such that $u > 1$ on $\partial\Omega \times [-T,0]$ and $u(0,0) = 0$. Let E be an ellipsoid centered at the origin such that $|E| \ge 2^{-j}\,|B_1|$ and*

$$E \subset \{\, u(x,0) < 1\,\} \subset \{\, u(x,-T) < 1\,\} \subset B_1(y).$$

Let δ_0, M be universal as they appear in Lemma 7.4 for $d = C_n$ the constant from Lemma 2.4. Then, there exists a constant $C(j)$ (depending on universal constants and j) such that if $k \geq 0$, l are integers and

$$(C_0^{-k}, (1+\delta_0)^{-l}) \in A_{-t_0} \quad \textit{for some } t_0 \in (0, T]$$

and m_0 is an integer satisfying

$$3m_0 \leq k - l - C(j), \qquad M^{m_0}\, t_0 \geq C(j)\, T,$$

then we can find $\tilde{x} \in \{\, u(x, -T) < 1 \,\}$ such that

$$u(x, -T) \geq u(\tilde{x}, -\tilde{t}) - C_0^{C(j)+l+3m_0-k} + \max_{i=1,2}\{q_i \cdot (x - \tilde{x})\},$$

with

$$\tilde{t} = T - \frac{T}{C(j)} \qquad (q_2 - q_1)\cdot e_n \geq (1+\delta_0)^{-l-C(j)}.$$

Remark 8.6. Another way of stating the conclusion of the lemma is that the translation

$$\tilde{u}(x,t) := u(x+\tilde{x}, t-\tilde{t}) - u(\tilde{x}, -\tilde{t}), \qquad \tilde{t} = T - \frac{T}{C(j)} \tag{8.4}$$

satisfies

$$\left(C_0^{C(j)+l+3m_0-k}, (1+\delta_0)^{-l-C(j)}\right) \in A_{-T/C(j)}(\tilde{u}).$$

Proof. The proof is by induction in j.

The case $j = 1$ is proved in Lemma 7.4. Indeed, since $B_{1/2} \subset E \subset B_1(y) \subset B_{d/2}$ we see that the hypothesis (7.2) is satisfied and the conclusion holds for $\tilde{x} = 0$.

For a general j we start the proof as before. The only difference here is that we cannot guarantee in the induction step $m \Rightarrow m+1$ that there exists a section at time $-t_m = -M^m t_0$ which is $C_n\, d = C_n^2$ balanced around the origin.

Let's assume this fails for a first integer m. By Lemma 2.4 we can find a $C_1(j)$-balanced section (with $C_1(j) > C_n^2$) at the time $-t_m$. The idea is to apply Proposition 6.4 as in the induction step and then to "replace" the origin with the center of mass x^* of this section. To be more precise, by Remark 6.7, the translation

$$\tilde{u}(x,t) = u(x^* + x, t - t_m) - u(x^*, -t_m)$$

satisfies

$$\left(C_2(j)\, C_0^{r_1-k}, (1+\delta_0)^{r_2-l}\, e^{-\delta_0}\right) \in A_{-\tilde{t}_0}(\tilde{u})$$

with

$$\tilde{t}_0 := c_1(j)\, t_m = c_1(j)\, M^m\, t_0$$

and from (7.4)-(7.6)

$$r_1 \leq 3m + r_2, \quad 0 \leq r_2 \leq l + C_3(j).$$

Here we assumed that $T > t_m + \tilde{t}_0$, otherwise the proof is the same as before by taking $\tilde{x} = 0$, and there is no need to change the origin. Notice that $m_0 > m$ if $C(j) > 1/c_1(j)$.

The above imply

$$(C_0^{-\tilde{k}}, (1+\delta_0)^{-\tilde{l}}) \in A_{-\tilde{t}_0}(\tilde{u})$$

with

$$\tilde{l} := l - r_2 + C_1 \quad \text{and} \quad \tilde{k} = k - (3m + r_2) - C_4(j).$$

Now we apply the induction $(j-1)$-step for $\tilde{u}$. First we set

$$\tilde{m}_0 := m_0 - m \quad \text{and} \quad \tilde{T} := T - t_m,$$

and we have $\tilde{T} \geq \tilde{t}_0 \geq c_1(j) t_m \geq c_2(j)\, T$.

By Lemma 2.4 the maximal ellipsoid centered at the origin and included in the set $\{\, \tilde{u}(x,0) < \tilde{a}\}$ has volume greater than $2^{j-1}\, |B_1|$. The constant $\tilde{a} = 1 - u(x^*, -t_m)$ and by Remark 2.3, $c_3(j) \leq \tilde{a} \leq 1/c_3(j)$. Thus in order to apply the rescaled induction step for $\tilde{u}$ we need to check that (see Remark 7.6)

$$\tilde{k} \geq C'(j), \quad 3\tilde{m}_0 \leq \tilde{k} - \tilde{l} - C'(j), \quad M^{\tilde{m}_0} \tilde{t}_0 \geq \tilde{T} C'(j)$$

for some large constant $C'(j)$.

If $C(j)$ is sufficiently large then

$$M^{\tilde{m}_0}\, \tilde{t}_0 = M^{m_0 - m}\, c_1(j) M^m\, t_0 \geq C(j) c_1(j)\, T \geq C'(j)\, \tilde{T},$$

and

$$\begin{aligned} \tilde{k} - (\tilde{l} + 3\tilde{m}_0) &= k - l - 3(m + \tilde{m}_0) - C_4(j) - C_1 \\ &= (k - l - 3m_0) - C_5(j) \\ &\geq C(j) - C_5(j) \geq C'(j). \end{aligned} \tag{8.5}$$

and also,

$$\tilde{k} \geq k - (3m + l) - C_3(j) - C_4(j)$$

$$\geq k - (3m_0 + l) - C_6(j) \geq C(j) - C_6(j) \geq C'(j).$$

From the equality in (8.5), $\tilde{T} \geq c_2(j) T$ and $\tilde{l} \leq l + C_1$ we clearly obtain the desired result when we apply the induction step by choosing $C(j)$ sufficiently large. □

Remark 8.7. If in addition to the hypothesis of the lemma we have

$$u(0,0) - u(0,-T) \geq a,$$

then

$$u(\tilde{x},-\tilde{t}) - u(\tilde{x},-T) \geq \frac{a}{C(j)} - C_0^{C(j)+l+3m_0-k}, \qquad \text{for } \tilde{t} = T - \frac{T}{C(j)}.$$

This and the conclusion of the lemma imply

$$a \leq C_0^{C(j)+l+3m_0-k},$$

with $C(j)$ a constant larger than the previous ones.

Proof of Remark 8.7. From the proof of Lemma 8.5 we see that when for a certain m we replace 0 with the center of mass x^* of the section $S_h := \{\, u(x,-t_m) \leq l(x)\}$ (for l linear) with $h = C_0^{r_1-k}$, then

$$u(x^*,-t_m) - l(x^*) \geq -C(j)h.$$

On the other hand, we have

$$u(0,-T) - l(0) \leq u(0,-T) - u(0,0) \leq -a,$$

and since $u(x,-T) - l(x)$ is negative in S_h, at x^* we have

$$u(x^*,-T) - l(x) \leq -\frac{a}{C(j)}.$$

In conclusion

$$u(x^*,-t_m) \geq u(x^*,-T) + \tilde{a}, \qquad \tilde{a} := \frac{a}{C(j)} - C(j)\, h.$$

Since we perform this change of origin at most j times we obtain the desired result. □

Proof of Theorem 8.1. Let

$$(C_0^{-k}, (1+\delta_0)^{-l}) \in A_0, \qquad \text{for some } k \geq 0$$

where C_0 and δ_0 are the constants taken from Lemma 8.5. Let E be an ellipsoid of volume 2^{-j} around the origin included in the set $\{x : u_0(x) < 1\}$ where j depends on $\text{dist}(k, \partial\{u_0(x) < 1\}$. In view of Lemma 7.2, it suffices to prove the existence of constants ϵ_0 and C_1 universal and $\tilde{C}(j)$ such that

$$l \geq \epsilon_0\,(k + C_1\,\log T) - \tilde{C}(j). \tag{8.6}$$

Since our assumption $(C_0^{-k}, (1+\delta_0)^{-l}) \in A_0$ implies that $l \geq -C_0(j)$ if $k \geq 0$, it follows that (8.6) is satisfied, for some $\tilde{C}(j)$, if

$$k \leq -C_1 \, \log T + C_1(j),$$

where $C_1(j)$ will be specified later. Assume, by contradiction that (8.6) does not hold. Thus, since $T \leq 1$,

$$\epsilon_0 \, k > l, \quad \text{for some} \quad k > -C_1 \, \log T + C_1(j) \, \geq C_1(j). \tag{8.7}$$

Using the Lipschitz continuity of u_0 we obtain, as in (7.7), that

$$(C_0^{-k}, (1+\delta_0)^{-l}) \in A_0 \Rightarrow (C_0^{-k+1}, (1+\delta_0)^{-l} - C(j)\, C_0^{-k}) \in A_{-t_0}$$

with $t_0 = c(j)\, C_0^{-k(np+1)}$ which implies that

$$(C_0^{-k+1}, (1+\delta_0)^{-l-1}) \in A_{-t_0}.$$

We now apply Lemma 8.5 with $m_0 = [\, \frac{k}{6} \,]$ and check that he hypotheses are satisfied. Recall that $M = C_0^{12(np+1)}$ hence

$$M^{m_0} \, t_0 \geq c(j)\, C_0^{(12m_0-k)(np+1)} \geq C(j) \geq C(j)\, T.$$

Also, $l < \epsilon_0 k$ implies that

$$k \geq \frac{2k}{3} \geq 3m_0 + l + C(j) \tag{8.8}$$

by choosing $C_1(j)$ sufficiently large.

Thus, Lemma 8.5 holds. Now we apply the estimate (6.1) for the translation function $\tilde{u}$ of (8.4) that appears in the conclusion of Lemma 8.5. In our case

$$\tilde{h} = C_0^{C(j)+3m_0+l-k}, \qquad \tilde{\alpha} = (1+\delta_0)^{-l-C(j)}, \qquad \tilde{t}_0 = \frac{T}{C(j)}.$$

Since $S'_{\tilde{h}} \subset B_1(y)$ we have $|S'_{\tilde{h}}| \leq C$, hence

$$\tilde{h}^{1-(n-2)p} \tilde{\alpha}^{-2p} \geq \frac{T}{C_2(j)}.$$

Using (8.8) we have

$$(1-(n-2)p)(-\frac{k}{3}) \, \log C_0 + 2p\, l \, \log(1+\delta_0) \, \geq \log T - C_3(j)$$

or

$$l - 2\epsilon_0 \, k \geq C \, \log T - C_4(j), \qquad \epsilon_0 := \frac{(1-(n-2)p) \, \log C_0}{12\, p \, \log(1+\delta_0)}$$

and C universal. We obtain the inequality

$$\epsilon_0 \, k \leq C \, |\log T| + C_4(j)$$

which contradicts our assumption (8.7) if we choose the constants C_1 and $C_1(j)$ appropriately. This concludes the proof of the theorem.

□

We will next sketch the proof of Theorem 8.2 for the case $p = \frac{1}{n-2}$.

Proof of Theorem 8.2. The proof is the same as above with the difference that we need to replace k by $\log k$ in (8.6), i.e. we need to show that there exists ϵ_0 and C_1 universal such that

$$l \geq \epsilon_0 \, (\log k + C_1 \, \log T) - \tilde{C}(j). \tag{8.9}$$

After we apply Lemma 8.5 we know that the translation $\tilde{u}$ is a above an angle of opening $\tilde{\alpha}$ at time $-\tilde{t}_0$ and it separates away from it at most a distance $\tilde{h}$ at time 0. Now we use the stronger estimate (rescaled) obtained in Proposition 4.7 instead of (6.1). We find

$$\tilde{h} \geq c(j) e^{-C\tilde{\alpha}^{-1}\tilde{t}_0^{-\frac{n-2}{2}}},$$

hence

$$C_0^{C(j)+3m_0+l-k} \geq e^{-\frac{C(j)(1+\delta_0)^l}{T^C}}.$$

We obtain

$$\frac{k}{3} \leq \frac{C(j)\,(1+\delta_0)^l}{T^C},$$

or

$$l \geq 2\epsilon_0 \, \log k + C \, \log T - C(j),$$

and we finish the proof as before. □

We will now proceed with the proof of Theorem 8.3.

Proof of Theorem 8.3. We begin by observing that since $u_0(0) = 0$, then

$$T \leq C(\, \|u(\cdot, -T)\|_{L^\infty(\bar{S})}).$$

We want to prove that if $(C_0^{-k}, (1+\delta_0)^{-l}) \in A_0$, for some $k \geq 0$, then

$$l \geq \epsilon_0 \, k + C(j, \, a) \quad \text{with} \quad a := \|\partial_{e_n} u(\cdot, -T)\|_{C^\beta(\bar{S})} \tag{8.10}$$

for some ϵ_0 depending on β and universal constants. To show (8.10) we argue similarly as before. If (8.10) doesn't hold, then

$$\epsilon_0 \, k > l, \quad \text{for some} \quad k > C_1(j, a).$$

We set $m_0 = [\,\frac{k}{6}\,]$ and that the hypotheses of Lemma 8.5 are clearly satisfies. We find that

$$\left(C_0^{C(j)+3m_0+l-k}, (1+\delta_0)^{-l-C(j)}\right) \in A_{-\tilde{T}}(\tilde{u})$$

from which we conclude that

$$\left(C_0^{-\frac{k}{3}}, (1+\delta_0)^{-l-C(j)}\right) \in A_{-\tilde{T}}(\tilde{u}).$$

Using that $\partial_{e_n} u(\cdot, -T) \in C^{\beta}$ at $\tilde{x}$ we obtain

$$\frac{\log(1+\delta_0)}{\log C_0}\,(l + C(j)) \geq \frac{\beta}{\beta+1}\,\frac{k}{3} - C(j,a)$$

from which we derive a contradiction if $\epsilon_0(\beta)$ is chosen sufficiently small and $C_1(j,a)$ is chosen large. This concludes the proof of our theorem. □

We finish with the proof of Theorem 8.4.

Proof of Theorem 8.4. We use the previous notation. It suffices to show that for some ϵ_0 universal

$$l \geq \epsilon_0 k - C(j,a).$$

From Proposition 3.12 we obtain the bound $T \leq C(j,a)$. Now the proof is the same as before. In view of the Remark 8.7 our hypothesis implies that

$$C_0^{C(j)+3m_0+l-k} \geq a,$$

and the conclusion clearly follows. □

References

[1] Andrews, B., *Motion of hypersurfaces by Gauss curvature*; Pacific J. Math. 195 (2000), no. 1, 1–34.

[2] Andrews, B., *Gauss curvature flow: the fate of the rolling stones*; Invent. Math. 138 (1999), no. 1, 151–161.

[3] Caffarelli, L. A., *A localization property of viscosity solutions to the Mongé-Ampere equation and their strict convexity,* Ann. of Math. **131** (1990), 129-134.

[4] Caffarelli, L. A., *Boundary regularity of maps with convex potentials,* Comm. Pure Appl. Math. **45** (1992), no. 9, 1141–1151.

[5] Chopp, D., Evans, L. C. and Ishii, H. , *Waiting time effects for Gauss curvature flows*; Indiana Univ. Math. J. 48 (1999), 311–334.

[6] B. Chow, *Deforming convex hypersurfaces by the nth-root of the Gaussian curvature*; J. Differential Geom. 22 (1985), 117–138.

[7] Daskalopoulos, P. and Hamilton R., *The Free Boundary on the Gauss Curvature Flow with Flat Sides*; J. Reine Angenw. Math. 510 (1999), 187–227.

[8] Daskalopoulos, P. and Lee, K., *Worn stones with flat Sides: all time regularity of the interface*; Invent. Math, 156 (2004), no. 3, 445–493.

[9] Firey, W.J., *Shapes of worn stones*; Mathematika 21 (1974), 1–11.

[10] Gutierrez, C. E., *The Monge-Ampére equation*; Progress in Nonlinear Differential Equations and their Applications. 44, Birkhauser Boston, Inc., Boston, MA, (2001).

[11] Gutiérrez, C.E., and Huang, Q., *$W^{2,p}$ estimates for the parabolic Monge-Ampére Equation*; Arch. Rational Mech. Anal. 159 (2001), 137–177.

[12] Hamilton, R., *Worn stones with flat sides*; in a tribute to Ilya Bakelman, Discourses Math. Appl. 3 (1993), 69–78.

[13] Huang, Q. and Lu, G., *On a priori $C^{1,\alpha}$ and $W^{2,p}$ estimates for a parabolic Monge-Ampre equation in the Gauss curvature flows*, Amer. J. Math. 128 (2006), no. 2, 453–480.

[14] Pogorelov, A. V *The Minkowski Multidimensional Problem,* John Wiley Sons, Washington DC, 1978.

[15] Tso, K., *Deforming a Hypersurface by its Gauss-Kronecker Curvature*; Comm. Pure and Appl. Math. XXXVIII (1985), 867–882.

Department of Mathematics, Columbia University, New York, USA
E-mail address: pdaskalo@math.columbia.edu

Department of Mathematics, Columbia University, New York, USA
E-mail address: savin@math.columbia.edu