$C^{1,\alpha}$ REGULARITY FOR INFINITY HARMONIC FUNCTIONS IN TWO DIMENSIONS

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Abstract. We propose a new method for showing $C^{1,\alpha}$ regularity for solutions of the infinity Laplacian equation and provide full details of the proof in two dimensions.

The proof for dimensions $n \geq 3$ depends upon some conjectured local gradient estimates for solutions of certain transformed PDE.

1. Introduction

This paper discusses the possible local $C^{1,\alpha}$ regularity of viscosity solutions u of the *infinity Laplacian PDE*

$$-\Delta u := -u_{x_i} u_{x_j} u_{x_i x_j} = 0 \tag{1.1}$$

within an open region $U \subseteq \mathbb{R}^n$. We refer the reader to the survey paper by Aronsson, Crandall and Juutinen [A-C-J], which explains the interest in this highly degenerate and highly nonlinear elliptic PDE, and just note here that (1.1) arises as a sort of Euler-Lagrange equation for a model problem in the "calculus of variations in the sup-norm".

We say that u is "infinity harmonic" if u is a viscosity solution of (1.1), the definition of which we next review.

Viscosity solutions, comparison with cones. Let us recall here that a continuous function u is called a *viscosity solution* of the infinity Laplacian PDE (1.1) provided for each smooth function ϕ,

(i) if $u - \phi$ has a local maximum at a point $x_0 \in U$, then

$$-\Delta_\infty \phi(x_0) \leq 0,$$

and (ii) if $u - \phi$ has a local minimum at a point $x_0 \in U$, then

$$-\Delta_\infty \phi(x_0) \geq 0.$$

We will in fact rarely invoke this characterization of viscosity solutions, but rather the equivalent *comparison with cones property*, as discussed in [C-E-G]. This states that for each open set $V \subseteq U$ and each point $x_0 \notin V$,

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(i) if $u \leq c$ on ∂V for the cone $c(x) = a|x - x_0| + b$, then

$$u \leq c \quad \text{within } V;$$

and (ii) if $u \geq c$ on ∂V for the cone $c(x) = a|x - x_0| + b$, then

$$u \geq c \quad \text{within } V.$$

In these formulas a and b are real numbers.

Differentiability, $\mathbf{C^{1,\alpha}}$ regularity. Since we can "touch the graph of u from above and below by cones", it is simple to show that bounded, viscosity solutions of (1.1) are locally Lipschitz continuous within U and are consequently differentiable almost everywhere.

Furthermore, some observations in [C-E-G] and [C-E] suggest that u is in fact differentiable everywhere. These papers prove that if u is infinity harmonic within say the unit ball $B = B(0,1)$, with $u(0) = 0$, then given any small number $\lambda > 0$, there exists a small constant $\tau > 0$ such that the rescaled function

$$u_\tau(x) := \frac{u(\tau x)}{\tau}$$

satisfies

$$|u_\tau - \mathbf{e}_\tau \cdot x| \leq \lambda \|u\|_{L^\infty(B)}$$

for some appropriate vector $\mathbf{e}_\tau$. The function u is consequently well approximated by a linear functions at small length scales. Unfortunately, this assertion alone does not mean u is necessarily differentiable at 0, since the methods of [C-E-G], [C-E] definitely do not imply that $\lim_{\tau \to 0} \mathbf{e}_\tau$ exists. We have in particular no way to compare the differing vectors $\mathbf{e}_\tau$ corresponding to approximation on differing length scales.

It therefore has been a major open problem to show an infinity harmonic function is everywhere differentiable, and perhaps even C^1. The second author in [S] has recently proved C^1 regularity in $n = 2$ dimensions, but with no estimate on the modulus of continuity of the gradient Du.

This paper carries forward the regularity program by (i) proving $C^{1,\alpha}$ smoothness in $n = 2$ dimensions for some small $\alpha > 0$, and (ii) proposing a general scheme to establish $C^{1,\alpha}$ for $n \geq 3$ dimensions. We are however not able to carry out all the steps of (ii) in general, and need some as yet unproved gradient estimates for solutions of a sequence of transformed PDE.

We discuss next our general strategy.

Approximation by planes. Almost all known methods for showing $C^{1,\alpha}$ regularity (or partial regularity) for a solution u of an elliptic

PDE turn upon showing that if $u(0) = 0$ and if u is somehow approximated by the linear mapping $l = \mathbf{e} \cdot x$ on a small ball $B(0, r)$, then u can be better approximated by a slightly different linear mapping on some smaller ball $B(0, \tau r)$, where $0 < \tau < 1$.

The fundamental point is to show that the error in the approximation improves by a multiplicative factor strictly less than one. Such an estimate can then be iterated, thereby providing control on the differences between the linear approximations at different length scales.

Typically such an assertion follows from a contradiction argument, which investigates a sequence $\{v^k\}_{k=1}^{\infty}$ of isotropic rescalings of u about the point 0. However naive versions of this procedure are known to fail for the infinity Laplacian: see the discussion and counterexample constructed in [E-Y].

We instead propose here a very highly anisotropic rescaling (2.5) and blow-up procedure, replacing balls by thin cylinders, oriented along the direction of the approximate gradient. We will need a small flatness condition to begin our iteration, but this is a consequence of the conclusions of [C-E-G] and [C-E], cited above. The idea is to show that if our solution u is sufficiently close to a plane in some cylinder, then it is even closer, by a factor strictly less than one, to a slightly different plane in a smaller and slightly tilted cylinder. That we must work in such highly nonisotropic cylinders, rather than round balls, is forced by the extremely degeneracy of our elliptic PDE (1.1).

To repeat, our method proves local $C^{1,\alpha}$ regularity, *provided* we can establish Lipschitz estimates for a certain sequence of appropriately rescaled functions. These estimates unfortunately so far remain unproved for dimensions $n \geq 3$. However, the last section of the paper, due to the second author, proves the requisite estimates for $n = 2$ dimensions.

2. Rescaling and blow up

2.1 An example. To begin, let us consider in $n = 2$ dimensions the square

$$Q := \{|x_1| \leq 1, |x_2| \leq 1\}$$

and solve the infinity-Laplacian PDE (1.1) in $U := Q - (0, 0)$, subject to the boundary conditions

$$u = x_2 \quad \text{on } \partial Q, \quad u(0, 0) = \lambda,$$

for a small, positive number λ. It is not hard to see that the set $\{u > x_2\}$ is approximately a vertical strip of width $\lambda^{\frac{1}{2}}$.

This example suggests that a perturbation of size λ influences a solution only a distance $\lambda^{\frac{1}{2}}$ in a direction perpendicular to the gradient.

2.2 A model problem. Motivated by this example, consider now in $n \geq 2$ variables a solution u of the infinity Laplacian PDE in a region containing the thin cylinder

$$Q_\lambda := \{|x'| \leq \lambda^{\frac{1}{2}}, |x_n| \leq 1\}, \tag{2.1}$$

where $\lambda > 0$ is small. Here and hereafter we write

$$x = (x_1, \dots, x_n) = (x', x_n) \quad \text{for } x' = (x_1, \dots, x_{n-1}).$$

We normalize by assuming

$$u(0) = 0. \tag{2.2}$$

We assume next the *flatness condition* that our solution u is very close in the sup-norm to the plane x_n:

$$\max_{Q_\lambda} |u - x_n| \leq \lambda. \tag{2.3}$$

Our additional *Fundamental Assumption* is that (2.3) implies for any solution of the infinity Laplacian equation (1.1) the interior gradient bounds

$$\begin{aligned} \sup_{\frac{1}{2}Q_\lambda} |D'u| &\leq C\lambda^{\frac{1}{2}}, \\ \sup_{\frac{1}{2}Q_\lambda} |1 - u_{x_n}| &\leq C\lambda \end{aligned} \tag{2.4}$$

for some constant C. Here

$$D'u := (u_{x_1}, \dots, u_{x_{n-1}})$$

denotes the gradient in the variables x'; and

$$\tfrac{1}{2}Q_\lambda := \{|x'| \leq \tfrac{1}{2}\lambda^{\frac{1}{2}}, |x_n| \leq \tfrac{1}{2}\}.$$

2.3 Rescaling and blow-up. Consider next a family of functions $\{u^k\}_{k=1}^\infty$ which satisfy (2.2), (2.3) and (2.4) for a sequence $\lambda = \lambda_k \to 0$.

Define then the highly nonisotropically rescaled functions

$$v^k(x) := \frac{1}{\lambda_k}(u^k(\lambda_k^{\frac{1}{2}} x', x_n) - x_n), \tag{2.5}$$

which, owing to (2.3) and (2.4), are bounded within the standard cylinder

$$Q := \{|x'| \leq 1, |x_n| \leq 1\}$$

and are uniformly Lipschitz continuous within

$$\tfrac{1}{2}Q := \{|x'| \leq \tfrac{1}{2}, |x_n| \leq \tfrac{1}{2}\}.$$

Passing as necessary to a subsequence, we may therefore assume that as $\lambda_k \to 0$ we have

$$v^k \to v, \tag{2.6}$$

uniformly in $\frac{1}{2}Q$.

2.4 The blown-up PDE. What PDE does v satisfy?

Theorem 2.1. *The limit function v is a viscosity solution of the PDE*

$$-\sum_{i,j=1}^{n-1} v_{x_i} v_{x_j} v_{x_i x_j} - 2\sum_{i=1}^{n-1} v_{x_i} v_{x_i x_n} - v_{x_n x_n} = 0 \tag{2.7}$$

inside $\frac{1}{2}Q$.

Proof. Assume first that each function u^k is smooth. Then according to the rescaling (2.5),

$$\begin{aligned} v^k_{x_i} &= \lambda_k^{-\frac{1}{2}} u^k_{x_i}(\lambda_k^{\frac{1}{2}} x', x_n) \qquad (i = 1, \dots, n-1), \\ v^k_{x_n} &= \lambda_k^{-1}(u^k_{x_n}(\lambda_k^{\frac{1}{2}} x', x_n) - 1), \end{aligned}$$

and

$$\begin{aligned} v^k_{x_i x_j} &= u^k_{x_i x_j}(\lambda_k^{\frac{1}{2}} x', x_n) \qquad (i, j = 1, \dots, n-1), \\ v^k_{x_i x_n} &= \lambda_k^{-\frac{1}{2}} u^k_{x_i x_n}(\lambda_k^{\frac{1}{2}} x', x_n) \qquad (i = 1, \dots, n-1), \\ v^k_{x_n x_n} &= \lambda_k^{-1} u^k_{x_n x_n}(\lambda_k^{\frac{1}{2}} x', x_n). \end{aligned}$$

Since u solves the infinity Laplacian PDE (1.1), we have

$$\begin{aligned} 0 &= -u^k_{x_i} u^k_{x_j} u^k_{x_i x_j} \\ &= -\sum_{i,j=1}^{n-1} \lambda_k^{\frac{1}{2}} v^k_{x_i} \lambda_k^{\frac{1}{2}} v^k_{x_j} v^k_{x_i x_j} \\ &\quad - 2\sum_{i=1}^{n-1} \lambda_k^{\frac{1}{2}} v^k_{x_i}(1 + \lambda_k v^k_{x_n}) \lambda_k^{\frac{1}{2}} v^k_{x_i x_n} - (1 + \lambda_k v^k_{x_n})^2 \lambda_k v^k_{x_n x_n}. \end{aligned}$$

We divide by $\lambda_k > 0$ and then send $\lambda_k \to 0$, thereby formally deriving the limit PDE (2.7).

If the functions u^k are not smooth, then standard viscosity solution methods, using the foregoing calculations, let us rigorously derive that the limit v is a viscosity solution of (2.7). We do not provide details of this routine argument, other than to note that the definition of viscosity solution lets us switch from the merely Lipschitz continuous

v to a smooth function ϕ, for which the preceding calculations are justified. □

2.5 Comparison with singular solutions. We next recast the comparison with cones property for infinity harmonic functions into a comparison property with certain singular solutions of the blown-up PDE (2.7), having the form

$$s(x) := ax_n + \frac{|x'|^2}{2x_n} \qquad \text{for } x_n \neq 0, \tag{2.8}$$

for $a \in \mathbb{R}$. A direct calculation shows that s does indeed solve (2.7) where $x_n \neq 0$.

Define for a fixed constant μ and small $r > 0$ the upper cylinder

$$C^+(r) := \{|x'| \leq \mu r, 0 \leq x_n \leq r\} \tag{2.9}$$

and the lower cylinder

$$C^-(r) := \{|x'| \leq \mu r, -r \leq x_n \leq 0\}. \tag{2.10}$$

We will always take $r > 0$ so small that $C^{\pm}(r) \subset \frac{1}{2}Q$.

Theorem 2.2. *Assume that v is a viscosity solution of* (2.7) *within the cylinder $\frac{1}{2}Q$.*

(i) *If*

$$v \leq s \quad \textit{on } \partial C^+(r),$$

then

$$v \leq s \quad \textit{within } C^+(r).$$

(ii) *Similarly, if*

$$v \geq s \quad \textit{on } \partial C^-(r),$$

then

$$v \geq s \quad \textit{within } C^-(r).$$

In other words, we have comparison from above by the singular solutions s in the small upper cylinders $C^+(r)$, and comparison from below in the lower cylinders $C^-(r)$.

Note that since $s \to \pm\infty$ as $x_n \to \pm 0$ for $x' \neq 0$, we need only check the value of $v(0)$ to see if v lies below or above s on $\partial C^{\pm}(r) \cap \{x_n = 0\}$. Furthermore, it makes no sense to talk about comparison from below by s on $C^+(r)$ or from above on $C^-(r)$.

Proof. Since each infinity-harmonic function u^k satisfies comparison with cone functions of the form

$$c(x) := (1 + a\lambda_k)|x|,$$

the rescaled functions v^k, defined by (2.5), satisfy comparison with the rescaled functions

$$c^k(x) := \frac{1}{\lambda_k}(c(\lambda_k^{\frac{1}{2}}x', x_n) - x_n).$$

We consequently deduce that for $x_n > 0$, the limit v satisfies comparison from above with the function

$$\begin{aligned}\lim_{k\to\infty} c^k(x) &:= \lim_{k\to\infty} \frac{1}{\lambda_k}\left((1+a\lambda_k)(\lambda_k|x'|^2 + x_n^2)^{\frac{1}{2}} - x_n\right) \\ &= \lim_{k\to\infty} \frac{x_n}{\lambda_k}\left((1+a\lambda_k)(1+\lambda_k\frac{|x'|^2}{x_n^2})^{\frac{1}{2}} - 1\right) \\ &= ax_n + \frac{|x'|^2}{2x_n} = s(x).\end{aligned}$$

We likewise see that v satisfies comparison with s from below if $x_n < 0$. □

3. Linear approximation

Our next goal is proving that a Lipschitz solution v of the blown-up PDE

$$-\sum_{i,j=1}^{n-1} v_{x_i}v_{x_j}v_{x_ix_j} - 2\sum_{i=1}^{n-1} v_{x_i}v_{x_ix_n} - v_{x_nx_n} = 0 \tag{3.1}$$

in $\frac{1}{2}Q$ can on each small cylinder

$$\tau Q = \{|x'| \le \tau, |x_n| \le \tau\}$$

be uniformly approximated by a linear function $l = \mathbf{e}_\tau \cdot x$. This is a kind of analog of assertions from the earlier papers [C-E-G] and [C-E] about infinity harmonic functions.

We assume hereafter the max-norm bound on the solution

$$\max_Q |v| \le 1 \tag{3.2}$$

and the interior gradient bound

$$\sup_{\frac{1}{2}Q} |Dv| \le C \tag{3.3}$$

for some constant C. We suppose also

$$v(0) = 0. \tag{3.4}$$

Note that our blow-up limit from Section 2 satisfies these hypothesis, provided the Fundamental Assumption is valid.

3.1 Comparison with singular solutions. We start by modifying from [C-E-G] and [C-E] some comparison with cones methods, but working instead with the singular solutions introduced above at (2.8).

For small $r > 0$, define

$$T^+(r) := \max_{|x'|\leq\frac{1}{2}} \frac{1}{r}\left[v(x',r) - \frac{|x'|^2}{2r}\right] \tag{3.5}$$

and

$$T^-(-r) := \max_{|x'|\leq\frac{1}{2}} \frac{1}{r}\left[-v(x',-r) - \frac{|x'|^2}{2r}\right]. \tag{3.6}$$

In view of (3.3) and (3.4), we have

$$|T^{\pm}(\pm r)| \leq C; \tag{3.7}$$

and furthermore the maxima in (3.5), (3.6) are attained at points $x' = x'(\pm r)$ satisfying

$$|x'(\pm r)| \leq Cr. \tag{3.8}$$

Theorem 3.1. (i) *The mappings $r \mapsto T^{\pm}(\pm r)$ are nondecreasing, and consequently the limits*

$$T^{\pm}(0) := \lim_{r\to 0} T^{\pm}(\pm r) \tag{3.9}$$

exist.

(ii) *Furthermore,*

$$T^+(0) = T^-(0). \tag{3.10}$$

We make no assertion about the sign of $T^+(0) = T^-(0)$.

Proof. 1. Let $\mu > 0$ be a large constant, to be selected later. Within the upper cylinder $C^+(r)$, defined by (2.9), we set

$$s(x) := T^+(r)x_n + \frac{|x'|^2}{2x_n}.$$

This is a singular solution of (2.7), having the requisite form (2.8) to which the comparison Theorem 2.2 applies.

On the top $\{x_n = r\}$ of the cylinder $C^+(r)$, the definition (3.5) implies

$$v(x',r) \leq T^+(r)r + \frac{|x'|^2}{2r} = s(x',r).$$

On the bottom $\{x_n = 0\}$, we note that $s \equiv \infty$, except at 0. On the vertical sides $\{|x'| = \mu r\}$, we have

$$v(x) \leq -Cr + \frac{\mu^2 r}{2} \leq T^+(r)x_n + \frac{\mu^2 r^2}{2x_n} = s(x),$$

provided we recall (3.3), (3.4) and (3.7) and fix μ sufficiently large.

According to Theorem 2.2, it follows that

$$v \leq s$$

inside $C^+(r)$. In particular, if $0 < t < r$, then

$$v(x', t) \leq s(x', t) = T^+(r)t + \frac{|x'|^2}{2t}$$

for $|x'| \leq \mu r$. Hence

$$T^+(t) \leq T^+(r),$$

and so $r \mapsto T^+(r)$ is nondecreasing with r.

2. We similarly define the lower cylinder $C^-(r)$ by (2.10) and set

$$s(x) := T^-(-r)x_n + \frac{|x'|^2}{2x_n}$$

for $x_n < 0$. As above, we deduce $w \leq s$ inside $C^-(r)$, if μ is large enough. For $0 < t < r$, we therefore have

$$w(x', -t) = -T^-(-r)t - \frac{|x'|^2}{2t} \leq s(x', -t)$$

for $|x'| \leq \mu r$. It follows that

$$T^-(-t) \leq T^-(-r);$$

and so $r \mapsto T^-(-r)$ is nondecreasing with r. This proves assertion (i).

3. To prove (3.10), let us assume for later contradiction that

$$T^-(0) > T^+(0).$$

Select $\delta > 0$ so small that

$$T^+(0) - T^-(0) + 2\delta < 0. \tag{3.11}$$

Next, take r_0 so small that

$$T^+(r_0) < T^+(0) + \delta. \tag{3.12}$$

Then

$$v(x', r_0) \leq (T^+(0) + \delta)r_0 + \frac{|x'|^2}{2r_0} \tag{3.13}$$

for all $|x'| \leq \mu r_0$.

4. We turn our attention now to the cylinder

$$C = C(r_0, r) := \{|x'| \leq \mu r_0, -r \leq x_n \leq r_0\},$$

the small number $r > 0$ to be selected.

Denote by $x'(-r)$ a value of x' where the maximum in (3.6) is attained. So

$$v(x'(-r),-r) = -rT^-(-r) - \frac{|x'(-r)|^2}{2r}; \tag{3.14}$$

and according to (3.8) we may assume

$$|x'(-r)| \leq Cr.$$

In view of (3.13), if $r > 0$ is small enough, then

$$v(x', r_0) \leq v(x'(-r),-r) + (T^+(0)+2\delta)(r_0+r) + \frac{|x'-x'(-r)|^2}{2(r_0+r)}$$

on the top $\{x_n = r_0\}$ of the cylinder C. We also observe that

$$v(x) \leq v(x'(-r),-r) + (T^+(0)+2\delta)(x_n+r) + \frac{|x'-x'(-r)|^2}{2(x_n+r)}$$

on the vertical sides $\{|x'| = \mu r_0\}$, again provided r is very small.

Thus from the comparison principle for v, we have

$$v(x) \leq v(x'(-r),-r) + (T^+(0)+2\delta)(x_n+r) + \frac{|x'-x'(-r)|^2}{2(x_n+r)}.$$

And then (3.14) implies

$$\begin{aligned} v(x) \leq & -rT^-(-r) + (T^+(0)+2\delta)(x_n+r) \\ & + \frac{|x'-x'(-r)|^2}{2(x_n+r)} - \frac{|x'(-r)|^2}{2r} \end{aligned} \tag{3.15}$$

inside the cylinder $C = C(r_0, r)$.

Put $x = 0$, to deduce from (3.11) that

$$v(0) \leq (-T^-(-r) + T^+(0) + 2\delta)r < 0$$

if r is small enough, a contradiction to (3.4).

The case that

$$T^-(0) < T^+(0)$$

leads likewise to a contradiction. □

Remark. For future reference, we extract from this proof the assertion that for any $\delta > 0$, we can select $r_0 \geq r_1 > 0$ (depending upon δ and v) such that we have the bound from above:

$$\begin{aligned} v(x) \leq & -r_1T^-(-r_1) + (T^+(0)+2\delta)(x_n+r_1) \\ & + \frac{|x'-x'(-r_1)|^2}{2(x_n+r_1)} - \frac{|x'(-r_1)|^2}{2r_1} \end{aligned} \tag{3.16}$$

within the cylinder

$$\{|x'| \leq \mu r_0, -r_1 \leq x_n \leq r_0\}.$$

Likewise, we can assume that for the same $r_0 \geq r_1 > 0$, we have the bound from below:

$$\begin{aligned} v(x) \geq & -r_1 T^+(r_1) + (T^-(0) + 2\delta)(x_n - r_1) \\ & + \frac{|x' - x'(r_1)|^2}{2(x_n - r_1)} + \frac{|x'(r_1)|^2}{2r_1} \end{aligned} \tag{3.17}$$

within the cylinder

$$\{|x'| \leq \mu r_0, -r_0 \leq x_n \leq r_1\}.$$

□

3.2 Approximation by linear functions. We now show that we can approximate v on a smaller cylinder

$$\tau Q = \{|x'| \leq \tau, |x_n| \leq \tau\}$$

by a linear function $l = \mathbf{e}_\tau \cdot x$.

The idea will be to utilize the one-sided estimate (3.16), which for small r bounds v from above by a smooth function of x. Since we can employ (3.17) to similarly bound v from below by a different smooth function, we will be able to build a two-sided linear approximation to v near 0.

We continue to assume that v is a viscosity solution of the PDE (3.1), satisfying (3.2), (3.3) and (3.4).

Theorem 3.2. *Given any $0 < \eta < 1$, there exists a constant*

$$0 < \tau \leq \frac{1}{2},$$

depending upon v and η, such that

$$\max_{\tau Q} |v - \mathbf{e}_\tau \cdot x| \leq \eta\tau, \tag{3.18}$$

for some vector $\mathbf{e}_\tau$ satisfying

$$|\mathbf{e}_\tau| \leq C_1 \tag{3.19}$$

for a constant C_1.

Note that the scaling factor τ may depend upon the particular solution v: we will later remove this restriction.

Proof: 1. Define $x'(-r)$ as in the previous proof, and write

$$y'(r) := \frac{x'(-r)}{r}.$$

Owing to (3.8), we have $|y'(r)| \le C$.

From the estimate (3.16), we see that for each small $\delta > 0$ there exist $r_0 \ge r_1 > 0$ such that

$$\begin{aligned} \frac{v(r_1 x)}{r_1} &\le -T^-(-r_1) + T^+(0) + 2\delta \\ &\qquad + (T^+(0)+2\delta)x_n + \frac{|x'-y'(r_1)|^2}{2(x_n+1)} - \frac{|y'(r_1)|^2}{2} \end{aligned} \tag{3.20}$$

inside the cylinder

$$\{|x'| \le \mu\frac{r_0}{r_1}, -1 \le x_n \le \frac{r_0}{r_1}\} \supset \tfrac{1}{2}Q,$$

the containment holding since $r_0 \ge r_1$ and μ is large.

Define

$$w(x) := (T^+(0)+2\delta)x_n + \frac{|x'-y'(r_1)|^2}{2(x_n+1)} - \frac{|y'(r_1)|^2}{2}.$$

Then $w(0) = 0$ and

$$|Dw|, |D^2 w| \le C_2 \quad \text{in } \tfrac{1}{2}Q,$$

for a universal constant C_2, which in particular does not depend upon δ, r_0 or r_1. Consequently, (3.20) implies

$$w(x) \le \mathbf{e}\cdot x + C_2|x|^2 \tag{3.21}$$

for the vector $\mathbf{e} = Dw(0)$; and therefore

$$\frac{v(r_1 x)}{r_1} \le -T^-(-r_1) + T^+(0) + 2\delta + \mathbf{e}\cdot x + C_2|x|^2 \quad \text{in } \tfrac{1}{2}Q.$$

2. Now let $\sigma > 0$. Then

$$\frac{1}{r_1\sigma}v(r_1\sigma x) - \mathbf{e}\cdot x \le (-T^-(-r_1) + T^+(0) + 2\delta)\sigma^{-1} + C_3\sigma \tag{3.22}$$

within the cylinder $\frac{1}{2\sigma}Q$.

Given $\eta > 0$, we first choose $0 < \sigma < \frac{1}{2}$ so small that

$$C_3\sigma < \eta.$$

Now pick first $\delta > 0$ and then $r_0 \ge r_1 > 0$ so small (3.20) holds and also

$$(-T^-(-r_1) + T^+(0) + 2\delta)\sigma^{-1} < \eta.$$

We conclude using estimate (3.22) that

$$\frac{1}{r_1\sigma}v(r_1\sigma x) - \mathbf{e}\cdot x \le 2\eta \tag{3.23}$$

within the cylinder $\frac{1}{2\sigma}Q \supset Q$.

Invoking the estimate (3.17), we can similarly estimate from below that

$$\frac{1}{r_1\sigma}v(r_1\sigma x) - \mathbf{e}' \cdot x \geq -2\eta \tag{3.24}$$

inside Q, for the same r_1 but some possibly different vector $\mathbf{e}'$.

3. Combining the inequalities (3.23) and (3.24), we deduce

$$(\mathbf{e}' - \mathbf{e}) \cdot x \leq 4\eta$$

for all $x \in Q$; whence

$$|\mathbf{e} - \mathbf{e}'| \leq 4\eta. \tag{3.25}$$

If we now put

$$\tau := r_1\sigma, \ \ \mathbf{e}_\tau := \mathbf{e},$$

then from (3.23), (3.24) and (3.25) we deduce that

$$\max_{\tau Q} |v - \mathbf{e}_\tau \cdot x| \leq 6\eta\tau.$$

□

Next, we employ a compactness argument to remove the restriction that τ may depend upon our particular solution v:

Theorem 3.3. *Given any* $0 < \eta < 1$, *there exists a constant* $\tau_1(\eta) > 0$ *such that if* v *is a viscosity solution of the PDE* (3.1), *satisfying* (3.2)–(3.4), *then for some*

$$0 < \tau_1(\eta) < \tau < \frac{1}{2}, \tag{3.26}$$

we have the estimate

$$\max_{\tau Q} |v - \mathbf{e}_\tau \cdot x| < \eta\tau, \tag{3.27}$$

for a vector $\mathbf{e}_\tau \in \mathbb{R}^n$ *satisfying*

$$|\mathbf{e}_\tau| \leq C_1. \tag{3.28}$$

Proof. Assume the statement above is false for some $\eta > 0$. Then we can find a sequence of $\tau_k \to 0$ and corresponding functions v^k that satisfy the hypothesis of the Theorem, but for which the conclusion is false for all values of $\tau_k < \tau < \frac{1}{2}$.

We may assume the functions v^k converge uniformly in Q to a function v^*. Then v^* satisfies the hypotheses of Theorem 3.2, according to which

$$|v^* - \mathbf{e}^* \cdot x| < \tau^*\eta \quad \text{in } \tau^*Q, \tag{3.29}$$

for some small $0 < \tau^* \leq \frac{1}{2}$. If k is large, then v^k satisfies (3.29) as well; and this is a contradiction. $\square$

4. Improving flatness, $C^{1,\alpha}$ regularity.

4.1 Improved flatness in tilted cylinders. Now assume that within a narrow cylinder the infinity harmonic function u differs from a linear function $l = \mathbf{e} \cdot x$ in the sup-norm by no more than a small number λ. We will show that on a smaller and possibly tilted cylinder, the distance of u in the sup-norm to a slightly different linear function is less than $\frac{1}{2}\lambda$.

In particular, the linear approximation of u on the smaller cylinder improves by a factor strictly less than one. As explained in the introduction, this is the key point.

Notation. (i) Given a nonzero vector $\mathbf{e} \in \mathbb{R}^n$ and $a, b > 0$, we define

$$Q(\mathbf{e}, a, b) := \left\{ x \in \mathbb{R}^n \mid |x \cdot \frac{\mathbf{e}}{|\mathbf{e}|}| \leq a; |x - \frac{x \cdot \mathbf{e}}{|\mathbf{e}|^2}\mathbf{e}| \leq b \right\}$$

to be the cylinder with center 0, axis $\mathbf{e}$, height $2a$ and radius b.

(ii) In this notation

$$Q_\lambda = \{|x_n| \leq 1, |x'| \leq \lambda^{\frac{1}{2}}\} = Q(\mathbf{e}_n, 1, \lambda^{\frac{1}{2}})$$

for the coordinate vector $\mathbf{e}_n = (0, \ldots, 0, 1)$.

(iii) We define as well the isotropic rescalings about the origin:

$$u_\tau(x) := \frac{u(\tau x)}{\tau}. \tag{4.1}$$

Fundamental Assumption. We repeat our primary assumption, that the flatness condition

$$\max_{Q_\lambda} |u - x_n| \leq \lambda \tag{4.2}$$

implies for any viscosity solution of the infinity Laplacian PDE (1.1) the gradient bounds

$$\begin{aligned} &\sup_{\frac{1}{2}Q_\lambda} |D'u| \leq C\lambda^{\frac{1}{2}}, \\ &\sup_{\frac{1}{2}Q_\lambda} |1 - u_{x_n}| \leq C\lambda \end{aligned} \tag{4.3}$$

for some constant C.

Given this hypothesis, here is our main assertion about improved linear approximation:

Theorem 4.1. *Suppose the above Fundamental Assumption.*

There exist constants $\lambda_0 > 0$ and $\frac{1}{2} > \tau_0 > 0$ such that if

$$0 < \lambda \leq \lambda_0, \tag{4.4}$$

u is infinity harmonic, $u(0) = 0$, and

$$|u - \mathbf{e} \cdot x| \leq \lambda \qquad \text{in the cylinder } Q(\mathbf{e}, 1, \lambda^{\frac{1}{2}}), \tag{4.5}$$

for some vector $\mathbf{e}$ of length

$$\frac{1}{2} \leq |\mathbf{e}| \leq 2,$$

then for some scaling factor

$$0 < \tau_0 < \tau < \tfrac{1}{2}, \tag{4.6}$$

the rescaled function u_τ satisfies

$$|u_\tau - \mathbf{e}_\tau \cdot x| \leq \lambda/2 \qquad \text{in the cylinder } Q(\mathbf{e}_\tau, 1, (\tfrac{\lambda}{2})^{\frac{1}{2}}), \tag{4.7}$$

for a vector $\mathbf{e}_\tau$ satisfying

$$|\mathbf{e}_\tau - \mathbf{e}| \leq C\lambda^{\frac{1}{2}}. \tag{4.8}$$

Proof. 1. Assume first that $\mathbf{e}$ is a unit vector, say $\mathbf{e} = \mathbf{e}_n = (0, \ldots, 0, 1)$. Suppose

$$|u - x_n| \leq \lambda \qquad \text{in } Q(\mathbf{e}_n, 1, \lambda^{\frac{1}{2}}) = Q_\lambda.$$

We will show in this case that

$$|u_\tau - \mathbf{e}_\tau \cdot x| \leq \lambda/16 \qquad \text{in } Q(\mathbf{e}_\tau, 1, \lambda^{\frac{1}{2}}) \tag{4.9}$$

for some appropriate vector $\mathbf{e}_\tau$, provided λ is sufficiently small.

2. Suppose this assertion is false, no matter how small λ is. Then for a sequence $\lambda_k \to 0$ there exist infinity harmonic functions u^k such that $u^k(0) = 0$ and

$$|u^k - x_n| \leq \lambda_k \qquad \text{in } Q_{\lambda_k},$$

but for which (4.9) fails.

Rescale each u^k according to (2.5). In light of the Fundamental Assumption, the rescaled functions v^k are uniformly Lipschitz continuous in $\frac{1}{2}Q$, and so have a subsequence that converges uniformly to a function v that satisfies the hypothesis of Theorem 3.3.

Let C_1 be the constant from Theorem 3.3 and take $\eta = \frac{1}{C_1 32}$. Then for small enough λ_k and for some $\frac{1}{2} > \tau_k \geq \tau_0 := \frac{\tau_1(\eta)}{2C_1} > 0$, we have

$$|\frac{1}{\lambda_k}[(u^k(\lambda_k^{\frac{1}{2}} x', x_n) - x_n) - \mathbf{e}_{\tau_k} \cdot x| \leq \tau_k/16$$

in $2C_1\tau_k Q$, for some vector $\mathbf{e}_{\tau_k} = (e', e_n)$ satisfying $|\mathbf{e}_{\tau_k}| \le C_1$.

Therefore

$$|u^k(x) - (\lambda_k^{\frac12} e', 1 + \lambda_k e_n) \cdot x| \le \tau_k \lambda_k / 16$$

in $Q(\mathbf{e}_n, 2C_1\tau_k, 2C_1\tau_k\lambda_k^{\frac12})$. Restating this last estimate, we have

$$|u^k_{\tau_k}(x) - \mathbf{e}^k \cdot x| \le \lambda_k/16$$

in $Q(\mathbf{e}_n, 2C_1, 2C_1\lambda_k^{\frac12})$, where

$$\mathbf{e}^k := (\lambda_k^{\frac12} e', 1 + \lambda_k e_n).$$

Since $|\mathbf{e}_{\tau_k}| \le C_1$ and we can assume $C_1 \ge 1$, it follows that

$$Q(\mathbf{e}^k, 1, \lambda_k^{\frac12}) \subset Q(\mathbf{e}_n, 2C_1, 2C_1\lambda_k^{\frac12});$$

We consequently derive a contradiction.

This proves (4.9), and consequently the Theorem in the case that $\mathbf{e}$ is a unit vector.

3. For the general case that $1/2 \le |\mathbf{e}| \le 2$, we observe that the rescaled function

$$\tilde{u}(x) := \frac{2}{|\mathbf{e}|} u(x/2)$$

satisfies the hypothesis of the Theorem for the unit vector $\mathbf{e}/|\mathbf{e}|$ and 4λ. That is,

$$|\tilde{u}(x) - \frac{\mathbf{e}}{|\mathbf{e}|} \cdot x| \le 4\lambda \quad \text{in } Q(\mathbf{e}, 1, (4\lambda)^{\frac12}).$$

By Steps 1 and 2 of the proof above, there exists $\tau_0 < \tau < \frac12$ such that

$$|\tilde{u}_\tau(x) - \mathbf{e}_\tau \cdot x| \le \lambda/4 \quad \text{in } Q(\mathbf{e}_\tau, 1, (4\lambda)^{\frac12}))$$

for some vector $\mathbf{e}_\tau$ satisfying

$$|\mathbf{e}_\tau - \frac{\mathbf{e}}{|\mathbf{e}|}| \le C\lambda^{\frac12}.$$

We conclude that

$$|u_{\tau/2}(x) - \mathbf{e}_{\frac{\tau}{2}} \cdot x| \le |\mathbf{e}|\lambda/4 \le \lambda/2$$

in $Q(\mathbf{e}_{\frac{\tau}{2}}, 1, (\lambda/2)^{\frac12})$, where

$$\mathbf{e}_{\frac{\tau}{2}} := |\mathbf{e}|\mathbf{e}_\tau.$$

And we have the required estimate

$$|\mathbf{e}_{\frac{\tau}{2}} - \mathbf{e}| \le C\lambda^{\frac12}.$$

$\square$

4.2 Local $\mathbf{C^{1,\alpha}}$ regularity. Finally we iterate the preceding assertion about improvement of linear approximations:

Theorem 4.2. *Let λ_0 be the constant from Theorem 4.1, and assume*

$$|u - \mathbf{e}\cdot x| \leq |\mathbf{e}|\lambda_0 \quad \textit{in } Q(\mathbf{e}, 1, \lambda_0^{\frac{1}{2}}). \tag{4.10}$$

Then

$$|Du(x) - Du(0)| \leq C|\mathbf{e}||x|^{\beta}, \tag{4.11}$$

for constants $C > 0$ and $0 < \beta < 1$.

Proof: Without loss of generality, we may suppose $|\mathbf{e}| = 1$.

1. We apply Theorem 4.1 repeatedly, to find scaling factors

$$0 < \dots r_k < \dots < r_1 < r_0 = 1,$$

such that

$$\tau_0 \leq \frac{r_{k+1}}{r_k} \leq \frac{1}{2}, \tag{4.12}$$

for which the rescaled functions u_{r_k} satisfy the estimates

$$|u_{r_k} - \mathbf{e}_{r_k}\cdot x| \leq \lambda_k \qquad \text{in the cylinders } Q(\mathbf{e}_{r_k}, 1, \lambda_k^{\frac{1}{2}}) \tag{4.13}$$

for $\lambda_k := 2^{-k}\lambda_0$ and vectors $\mathbf{e}_{r_k}$ satisfying

$$|\mathbf{e}_{r_k} - \mathbf{e}_{r_{k-1}}| \leq C\lambda_k^{\frac{1}{2}}. \tag{4.14}$$

2. Applying the Fundamental Assumption to the infinity harmonic functions u_{r_k}, we find

$$|Du_{r_k}(x) - Du_{r_k}(0)| \leq C\lambda_k^{\frac{1}{2}}$$

for $|x| \leq \frac{1}{2}\lambda_k^{\frac{1}{2}}$. In view of (4.12), this implies

$$|Du(x) - Du(0)| \leq C_1 2^{-k/2},$$

for $|x| \leq C\tau_0^k 2^{-k/2}$; and estimate (4.11) follows. □

We next observe that for each point in our domain, we can achieve the starting flatness condition (4.10), if we look on a sufficiently small length scale. This follows from the next lemma, which follows from [C-E].

Lemma 4.3. *There exist a universal constant $\tau_1 > 0$, such that if u is infinity harmonic in the unit ball $B := B(0,1) \subset \mathbb{R}^n$,*

$$|u| \leq 1$$

and

$$u(0) = 0,$$

then there exists

$$\tau_1 \leq \tau \leq \frac{1}{2},$$

such that

$$\max_B |u_\tau - \mathbf{e} \cdot x| \leq \frac{\lambda_0}{4} \|u\|_{L^\infty(B)}. \tag{4.15}$$

for the rescaled function u_τ and some vector $\mathbf{e}_\tau$ satisfying

$$|\mathbf{e}_\tau| \leq C\|u\|_{L^\infty(B)}. \tag{4.16}$$

Notice carefully that this Lemma says that u can be well approximated by a linear mapping on sufficiently small balls, although our iteration procedure above works on small cylinders.

Next is the key assertion of local $C^{1,\alpha}$ regularity.

Theorem 4.4. *Suppose that the Fundamental Assumption holds. Let u be infinity harmonic in the unit ball $B \subset \mathbb{R}^n$, with the bound*

$$|u| \leq 1.$$

Then

$$|Du(x) - Du(0)| \leq C|x|^\alpha \tag{4.17}$$

for constants C and $\alpha > 0$.

In particular, u is $C^{1,\alpha}$ in the interior of B.

Proof: 1. We repeatedly apply Lemma 4.3, to find scaling factors

$$0 < \dots s_k < \cdots < s_1 < s_0 = 1,$$

such that

$$\tau_1 \leq \frac{s_{k+1}}{s_k} \leq \frac{1}{2},$$

for which the rescaled functions u_{s_k} satisfy

$$\|u_{s_{k+1}} - \mathbf{e}_{s_{k+1}} \cdot x\|_{L^\infty(B_1)} \leq \frac{\lambda_0}{4} \|u_{s_k}\|_{L^\infty(B_1)}. \tag{4.18}$$

for vectors $\mathbf{e}_{s_{k+1}}$ satisfying

$$|\mathbf{e}_{s_{k+1}}| \leq C\|u_{s_k}\|_{L^\infty(B_1)}$$

Observe also that if

$$|\mathbf{e}_{s_{k+1}}| \leq \frac{1}{4} \|u_{s_k}\|_{L^\infty(B_1)}, \tag{4.19}$$

then

$$\|u_{s_{k+1}}\|_{L^\infty(B_1)} \leq \frac{1}{2} \|u_{s_k}\|_{L^\infty(B_1)}.$$

2. Suppose first that (4.19) holds for all $k = 1, 2, \dots$. Then if

$$s_{k+1}/2 \leq |x| \leq s_k/2,$$

we have

$$|Du(x)| \leq C\|u_{s_k}\|_{L^\infty(B_1)} \leq C2^{-k} \leq C|x|^\alpha \tag{4.20}$$

for some appropriate $\alpha > 0$; and the Theorem is proved in this case.

3. In the case when (4.19) is not always satisfied, let k denote the first index for which (4.19) fails; so that

$$|\mathbf{e}_{s_{k+1}}| > \frac{1}{4}\|u_{s_k}\|_{L^\infty(B_1)}.$$

This and (4.18) imply

$$\|u_{s_{k+1}} - \mathbf{e}_{s_{k+1}} \cdot x\|_{L^\infty(B_1)} \leq \lambda_0|\mathbf{e}_{s_{k+1}}|.$$

We can therefore apply Theorem 4.2 to $u_{s_{k+1}}$. We deduce that for all $x \in B_1$,

$$\begin{aligned} |Du(s_{k+1}x) - Du(0)| &\leq C\|u_{s_k}\|_{L^\infty(B_1)}|x|^\beta \\ &\leq C2^{-k}|x|^\beta \leq C|s_{k+1}x|^\beta. \end{aligned}$$

And from Steps 2 above, we see that (4.20) holds also for $|x| \geq s_{k+1}$. The Theorem is proved. □

5. Two dimensions

In this section we verify that the Fundamental Assumption, and therefore $C^{1,\alpha}$ regularity, holds in

$$n = 2$$

dimensions. The following arguments are due to the second author and sharpen some insights from his earlier paper [S].

Definition: The plane $p := a + \mathbf{e} \cdot x$ is called a *crossing tangent plane* for u in an η-neighborhood of 0 if either the open set $\{u > p\}$ or the open set $\{u < p\}$ has at least two distinct connected components which intersect the disk $B(0, \eta)$.

The paper [S] proves

Theorem 5.1. *Let u be infinity harmonic in a convex domain $U \subset \mathbb{R}^2$ and assume u is not identically equal to a plane in an η-neighborhood of* 0.

Then u admits a crossing tangent plane in an η-neighborhood of 0.

The next estimates improves and simplifies the results of Proposition 2.4 of [S], to verify the Fundamental Assumption.

Theorem 5.2. *Assume*

$$|u - x_2| \le \lambda \quad \textit{in } Q_\lambda. \tag{5.1}$$

(i) *Suppose the plane P of slope $\mathbf{e} = (e_1, e_2)$ is a crossing tangent plane in a small η-neighborhood of 0. Then for some constant C we have the estimates*

$$|e_1| \le C\lambda^{\frac{1}{2}} \quad \textit{and} \quad |e_2 - 1| \le C\lambda. \tag{5.2}$$

(ii) *Furthermore,*

$$|u_{x_1}| \le C\lambda^{\frac{1}{2}}, \quad |u_{x_2} - 1| \le C\lambda \quad \textit{in } \tfrac{1}{2}Q_\lambda \tag{5.3}$$

for some constant C.

Proof: 1. It is easy to show that

$$|\mathbf{e}| \le 1 + C\lambda \tag{5.4}$$

for a constant C.

Next we assume for later contradiction that both of the following inequalities hold:

$$1 - e_2 \ge C_1\lambda, \qquad |e_1| \le C_1^{-1}\lambda^{-1/2}(1 - e_2), \tag{5.5}$$

the large constant C_1 to be chosen later.

2. We now *claim* that

$$\mathbf{e} \cdot x \le x_2 - \lambda \quad \text{if } x_2 \ge 2C_1^{-1},\ |x_1| \le \lambda^{\frac{1}{2}} \tag{5.6}$$

and

$$\mathbf{e} \cdot x \ge x_2 + \lambda \quad \text{if } x_2 \le -2C_1^{-1},\ |x_1| \le \lambda^{\frac{1}{2}}. \tag{5.7}$$

To see this, note that

$$\begin{aligned} x_2 - \mathbf{e} \cdot x &= -e_1 x_1 + (1 - e_2)x_2 \\ &\ge -\lambda^{\frac{1}{2}}|e_1| + (1 - e_2)x_2 \\ &\ge (1 - e_2)(x_2 - C_1^{-1}) \\ &\ge \lambda \end{aligned}$$

provided $x_2 \ge 2C_1^{-1}$. Similarly

$$x_2 - \mathbf{e} \cdot x \le -\lambda,$$

provided $x_2 \le -2C_1^{-1}$.

The inequalities (5.6) and (5.7) imply that

$$\{x_2 \ge 4C_1^{-1}\} \subset \{u > p\}$$

and

$$\{x_2 \le -4C_1^{-1}\} \subset \{u < p\}.$$

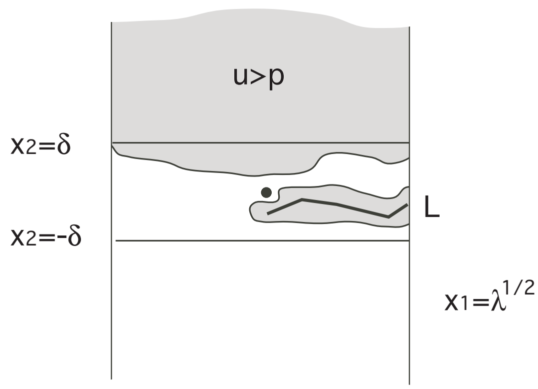

FIGURE 1. The regions where $u > p$

Let $\delta > 0$ be a small number, to be determined later. We now fix the constant C_1 in (5.5) so large that

$$\{x_2 \geq \delta\} \subset \{u > p\} \tag{5.8}$$

and

$$\{x_2 \leq -\delta\} \subset \{u < p\}. \tag{5.9}$$

3. Since $p := a + \mathbf{e} \cdot x$ is a crossing tangent plane, either the open set $\{u > p\}$ or the open set $\{u < p\}$ has at least two distinct connected components that intersect the small ball $B(0, \eta)$. Consequently we can find a connected component of, say, $\{u > p\}$ that is included in $|x_2| < \delta$.

Notice that this component cannot be compactly included in $|x_1| < \lambda^{\frac{1}{2}}$, since otherwise we would contradict the comparison principle. Therefore this connected component of $\{u > p\}$ contains a polygonal line L, that starts from an η-neighborhood of the origin and ends, say, on the line segment $\{x_1 = \lambda^{\frac{1}{2}}, |x_2| < \delta\}$. See Figure 1.

4. Denote by S the strip

$$S := \{\lambda^{\frac{1}{2}}/4 \leq x_1 \leq 3\lambda^{\frac{1}{2}}/4\} \times \{|x_2| \leq 1\}.$$

Let U be the connected component of $\{u > p\}$ in S that contains $\{x_2 \geq \delta\}$. Note that $L \cap S \subset S \setminus U$.

5. We are next going to compare u with a suitable cone function v in $S \setminus U$.

Consider the family of cones with vertex at $y = y_\delta := (\lambda^{\frac{1}{2}}/2, -10\delta)$, height $y_2 + \lambda$ and slope c. That is, define

$$v_{y,c}(x) := y_2 + \lambda + c|x - y|.$$

We now *claim* that for

$$c_0 = 1 - \frac{\lambda}{\delta},$$

the cone function $v_0 = v_{y,c_0}$ satisfies

$$v_0 > x_2 + \lambda \quad \text{on } \partial S \cap \{x_2 \leq \delta\} \tag{5.10}$$

and

$$v_0 < x_2 - \lambda \quad \text{on the segment } \{(\lambda^{\frac{1}{2}}/2, x_2) \mid |x_2| \leq \delta\}. \tag{5.11}$$

To prove (5.10), observe that $v_0(x) > x_2 + \lambda$, provided

$$c_0 > \frac{x_2 - y_2}{|x - y|}.$$

This holds if the angle α between $x - y$ and the positive x_2 axis satisfies

$$\cos \alpha < 1 - \frac{\lambda}{\delta}. \tag{5.12}$$

Now points in the set $\partial S \cap \{x_2 \leq \delta\}$ satisfy

$$\tan \alpha \geq \frac{1}{11\delta} \frac{\lambda^{\frac{1}{2}}}{4} : \tag{5.13}$$

see Figure 2.

Consequently, the assertion (5.10) follows if (5.13) implies (5.12). This is indeed so, if we fix δ to be small enough.

To prove the upper bound (5.11), note that if $x_2 \geq y_2 = -10\delta$, then

$$v_0(\lambda^{\frac{1}{2}}/2, x_2) = y_2 + \lambda + c_0(x_2 - y_2).$$

Consequently, (5.11) holds provided

$$y_2 + \lambda + (1 - \frac{\lambda}{\delta})(x_2 - y_2) < x_2 - \lambda.$$

And this is valid for $|x_2| \leq \delta$.

6. Now we compare the cones $v_{y,c}$ with the solution u in the region $S \setminus U$.

When $c > 1$, v is above u. Decrease c continually until v touches u restricted to $\partial(S \setminus U)$ for the first time, at a point x^*. Denote by c^* the value of c for which this happens.

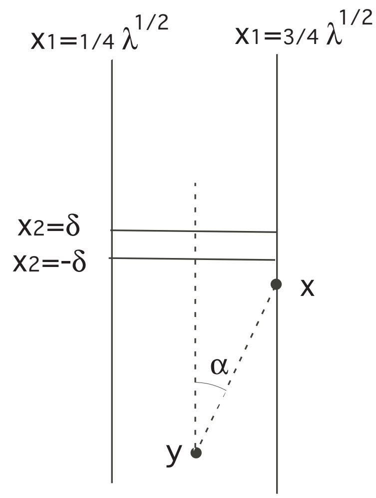

FIGURE 2. The angle α

According to (5.10) and (5.11), we have

$$c^* > c_0.$$

Therefore

$$x^* \in \{|x_2| < \delta, \lambda^{\frac{1}{2}}/4 < x_1 < 3\lambda^{\frac{1}{2}}/4\}.$$

Now

$$v_{y,c^*}(x^*) = u(x^*) = p(x^*), \tag{5.14}$$

and

$$v_{y,c^*} \geq u \quad \text{on } [y, x^*] \cap (S \setminus U).$$

$[y, x^*]$ denoting the line segment from y to x^*.

7. Let z be a point of intersection of the polygonal line L with the segment $[y, x^*]$. Then

$$v_{y,c^*}(z) \geq u(z) > p(z). \tag{5.15}$$

From (5.14), (5.15) we find

$$e \cdot \frac{x^* - y}{|x^* - y|} > c^* > c_0.$$

Since

$$\frac{\lambda^{\frac{1}{2}}}{\delta}|e_1| + e_2 \geq e \cdot \frac{x^* - y}{|x^* - y|},$$

we can employ (5.5) to find

$$\frac{1-e_2}{C_1\delta}+e_2 \geq 1-\frac{\lambda}{\delta}.$$

Therefore

$$\frac{\lambda}{\delta} \geq (1-e_2)(1-\frac{1}{C_1\delta}) \geq C_1\lambda(1-\frac{1}{C_1\delta}).$$

But this is a contradiction if C_1 is chosen large enough.

8. Consequently, (5.5) is false, and therefore at least one of the two stated inequality fails. In either case, it follows that

$$e_2 \geq 1-C_1\lambda - C_1\lambda^{\frac{1}{2}}|e_1|. \tag{5.16}$$

But then (5.4) implies

$$\begin{aligned} e_1^2 &\leq (1+C_1\lambda)^2 - e_2^2 \\ &\leq C\lambda + (1+e_2)(1-e_2) \\ &\leq C\lambda + C\lambda^{\frac{1}{2}}|e_1|; \end{aligned}$$

whence

$$|e_1| \leq C\lambda^{\frac{1}{2}}.$$

Then (5.16) and (5.4) yield the inequality

$$|1-e_2| \leq C\lambda.$$

We have at last proved estimate (5.2); and the paper [S] shows that (5.2) then implies the gradient bounds (5.3). □

Theorem 5.2 confirms that the Fundamental Assumption is valid for $n=2$ dimensions. We may therefore invoke the theory from Sections 2-4, to establish:

Theorem 5.3. *There exists a constant $\alpha > 0$ such that if u is a bounded viscosity solution of the infinity Laplacian PDE* infinity Laplacian PDE *in a open set $U \subseteq \mathbb{R}^2$, then*

$$u \in C^{1,\alpha}_{loc}(U).$$

Furthermore, for each open set $V \subset\subset U$, there exists a constant C, depending only on V, such that

$$||u||_{C^{1,\alpha}(V)} \leq C||u||_{L^\infty(U)}. \tag{5.17}$$

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