# Asymptotic behavior of infinity harmonic functions near an isolated singularity

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**Abstract**

In this paper, we prove if $n \geq 2$ and $x_0$ is an isolated singularity of a nonnegative infinity harmonic function $u$, then either $x_0$ is a removable singularity or $u(x) = u(x_0) + c|x - x_0| + o(|x - x_0|)$ near $x_0$ for some fixed constant $c$. Especially, if $x_0$ is not removable, it implies that $x_0$ is either a local maximum or local minimum. We will also prove an a closely related Bernstein type result which says that if $u$ a uniformly Lipschitz continuous, one-side bounded infinity harmonic function in $\mathbb{R}^n \backslash \{0\}$, then it must be a cone centered at 0.

## 1 Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^n$. Throughout this paper, we assumet that $n \geq 2$. We say that $u \in C(\Omega)$ is an *infinity harmonic function* in $\Omega$ if it is a viscosity solution of the following infinity Laplacian equation.

$$\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in } \Omega. \tag{1.1}$$

See Appendix I for the definition of viscosity supersolution, subsolution and solution of an elliptic equation. It is well-known that an infinity harmonic in $\Omega$ is a local minimizer of the supremum norm of the gradient in the sense that for any open set $V \subset \Omega$ and $v \in W^{1,\infty}(V)$,

$$u|_{\partial V} = v|_{\partial V}$$

implies that

$$\mathrm{esssup_V}|Du| \leq \mathrm{esssup_V}|Dv|.$$

An infinity harmonic function is also an *absolute Lipschitz extension* in $\Omega$, i.e, for for any open set $V \subset \bar{V} \subset \Omega$,

$$\sup_{x \neq y \in \partial V} \frac{|u(x) - u(y)|}{|x - y|} = \sup_{x \neq y \in \bar{V}} \frac{|u(x) - u(y)|}{|x - y|}. \tag{1.2}$$

The infinity Laplacian equation can be viewed as the limiting equation of $p$-Laplacian equations as $p \to +\infty$. Precisely speaking, for $p > 0$, assume that $u_p$ is a $p$-harmonic function in $\Omega$, i.e, $u_p$ is a solution of the $p$-Laplacian equation

$$\Delta_p u_p = \mathrm{div}(|Du_p|^{p-2} Du_p) = 0 \quad \text{in } \Omega. \tag{1.3}$$

If $u_p \to u_\infty$ uniformly in $\Omega$ as $p \to +\infty$, then $u_\infty$ is an infinity harmonic function in $\Omega$. We refer to a nice survey note by Crandall [C] for more backgrounds and information of equation (1.1). For $0 < p \leq +\infty$, if $u_p$ is a p-harmonic function in $\Omega\backslash\{x_0\}$, then $x_0$ is called an *isolated singularity* of $u_p$. $x_0$ is called a *removable singularity* if $u_p$ can be extended to be a $p$-harmonic function in $\Omega$. Otherwise $x_0$ is called a *nonremovable singularity*. When $1 < p \leq n$, a classical result of Serrin [SE] says that a nonnegative $p$-harmonic function $u_p$ is comparable to the fundamental solution of $p$-Laplacian equation near its nonremovable isolated singularity. When $n = 2$ and $2 < p < +\infty$, Manfredi[M] derived an asymptotic representation of $u_p$ near the singularity. In this paper, we will show that a nonegative infinity harmonic function is asymptotically a cone near its nonremovable isolated singularity. Especially, it implies that an infinity harmonic function has local maximum or minimum value at nonremovable isolated singularity. This is quite surprising and is largely due to the high nonlinearity of the mysterious infinity Laplacian equation. Note that cones are fundamental solutions of the infinity Laplacian equation. The following is our main result.

**Theorem 1.1** *Suppose that $n \geq 2$ and $u \in C(B_1(x_0)\backslash\{0\})$ is a nonnegative infinity harmonic function in $B_1(x_0)\backslash\{x_0\}$. Then $u \in W^{1,\infty}_{\mathrm{loc}}(B_1(x_0))$ and one of the following holds:*
*(i) $x_0$ is a removable singularity; or*
*(ii) there exists a fixed constant $c \neq 0$ such that*

$$u(x) = u(x_0) + c|x - x_0| + o(|x - x_0|),$$

*i.e,*

$$\lim_{x \to x_0} \frac{|u(x) - u(x_0) - c|x - x_0||}{|x - x_0|} = 0.$$

*Especially in case (ii), $u$ has local maximum or local minimum value at $x_0$. Moreover, we have that*

$$|c| = \mathrm{esssup}_V |Du|,$$

*where $V$ is some neighbourhood of $x_0$.*

We want to remark here that the above theorem is not correct when $n = 1$. For example, for any $t \in (0, 1]$, $u_t = t(-|x|) + (1 - t)x$ is an infinity harmonic function on $(-1, 1)\backslash\{0\}$ and 0 is an isolated singularity. When $t \neq 1$, (ii) in Theorem 1.1 is not satisfied.

As an application of Theorem 1.1, we can construct a family of nonclassical infinity harmonic functions in $\mathbb{R}^2$. Precisely speaking,

**Corollary 1.2** *Suppose that $\Omega$ is an bounded open subset of $\mathbb{R}^2$ and $x_0 \in \Omega$. Assume that $u \in C(\bar{\Omega})$ is an infinity harmonic function in $\Omega\backslash\{x_0\}$ and satisfies $u|_{\partial\Omega} = 0$ and $u(x_0) = 1$. Then $u \in C^2(\Omega\backslash\{x_0\})$ if and only if $\Omega = B_r(x_0)$ and $u(x) = 1 - \frac{|x-x_0|}{r}$ for some $r > 0$.*

We will also prove a closely related Bernstein type theorem about uniformly Lipschtz continuous infinity harmonic functions in $\mathbb{R}^n\backslash\{0\}$.

**Theorem 1.3** *If $u$ satisfies the following:*
*(i)* $\mathrm{esssup}_{\mathbb{R}^n}|Du| = 1$;
*(ii) for some $M \in \mathbb{R}$ and $\epsilon > 0$, $u(x) \le M + (1-\epsilon)|x|$ for all $x \in \mathbb{R}^n$;*
*(iii) $u$ is an infinity harmonic function in $\mathbb{R}^n\backslash\{0\}$.*
*Then*

$$u(x) = u(0) - |x|.$$

The first author proved in [SA] that if $u$ is a uniformly Lipschitz continuous infinity harmonic function in $\mathbb{R}^2$, then $u$ must be linear, i.e, $u = p \cdot x + c$ for some $p \in \mathbb{R}^2$ and $c \in \mathbb{R}$. In general, a uniformly Lipschitz continuous infinity harmonic function in $\mathbb{R}^n\backslash\{0\}$ might be neither be linear nor a cone. The following is a family of such functions. For $R > 0$ and $0 < \alpha < 1$, let $u_{R,\alpha}$ be the solution of the following equation

$$\Delta_\infty u_{R,\alpha} = 0 \quad \text{on } B_R(0)\backslash\{0\},$$

$$u_{R,\alpha}(0) = 0$$

and

$$u_{R,\alpha}|_{\partial B_R(0)} = \alpha x_n - (1-\alpha)R.$$

It is clear that

$$u_{R,0}(x) = -|x|$$

and

$$u_{R,1}(x) = x_n.$$

Hence for each R, there exists $0 < \alpha(R) < 1$ such that

$$u_{R,\alpha(R)}(0,..,0,1) = 0.$$

Now suppose $u = \lim_{R\to+\infty} u_{R,\alpha(R)}$. Then $u$ is an infinity harmonic function in $\mathbb{R}^n\backslash\{0\}$ and $\mathrm{esssup}_{\mathbb{R}^n}|Du| = 1$. Moreover, $u$ is neither linear nor a cone since $u(0,...,0,1) = 0$ and $u(0,...,0,t) = t$ for $t \le 0$. Using Theorem 1.1 and the fact that $u(x', x_n) = u(-x', x_n)$, it is not hard to see that the $u$ constructed above is not $C^2$ in $\mathbb{R}^n\backslash\{0\}$. See Corollary 3.2 for the proof. When $n = 2$, using some techniques developed by the first author in [SA], we can show that any uniformly Lipschitz continuous infinity harmonic function in $\mathbb{R}^2\backslash\{0\}$ must be bounded by a linear function and a cone. We conjecture that if $u$ is $C^2$ in $\mathbb{R}^n\backslash\{0\}$, then it must be linear or a cone. We say that $u$ is an *entire infinity harmonic function* if it is a viscosity solution of equation (1.1) in $\mathbb{R}^n$. Here we want to mention that Aronsson proved in [A] that a $C^2$ entire infinty harmonic function must be linear when $n = 2$. Estimates derived by Evans [E] implies that this conclusion is true for a $C^4$ entire infinity harmonic function in any dimension. It remains an interesting question whether the $C^4$ assumption in [E] can be reduced to $C^2$.

**Outline of the paper.** In section 2, we will review some preliminary facts of infinity harmonic functions. In section 3, we will prove our results listed in the introduction. In Appendix I, we will prove a simple lemma of isolated singularities of viscosity solutions of fully non-linear elliptic equations. Similar arguments can also be found in [D]. In Appendix II, we will present the tightness argument which is well kown to specialists.

## 2 Preliminary

For $x_0 \in \Omega$ and $0 < r < d(x_0, \partial\Omega)$, we denote

$$B_r(x_0) = \{x \in \mathbb{R}^n |\ |x - x_0| < r\}.$$

and

$$S_1 = \{x \in \mathbb{R}^n |\ |x| = 1\}.$$

Also, we set

$$S^+_{u,r}(x_0) = \max_{x \in \partial B_1(x_0)} \frac{u(x) - u(x_0)}{r}$$

and

$$S^-_{u,r}(x_0) = \frac{u(x_0) - \min_{x \in \partial B_1(x_0)} u(x)}{r}.$$

It is obvious that

$$\max\{S^+_{u,r}(x_0), S^-_{u,r}(x_0)\} \leq \text{esssup}_{B_r(x_0)}|Du|. \tag{2.1}$$

The following theorem is due to Crandall-Evans-Gariepy [CEG].

**Theorem 2.1** ([CEG]). *If $u \in C(\Omega)$ is a viscosity subsolution of equation (1.1), $S^+_{u,r}(x_0)$ is nondecreasing with respect to $r$. We denote*

$$S^+_u(x_0) = \lim_{r \to 0^+} S^+_{u,r}(x_0).$$

*For $x_r \in \partial B_r(x_0)$ such that $u(x_r) = \max_{\partial B_r(x_0)} u$, the following endpoint estimate holds*

$$S^+_u(x_r) \geq S^+_{u,r}(x_0) \geq S^+_u(x_0). \tag{2.2}$$

*If $u \in C(\Omega)$ is a viscosity supersolution of equation (1.1), $S^-_{u,r}(x_0)$ is nondecreasing with respect to $r$. We denote*

$$S^-_u(x_0) = \lim_{r \to 0^+} S^-_{u,r}(x_0).$$

*For $x_r \in \partial B_r(x_0)$ such that $u(x_r) = \min_{\partial B_r(x_0)} u$, the following endpoint estimate holds*

$$S^-_u(x_r) \geq S^-_{u,r}(x_0) \geq S^-_u(x_0). \tag{2.3}$$

*If $u \in C(\Omega)$ is a viscosity solution of equation (1.1), then $S^+_u(x_0) = S^-_u(x_0)$. We denote*

$$S_u(x_0) = S^+_u(x_0) = S^-_u(x_0).$$

*If $u$ is differentiable at $x_0$, then*

$$|Du(x_0)| = S^+_u(x_0) = S^-_u(x_0). \tag{2.4}$$

Owing to the above theorem, if $u$ is a viscosity subsolution, then $S_u^+(x)$ is uppersemicontinuous. Combining (2.1) and (2.4), we immediately have that

$$S_u^+(x) = \lim_{r\to+\infty} \mathrm{esssup}_{B_r}|Du|. \tag{2.5}$$

Similar conclusion holds for $S_u^-(x)$ when $u$ is a viscosity supersolution. Moreover, the following is a well known result which follows immediately from (2.1) , Theorem 2.1 and Lemma 5.2 in Appendix II.

**Theorem 2.2** *Suppose that $u \in W^{1,\infty}(B_1(0))$ is a viscosity subsolution of equation (1.1) in $B_1(0)$. Assume that*

$$S_u^+(0) = \mathrm{esssup}_{B_1(0)}|Du|;$$

*Then there exists some $e \in \partial B_1(0)$ such that for $t \in [0,1)$,*

$$u(te) = u(0) + tS_u^+(0).$$

*Also $Du(te)$ exists and $Du(te) = eS_u^+(0)$ for $t \in (0,1)$.*

When $n = 2$, Savin proved in [SA] that any infinity harmonic function is $C^1$. Moreover, the following uniform estimate holds.

**Theorem 2.3** ([SA]) *Suppose that $n = 2$. If $u$ is infinity harmonic function in $B_1(0)$ and for some $e \in B_1(0)$*

$$\max_{\bar{B}_1(0)} |u - e\cdot x| \leq \epsilon.$$

*Then for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that if $\epsilon < \epsilon(\delta)$, then*

$$|Du(0) - e| \leq \delta.$$

According to Theorem 2.1, it is easy to see that if $u \in C(B_1(0))$ is a viscosity subsolution or supersolution of equation (1.1) in $B_1(0)$, then, for $0 < r < \frac{1}{2}$,

$$\mathrm{esssup}_{\mathrm{B_r(0)}}|Du| \leq \frac{2}{1-r} \sup_{B_1(0)} |u|.$$

Hence by Theorem 2.3, a simple compactness argument implies the following result which we will use in the proof of Corollary 1.2.

**Theorem 2.4** *Let $n = 2$. Suppose that $u$ and $v$ are two infinity harmonic functions in $B_1(0)$ satisfying $|u|, |v| \leq 1$ and*

$$\max_{\bar{B}_1(0)} |u - v| \leq \epsilon.$$

*Then for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that if $\epsilon < \epsilon(\delta)$, then*

$$|Du(0) - Dv(0)| \leq \delta.$$

# 3 Proofs

**Lemma 3.1** *Assume that* $u : \mathbb{R}^n \to \mathbb{R}$ *satisfies the following:*
*(i)* $\mathrm{esssup}_{\mathbb{R}^n}|Du| \le 1$*;*
*(ii)* $u(0) = 0$ *and* $u(x) \le (1-\epsilon)|x|$ *for some* $\epsilon > 0$
*(iii)* $u$ *is a viscosity subsolution of equation (1.1) in* $\mathbb{R}^n \backslash \{0\}$.
*(iv) There exists* $e \in S_1$ *such that*

$$u(-te) = -t \quad \text{for all } t \ge 0.$$

*Then*

$$u(x) = -|x|.$$

*Proof.* Note that (i) and (iv) imply that $\mathrm{esssup}_{\mathbb{R}^n}|Du| = 1$. Without loss of generality, we assume that $e = (0, ..., 0, 1)$. Let

$$\mathcal{S} = \{\text{all the functions satisfying (i)-(iv)}\}.$$

Denote

$$w = \sup_{v \in \mathcal{S}} v.$$

It is clear that $w \in \mathcal{S}$. For any $\lambda > 0$. We have $w_\lambda = \frac{w(\lambda x)}{\lambda} \in \mathcal{S}$. Hence for all $\lambda > 0$

$$w \ge w_\lambda.$$

This implies that $w = w_\lambda$ for all $\lambda > 0$, i.e, $w$ is homogeneous of degree 1. Owing to (i), (iv) and Lemma 5.1 in Appendix II, we have that for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$w(x', x_n) \le x_n. \tag{3.1}$$

If there exists a point $(\bar{x}', 0)$ such that $|\bar{x}'| = 1$ and $w(\bar{x}', 0) = 0$, then owing to (i) and (3.1), for all $t \ge 0$.

$$w(\bar{x}', -t) = -t.$$

I claim that all $t \in \mathbb{R}$,

$$w(\bar{x}', t) = t.$$

In fact, denote $T = \sup\{t |\ w(\bar{x}', s) = s \quad \text{for all } s \le t\}$. It is clear that $T \ge 0$. If $T < +\infty$, by (i) and (2.5), We have that

$$S_w^+(\bar{x}', T - \frac{1}{2}) = 1.$$

Hence by (i), Theorem 2.2 and triangle inequality

$$w(\bar{x}', t) = t \quad \text{for } T - \frac{1}{2} \le t \le T + \frac{1}{2}.$$

This contradicts to the definiton of $T$. So $T = +\infty$. Hence our claim holds. Therefore by (i) and Lemma 5.1,

$$w(x) = e \cdot x.$$

This contradicts to (ii). Hence

$$\max_{\{(x',0)|\ |x'|=1\}} w < 0.$$

So combining with the homogeneity of $w$, there exists $\epsilon > 0$ such that

$$w \le 0 \quad \text{on } \Gamma_\epsilon, \tag{3.2}$$

where

$$\Gamma_\epsilon = \{(x', x_n)|\ x_n \le 0 \text{ or } x_n^2 \le \epsilon |x'|^2\}.$$

Denote

$$C_\epsilon = \{(x', x_n)|\ x_n \le 0 \text{ and } x_n^2 > \frac{1}{\epsilon}|x'|^2\}.$$

Geometrically, it is clear that for all $x \in C_\epsilon$, $B_{|x|}(x) \subset \Gamma_\epsilon$. Therefore for $x \in C_\epsilon$, owing to (3.2) and Theorem 2.1,

$$S_w^+(x) \le \frac{-w(x)}{|x|}. \tag{3.3}$$

Suppose that $u$ is differentiable at $x_0 \in C_\epsilon$. By the homogeneity,

$$Dw(x_0) \cdot x_0 = w(x_0).$$

Hence by (2.4) and (3.3),

$$Dw(x_0) = \frac{w(x_0)}{|x_0|^2} x_0.$$

Since $w$ is Lipschitz continuous, we derive that $w \in C^\infty(C_\epsilon)$ and

$$Dw(x) = \frac{w(x)}{|x|^2} x. \tag{3.4}$$

Hence

$$|Dw(x)| = \frac{-w(x)}{|x|} \quad \text{in } C_\epsilon. \tag{3.5}$$

Owing to (3.4),

$$D(|Dw(x)|) = 0 \quad \text{in } C_\epsilon$$

By (i) and (iv), $|Dw(x)| \equiv 1$ in $C_\epsilon$. So by (3.5),

$$w(x) = -|x| \quad \text{in } C_\epsilon$$

Since

$$-|x| \le u(x) \le w(x),$$

we have that

$$u(x) = -|x| \quad \text{in } C_\epsilon.$$

Now we denote

$$\mathcal{A} = \{\alpha \in S_1|\ u(-t\alpha) = -t \quad \text{for all } t \ge 0\}.$$

Obviously, $\mathcal{A}$ is closed and nonempty. Owing to the above proof, $\mathcal{A}$ is an open set of $S_1$. For $n \geq 2$, $S_1$ is connected. Hence $\mathcal{A} = S_1$. So

$$u(x) = -|x|.$$

$\square$

**Proof of Theorem 1.1.** According to [B], $\lim_{x\to x_0} u(x)$ exists. Hence by defining $u(x_0) = \lim_{x\to x_0} u(x)$, $u \in C(B_1)$. Suppose that $x_0$ is a nonremovable singularity. Owing to Lemma 4.2 in Appendix I, we may assume that $u$ is viscosity supersolution in $B_1(x_0)$. Hence $u \in W^{1,\infty}_{\text{loc}}(B_1(0))$. Without loss of generality, we assume that $x_0 = 0$ and $u(0) = 0$. Since $u$ is not a subsolution at $x_0$, According to (4.4) and Remark 4.3 in the Appendix I, there exists a $0 \neq p \in \mathbb{R}^n$ and $\epsilon > 0$ such that in a neighborhood of 0,

$$u(x) \leq p \cdot x - \epsilon|x|.$$

Hence there exists $\delta > 0$ and a smaller neighborhood $V \subset \bar{V} \subset B_1(0)$ of 0 such that

$$u(x) \geq p \cdot x - \delta \quad \text{in } V$$

and

$$u(x) = p \cdot x - \delta \quad \text{on } \partial V.$$

Denote

$$\bar{t} = \sup\{t \geq 0 | [0, -tp] \subset V\},$$

where $[0, -tp]$ denote the line segment connecting 0 and $-tp$. Hence

$$c = \text{esssup}_V |Du| \geq \frac{u(0) - u(-\bar{t}p)}{|\bar{t}p|} = \frac{\delta}{|\bar{t}p|} + |p| > |p|.$$

Denote

$$K = \sup_{x \neq y \in \partial(V\setminus\{0\})} \frac{|u(x) - u(y)|}{|x - y|}.$$

Since $u$ is an absolute Lipschitz extension in $B_1(0)\setminus\{0\}$, owing to (1.2), we have that

$$K \geq c > |p|. \tag{3.6}$$

Also,

$$u(x) \leq p \cdot x - \epsilon|x| \leq |p||x| \quad \text{in V}. \tag{3.7}$$

Combining (3.6) and (3.7), we derive that

$$K = \max_{x \in \partial V} \frac{u(0) - u(x)}{|x|}.$$

Choose $\bar{x} \in \partial V$ such that

$$-u(\bar{x}) = u(0) - u(\bar{x}) = K|\bar{x}|.$$

Since $K > |p|$, by the triangle inequality and the definition of $K$, we have

$$\{t\bar{x} |\ 0 \le t < 1\} \subset V.$$

This implies $K \le c$. Therefore $K = c$ and

$$u(t\bar{x}) = -tu(\bar{x}) = -tc|\bar{x}| \quad \text{for } 0 \le t \le 1. \tag{3.8}$$

Note that $|p| < c = K$. Now suppose that $\lambda_m \to 0^+$ as $m \to +\infty$ and

$$\lim_{m\to+\infty} \frac{u(\lambda_m x)}{\lambda_m} = w(x).$$

Owing to (3.7) and (3.8), it is easy to see that $\frac{w(x)}{c}$ satisfies the assumptions in Lemma 3.1 with $e = \frac{-\bar{x}}{|\bar{x}|}$. So

$$w(x) = -c|x|.$$

Since this is true for any sequence $\{\lambda_m\}$, we derive that

$$\lim_{\lambda\to0^+} \frac{u(\lambda x)}{\lambda} = -c|x|.$$

Hence Theorem 1.1 holds. $\square$

**Proof of Theorem 1.3.** It is clear that when $R$ is large enough, by (ii), we have that

$$u(x) \le M + (1-\epsilon)R \le u(0) + (1 - \frac{\epsilon}{2})R.$$

Hence, by comparison with cones, we have

$$u(x) \le u(0) + (1 - \frac{\epsilon}{2})|x| \quad \text{on } B_R(0).$$

Sending $R \to +\infty$, we derive that for all $x \in \mathbb{R}^n$,

$$u(x) \le u(0) + (1 - \frac{\epsilon}{2})|x|.$$

Without loss of generality, we may assume that $u(0) = 0$. Therefore by Lemma 3.1, to prove Theorem 1.3, it suffices to show that there exists $e \in \partial B_1(0)$ such that for all $t \ge 0$,

$$u(-te) = -t. \tag{3.9}$$

We claim that

$$\lim_{r\to0} \text{esssup}_{B_r(0)}|Du| = 1. \tag{3.10}$$

If not, let us assume that there exists $r > 0$ and $\delta \in (0, \epsilon)$ such that

$$\text{esssup}_{B_r(0)}|Du| \le 1 - \delta. \tag{3.11}$$

Choose $x_0 \in \mathbb{R}^n$ such that

$$S_u(x_0) \ge 1 - \frac{\delta}{2}.$$

Owing to (2.5), $x_0 \notin \bar{B}_{\frac{r}{2}}(0)$. Hence by endpoint estimate (2.2), there exists a sequence $\{x_m\}_{m\geq 0}$ such that
(1)
$$|x_m - x_{m-1}| = \frac{r}{2};$$
(2)
$$u(x_m) - u(x_{m-1}) \geq S_u(x_0)\frac{r}{2}.$$
(3)
$$S_u(x_m) \geq 1 - \frac{\delta}{2}.$$
Therefore $\lim_{m\to+\infty} |x_m| = +\infty$ and
$$u(x_m) \geq u(x_0) + (1 - \frac{\delta}{2})|x_m - x_0| \geq u(x_0) + (1 - \frac{\epsilon}{2})|x_m - x_0|.$$
This contradicts to (ii) when $m$ is sufficiently large. Hence (3.10) holds. Owing to Lemma 4.2 in Appendix I, $u$ is either a viscosity supersolution or viscosity subsolution in $\mathbb{R}^n$. If $u$ is a viscosity subsolution in $\mathbb{R}^n$, then
$$S_u^+(0) = 1.$$
Hence by Theorem 2.2, there exists $e \in S_1$ such that for all $t \geq 0$,
$$u(te) = t.$$
This contradicts to (ii) when $t$ is suffcienty large. So $u$ must by a viscosity supersolution in $\mathbb{R}^n$. Then
$$S_u^-(0) = 1.$$
Hence by considering $-u$ and Theorem 2.2, there exists $e \in S_1$ such that for all $t \geq 0$,
$$u(-te) = -t.$$
□

**Proof of Corollary 1.2.** Without loss of generality, we assume that $x_0 = 0$. Choose $\bar{x} \in \partial\Omega$ such that $|\bar{x}| = d(0, \partial\Omega) = r$. Then by (1.2)
$$\mathrm{esssup}_{\Omega}|Du| = \frac{1}{r}$$
and for $0 \leq t \leq 1$,
$$u(t\bar{x}) = 1 - \frac{t}{r}.$$
Owing to Theorem 1.1,
$$\lim_{\lambda\to 0^+} \frac{u(\lambda x) - u(0)}{\lambda} = -\frac{|x|}{r}.$$
Therefore by Theorem 2.4,
$$\lim_{x\to 0} |Du(x) - \frac{x}{r|x|}| = 0.$$

Especially,

$$\lim_{x\to 0}|Du(x)| = \frac{1}{r}. \tag{3.12}$$

Choose $y_0 \in \Omega\backslash\{0\}$ such that $Du(y_0) \neq 0$. Then there exists $\delta > 0$ and $\xi : [0,\delta] \to \Omega$ such that

$$\dot{\xi}(t) = Du(\xi(t))$$

and

$$\xi(0) = y_0,\ \xi(\delta) = 0.$$

Since $|Du(\xi(t))| \equiv |Du(y_0)|$, (3.12) implies that

$$|Du(y_0)| = \frac{1}{r}.$$

Hence for $x \in \Omega\backslash\{0\}$,

$$|Du(x)| \equiv \frac{1}{r}.$$

So $u(x) = 1 - \frac{1}{r}|x|$. Since $u|_{\partial\Omega} = 0$, $\Omega = B_r(0)$. $\square$

**Corollary 3.2** *The uniformly Lipschitz continuous function constructed in the introduction is not $C^2$ in $\mathbb{R}^n\backslash\{0\}$.*

*Proof.* By the construction and comparison principle, $u$ satisfies the following
(1)

$$\text{esssup}_{\mathbb{R}^n}|Du| = 1,$$

(2) for $t \geq 0$,

$$u(0,-t) = -t.$$

(3)

$$u(x', x_n) = u(-x', x_n).$$

Owing to (1), (2) and Theorem 1.1, for $x$ near 0,

$$u(x) = -|x| + o(|x|). \tag{3.13}$$

Assume that $u \in C^2(\mathbb{R}^n\backslash\{0\})$. Since $|Du|$ is preserved along the gradient follow, due to (3), $u$ is linear along the half line $\{(0,...,0,t)|\ t \geq 0\}$. Since $u(0,...,0,1) = u(0)$, we have that for $t > 0$,

$$u(0,...,0,t) \equiv 0.$$

This is contradictory to (3.13). $\square$

**Definition 3.3** Let $F$ be a closed set and $g$ a uniformly Lipschtz continuous function on $F$, we say that $u$ is an absolute Lipschitz extension of $(F,g)$ if $u|_F = g$ and for any open subset $U \subset \mathbb{R}^n\backslash F$,

$$\sup_{x,y\in\bar{U},x\neq y}\frac{u(x)-u(y)}{|x-y|} = sup_{x,y\in\partial U,x\neq y}\frac{u(x)-u(y)}{|x-y|}. \tag{3.14}$$

In general, the uniqueness of absolute Lipschitz extension is an open problem. In the following, as an application of Lemma 3.1, we will prove the uniqueness of Lipschitz extension for a special pair of $(F,g)$. Fix $e\in S_1$, we choose $F=\{te||t\leq 0\}$ and $g(x)=e\cdot x$. When $n\geq 2$, we can see that $u(x)=-|x|$ is an absolute Lipschitz extension of $(F,g)$. Moreover, Definition 3.3 implies that any absolute Lipschitz extension $u$ of $(F,g)$ satisfies
(1) $\mathrm{esssup}_{\mathbb{R}^n}|Du|=1$;
(2) $u\leq 0$;
(3) $u$ is an infinity harmonic function in $\mathbb{R}^n\backslash\{0\}$.
(4) $u(-te)=-t$ for $t\geq 0$.
(1) and (4) are obvious. (2) follows form applying (3.14) to the open set $U_\epsilon=\{x\in\mathbb{R}^n|\ u(x)>\epsilon\}$ for any $\epsilon>0$. We want to say a little bit about (3). It is clear that $u$ is an infinity harmonic function in $\mathbb{R}^n\backslash F$. Owing to Lemma 5.2 in Appendix II, for $x\in F\backslash\{0\}$, $Du(x)=e$. Hence by the definition of viscosity solutions and (4), $u$ is an infinity harmonic function on $F\backslash\{0\}$. The following corollary is an immediate result of Lemma 3.1.

**Corollary 3.4** $u(x)=-|x|$ *is the unique absolute Lipschitz extension of* $(F,g)$.

# 4 Appendix I: A simple lemma of isolated singularity of fully nonlinear elliptic equations

Denote $S^{n\times n}$ as the collection of all symmetric matrices. Suppose that $F\in C(S^{n\times n}\times\mathbb{R}^n\times\mathbb{R}\times\Omega)$ and satisfies that

$$F(M_1,p,z,x)\geq F(M_2,p,z,x)$$

if all the eigenvalues of $M_1-M_2$ are nonnegative.

**Definition 4.1** We say that $u\in C(\Omega)$ is a *viscosity supersolution (subsolution)* of

$$F(D^2u,Du,u,x)=0$$

if for any $\phi\in C^2(\Omega)$ and $x_0\in\Omega$,

$$\phi(x)-u(x)\leq(\geq)\phi(x_0)-u(x_0)=0$$

implies that

$$F(D^2\phi(x_0),D\phi(x_0),\phi(x_0),x_0)\leq(\geq)0.$$

$u$ is *viscosity solution* if it is both a supersolution and a subsolution.

The following is a simple lemma. Similar argument can be found in [D].

**Lemma 4.2** *Suppose that* $u\in C(B_1)$ *and is viscosity solution of the equation*

$$F(D^2u,Du,u,x)=0\quad\textit{in } B_1(0)\backslash\{0\}.$$

*Then $u$ is either a viscosity supersolution or a viscosity subsolution in the entire ball. Especially, if $u$ is differentiable at* 0*, then $u$ is solution in the entire ball, i.e,* 0 *is a removable singularity.*

*Proof.* I claim that if $u$ is not a viscosity supersolution then there exists $\epsilon > 0$ and $p \in \mathbb{R}^n$ such that

$$u(x) \geq p \cdot x + \epsilon|x| \quad \text{in } \bar{B}_\epsilon(0). \tag{4.1}$$

In fact, if $u$ is not a viscosity supersolution in the entire ball, then there exists $\phi \in C^2(B_1(0))$ such that

$$\phi(x) - u(x) < \phi(0) - u(0) = 0 \quad \text{for } x \in B_1(0)\backslash\{0\} \tag{4.2}$$

and

$$F(D^2\phi(0), D\phi(0), u(0), 0) > 0.$$

Let us choose $p = D\phi(0)$. If (4.1) is not true, then for any $m \in \mathbb{N}$, there exists $x_m \in B_{\frac{1}{m}}(0)$ such that

$$u(x_m) < \phi(x_m) + \frac{1}{m}|x_m|. \tag{4.3}$$

It is clear that $x_m \neq 0$. Denote

$$\phi_m(x) = \phi(x) + \frac{x_m}{m|x_m|} \cdot x.$$

Choose $y_m \in \bar{B}_1(0)$ such that

$$u(y_m) - \phi_m(y_m) = \min_{\bar{B}_1(0)} (u - \phi_m).$$

Owing to (4.2) and (4.3), $y_m \neq 0$ and $\lim_{m \to +\infty} y_m = 0$. Hence

$$F(D^2\phi(y_m), D\phi(y_m) + \frac{x_m \cdot y_m}{m|x_m|}, u(y_m), y_m) \leq 0.$$

Sending $m \to +\infty$, we derive that

$$F(D^2\phi(0), D\phi(0), u(0), 0) \leq 0.$$

This is a contradiction. Hence (4.1) holds. Similarly, we can show that if $u$ is not a viscosity subsolution at 0 then there exists $\epsilon > 0$ and $p \in \cdot\mathbb{R}^n$ such that

$$u(x) \leq p \cdot x - \epsilon|x| \quad \text{in } \bar{B}_\epsilon(0). \tag{4.4}$$

Note that (4.1) and (4.4) can not happen simultaneously. Especially, if $u$ is differentiable at 0, neither can happen. Hence Lemma 4.2 holds. □

**Remark 4.3** If $F$ is the infinity Laplacian operator, i.e, $F = p \cdot M \cdot p$, then the vector $p \in \mathbb{R}^n$ in (4.1) and (4.4) is not 0 since $p = D\phi(0)$.

# 5 Appendix II: Tightness argument and conclusions

The results in the section are well known. We present here for reader's convenience.

**Lemma 5.1** *Suppose that* $u \in W^{1,\infty}(\mathbb{R}^n)$ *and satisfies that*
*(1)* $\mathrm{esssup}_{\mathbb{R}^n}|Du| \leq 1$*;*
*(2) for* $t \geq 0$*,*

$$u(0, ..., 0, -t) = -t.$$

*Then for* $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$*,*

$$u(x) \leq x_n. \tag{5.1}$$

*Especially, if (2) is true for all* $t \in \mathbb{R}$*, then* $u = x_n$*.*

*Proof.* Note that (1) and (2) imply that $\mathrm{esssup}_{\mathbb{R}^n}|Du| = 1$. Owing to (1) and (2), for $t > 0$,

$$|u(x) + t| = |u(x) - u(0, ...0, -t)| \leq \sqrt{|x'|^2 + (x_n + t)^2}.$$

Hence

$$(u(x))^2 + 2tu(x) \leq |x'|^2 + x_n^2 + 2tx_n.$$

So

$$\frac{(u(x))^2}{2t} + u(x) \leq \frac{|x'|^2 + x_n^2}{2t} + x_n.$$

Sending $t \to +\infty$, we derive (5.1). $\Box$

**Lemma 5.2** *Suppose that* $u \in W^{1,\infty}(B_1(0))$ *and satisfies that*
*(1)*

$$\mathrm{esssup}_{B_1(0)}|Du| \leq 1; \tag{5.2}$$

*(2) for some* $e \in \partial B_1(0)$*,*

$$u(e) - u(0) = 1 \tag{5.3}$$

*Then for* $0 < t < 1$*,*

$$u(te) = u(0) + t \tag{5.4}$$

*and*

$$Du(te) = e. \tag{5.5}$$

Proof Owing to (5.2), for any $x$, $y \in B_1(0)$,

$$|u(x) - u(y)| \leq |x - y|.$$

Hence (5.4) follows from (5.3) and triangle inequality. Now choose $x_0 \in \{te|\ 0 < t < 1\}$. Suppose that $\lambda_m \to 0$ as $m \to +\infty$ and

$$\lim_{m\to+\infty} \frac{u(\lambda_m x + x_0) - u(x_0)}{\lambda_m} = w(x).$$

By (5.2) and (5.4), $w(x)$ satisfies that

$$\mathrm{esssup}_{\mathbb{R}^n}|Dw| \leq 1$$

and for all $t \in \mathbb{R}$

$$w(te) = t.$$

Hence by Lemma 5.1, $w(x) = e \cdot x$. Since this is true for sequence $\{\lambda_m\}$, we get that

$$\lim_{\lambda \to 0} \frac{u(\lambda x + x_0) - u(x_0)}{\lambda} = e \cdot x.$$

Therefore (5.5) holds. $\square$

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