SOME MINIMIZATION PROBLEMS IN THE CLASS OF CONVEX FUNCTIONS WITH PRESCRIBED DETERMINANT

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Abstract. We consider minimizers of linear functionals of the type

$$L(u) = \int_{\partial\Omega} u\, d\sigma - \int_{\Omega} u\, dx$$

in the class of convex functions u with prescribed determinant $\det D^2u = f$.

We obtain compactness properties for such minimizers and discuss their regularity in two dimensions.

1. Introduction

In this paper, we consider minimizers of certain linear functionals in the class of convex functions with prescribed determinant. We are motivated by the study of convex minimizers u for convex energies E of the type

$$E(u) = \int_{\Omega} F(\det D^2u)\, dx + L(u), \qquad \text{with } L \text{ a linear functional,}$$

which appear in the work of Donaldson [D1]-[D4] in the context of existence of Kähler metrics of constant scalar curvature for toric varieties. The minimizer u solves a fourth order elliptic equation with two nonstandard boundary conditions involving the second and third order derivatives of u (see (1.4) below). In this paper, we consider minimizers of L (or E) in the case when the determinant $\det D^2u$ is prescribed. This allows us to understand better the type of boundary conditions that appear in such problems and to obtain estimates also for unconstrained minimizers of E.

The simplest minimization problem with prescribed determinant which is interesting in its own right is the following

$$\text{minimize } \int_{\partial\Omega} u\, d\sigma, \qquad \text{with } u \in \mathcal{A}_0,$$

where Ω is a bounded convex set, $d\sigma$ is the surface measure of $\partial\Omega$, and $\mathcal{A}_0$ is the class of nonnegative solutions to the Monge-Ampère equation $\det D^2u = 1$:

$$\mathcal{A}_0 := \{u : \bar{\Omega} \to [0, \infty) |\, u \text{ convex}, \quad \det D^2u = 1\}.$$

Question: Is the minimizer u smooth up to the boundary $\partial\Omega$ if Ω is a smooth, say uniformly convex, domain?

In the present paper, we answer this question affirmatively in dimensions $n = 2$. First, we remark that the minimizer must vanish at x_0, the center of mass of $\partial\Omega$:

$$x_0 = \fint_{\partial\Omega} x\, d\sigma.$$

This follows easily since

$$u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \in \mathcal{A}_0$$

and

$$\int_{\partial\Omega} [u(x) - u(x_0) - \nabla u(x_0)(x - x_0)] d\sigma = \int_{\partial\Omega} [u - u(x_0)]\, d\sigma \leq \int_{\partial\Omega} u\, d\sigma,$$

with strict inequality if $u(x_0) > 0$. Thus we can reformulate the problem above as minimizing

$$\int_{\partial\Omega} u\, d\sigma - \mathcal{H}^{n-1}(\partial\Omega)\ u(x_0)$$

in the set of all solutions to the Monge-Ampère equation $\det D^2 u = 1$ which are not necessarily nonnegative. This formulation is more convenient since now we can perturb functions in all directions.

More generally, we consider linear functionals of the type

$$L(u) = \int_{\partial\Omega} u\, d\sigma - \int_{\Omega} u\, dA,$$

with $d\sigma$, dA nonnegative Radon measures supported on $\partial\Omega$ and Ω respectively. In this paper, we study the existence, uniqueness and regularity properties for minimizers of L. i.e.,

$$\text{(P)} \qquad \text{minimize } L(u) \text{ for all } u \in \mathcal{A}$$

in the class $\mathcal{A}$ of subsolutions (solutions) to a Monge-Ampère equation $\det D^2 u \geq f$:

$$\mathcal{A} := \{u : \overline{\Omega} \to \mathbb{R} |\, u \text{ convex}, \quad \det D^2 u \geq f\}.$$

Notice that we are minimizing a linear functional L over a convex set $\mathcal{A}$ in the cone of convex functions.

Clearly, the minimizer of the problem (P) satisfies $\det D^2 u = f$ in Ω. Otherwise we can find $v \in \mathcal{A}$ such that $v = u$ in a neighborhood of $\partial\Omega$, and $v \geq u$ in Ω with strict inequality in some open subset, thus $L(v) < L(u)$.

We assume throughout that the following 5 conditions are satisfied:

1) Ω is a bounded, uniformly convex, $C^{1,1}$ domain.

2) f is bounded away from 0 and ∞.

3)

$$d\sigma = \sigma(x)\, d\mathcal{H}^{n-1} \lfloor \partial\Omega,$$

with the density $\sigma(x)$ bounded away from 0 and ∞.

4)

$$dA = A(x)\, dx \quad \text{in a small neighborhhod of } \partial\Omega$$

with the density $A(x)$ bounded from above.

5)

$$L(u) > 0 \text{ for all } u \text{ convex but not linear.}$$

The last condition is known as *the stability of* L (see [D1]) and in 2D, is equivalent to saying that, for all linear functions l, we have

$$L(l) = 0 \qquad \text{and} \quad L(l^+) > 0 \quad \text{if} \quad l^+ \not\equiv 0 \quad \text{in } \Omega,$$

where $l^+ = \max(l, 0)$ (see Proposition 2.4).

Notice that the stability of L implies that $L(l) = 0$ for any linear function l, hence $d\sigma$ and dA must have the same mass and the same center of mass.

A minimizer u of the functional L is determined up to linear functions since both L and $\mathcal{A}$ are invariant under addition with linear functions. We "normalize" u by subtracting its the tangent plane at, say the center of mass of Ω. In Section 2, we shall prove in Proposition 2.5 that there exists a unique normalized minimizer to the problem (P).

We also prove a compactness theorem for minimizers.

Theorem 1.1 (Compactness)**.** *Let u_k be the normalized minimizers of the functionals L_k with data $(f_k, d\sigma_k, dA_k, \Omega)$ that has uniform bounds in k. Precisely, the inequalities* (2.1) *and* (2.4) *below are satisfied uniformly in k and $\rho \le f_k \le \rho^{-1}$. If*

$$f_k \rightharpoonup f, \ d\sigma_k \rightharpoonup d\sigma, \ dA_k \rightharpoonup dA,$$

then $u_k \to u$ uniformly on compact sets of Ω where u is the normalized minimizer of the functional L with data $(f, d\sigma, dA, \Omega)$.

If u is a minimizer, then the Euler-Lagrange equation reads (see Proposition 3.6)

$$\text{if } \varphi : \Omega \to \mathbb{R} \text{ solves } U^{ij}\varphi_{ij} = 0 \text{ then } L(\varphi) = 0,$$

where U^{ij} are the entries of the cofactor matrix U of the Hessian D^2u. Since the linearized Monge-Ampère equation is also an equation in divergence form, we can always express the Ω-integral of a function φ in terms of a boundary integral. For this, we consider the solution v to the Dirichlet problem

$$U^{ij}v_{ij} = -dA \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

Integrating by parts twice and using $\partial_i(U^{ij}) = \partial_j(U^{ij}) = 0$, we can compute

$$\begin{aligned} \int_\Omega \varphi \, dA &= -\int_\Omega \varphi \, U^{ij} v_{ij} \\ &= \int_\Omega \varphi_i \, U^{ij} v_j \, - \int_{\partial\Omega} \varphi U^{ij} v_j \nu_i \\ &= -\int_\Omega (U^{ij}\varphi_{ij}) v \, + \int_{\partial\Omega} \varphi_i U^{ij} v \nu_j - \int_{\partial\Omega} \varphi U^{ij} v_j \nu_i \\ &= -\int_{\partial\Omega} \varphi \; U^{ij} v_i \nu_j. \end{aligned} \tag{1.1}$$

From the Euler-Lagrange equation, we obtain

$$U^{ij} v_i \nu_j = -\sigma \quad \text{on } \partial\Omega.$$

Since $v = 0$ on $\partial\Omega$, we have $v_i = v_\nu \nu_i$, and hence

$$U^{ij} v_i \nu_j = U^{ij} \nu_i \nu_j v_\nu = U^{\nu\nu} v_\nu = \left(\det D^2_{x'} u\right) v_\nu$$

with $x' \perp \nu$ denoting the tangential directions along $\partial\Omega$. In conclusion, if u is a smooth minimizer then there exists a function v such that (u, v) solves the system

$$\begin{cases} \det D^2 u \ = f & \text{in } \Omega, \\ U^{ij} v_{ij} = -dA & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ U^{\nu\nu} v_\nu = -\sigma & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This system is interesting since the function v above satisfies two boundary conditions, Dirichlet and Neumann, while u has no boundary conditions. Heuristically, the boundary values for u can be recovered from the term $U^{\nu\nu} = \det D^2_{x'} u$ which appears in the Neumann boundary condition for v.

Our main regularity results for the minimizers u are in two dimensions.

Theorem 1.2. *Assume that $n = 2$, and the conditions 1)-5) hold. If $\sigma \in C^\alpha(\partial\Omega)$, $f \in C^\alpha(\overline{\Omega})$, and $\partial\Omega \in C^{2,\alpha}$, then the minimizer $u \in C^{2,\alpha}(\overline{\Omega})$ and the system* (1.2) *holds in the classical sense.*

We obtain Theorem 1.2 by showing that u separates quadratically on $\partial\Omega$ from its tangent planes and then we apply the boundary Hölder gradient estimates for v which were obtained in [LS].

As a consequence of Theorem 1.2, we obtain higher regularity if the data $(f, d\sigma, dA, \Omega)$ is more regular.

Theorem 1.3. *Assume that $n = 2$ and the conditions 1)-5) hold. If $\sigma \in C^\infty(\partial\Omega)$, $f \in C^\infty(\overline{\Omega})$, $A \in C^\infty(\overline{\Omega})$, $\partial\Omega \in C^\infty$, then $u \in C^\infty(\overline{\Omega})$.*

In Section 6, we provide an example of Pogorelov type for a minimizer in dimensions $n \geq 3$ that shows that Theorem 1.3 does not hold in this generality in higher dimensions.

We explain briefly how Theorem 1.3 follows from Theorem 1.2. If $u \in C^{2,\alpha}(\overline{\Omega})$, then $U^{ij} \in C^\alpha(\overline{\Omega})$ and Schauder estimates give $v \in C^{2,\alpha}(\overline{\Omega})$, thus $v_\nu \in C^{1,\alpha}(\partial\Omega)$. From the last equation in (1.2) we obtain $U^{\nu\nu} = \det D^2_{x'} u \in C^{1,\alpha}(\partial\Omega)$. This implies $u \in C^{3,\alpha}(\partial\Omega)$ and from the first equation in (1.2) we find $u \in C^{3,\alpha}(\overline{\Omega})$. We can repeat the same argument and obtain that $u \in C^{k,\alpha}$ for any $k \geq 2$.

As we mentioned above, our constraint minimization problem is motivated by the minimization of the Mabuchi energy functional from complex geometry in the case of toric varieties

$$M(u) = \int_\Omega -\log\det D^2 u + \int_{\partial\Omega} u d\sigma - \int_\Omega u dA.$$

In this case, $d\sigma$ and dA are canonical measures on $\partial\Omega$ and Ω. Minimizers of M satisfy the following fourth order equation, called Abreu's equation [A]

$$u_{ij}^{ij} := \sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -A,$$

where u^{ij} are the entries of the inverse matrix of D^2u. This equation and the functional M have been studied extensively by Donaldson in a series of papers [D1]-[D4] (see also [ZZ]). In these papers, the domain Ω was taken to be a polytope $P \subset \mathbb{R}^n$ and A was taken to be a positive constant. The existence of smooth solutions with suitable boundary conditions has important implications in complex geometry. It says that we can find Kähler metrics of constant scalar curvature for toric varieties.

More generally, one can consider minimizers of the following convex functional

$$E(u) = \int_\Omega F(\det D^2 u) + \int_{\partial\Omega} u d\sigma - \int_\Omega u dA \tag{1.3}$$

where $F(t^n)$ is a convex and decreasing function of $t \geq 0$. The Mabuchi energy functional corresponds to $F(t) = -\log t$ whereas in our minimization problem (P) (with $f \equiv 1$)

$$F(t) = \begin{cases} \infty & \text{if } t < 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

Minimizers of E satisfy a system similar to (1.2):

$$\begin{cases} -F'(\det D^2 u) = v & \text{in } \Omega, \\ U^{ij} v_{ij} = -dA & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ U^{\nu\nu} v_\nu = -\sigma & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

A similar system but with different boundary conditions was investigated by Trudinger and Wang in [TW2]. If the function F is strictly decreasing then we see from the first and third equations above that $\det D^2u = \infty$ on $\partial\Omega$, and therefore we cannot expect minimizers to be smooth up to the boundary (as is the case with the Mabuchi functional $M(u)$).

If F is constant for large values of t (as in the case we considered) then $\det D^2 u$ becomes finite on the boundary and smoothness up to the boundary is expected. More precisely assume that

$$F \in C^{1,1}((0,\infty)), \quad G(t) := F(t^n) \quad \text{is convex in } t, \quad \text{and} \quad G'(0^+) = -\infty,$$

and there exists $t_0 > 0$ such that

$$F(t) = 0 \quad \text{on } [t_0, \infty), \qquad F''(t) > 0 \quad \text{on } (0, t_0].$$

Theorem 1.4. *Assume $n = 2$, and the conditions 1)-5) and the above hypotheses on F are satisfied. If $\sigma \in C^\alpha(\partial\Omega)$, $A \in C^\alpha(\overline{\Omega})$, $\partial\Omega \in C^{2,\alpha}$ then the normalized minimizer u of the functional E defined in* (1.3) *satisfies $u \in C^{2,\alpha}(\overline{\Omega})$ and the system* (1.4) *holds in the classical sense.*

The paper is organized as follows. In Section 2, we discuss the notion of stability for the functional L and prove existence, uniqueness and compactness of minimizers of the problem (P). In Section 3, we state a quantitative version of Theorem 1.2, Proposition 3.1, and we also obtain the Euler-Lagrange equation. Proposition 3.1 is proved in sections 4 and 5, first under the assumption that the density A is bounded from below and then in the general case. In Section 6, we give an example of a singular minimizer in dimension $n \geq 3$. Finally, in Section 7, we prove Theorem 1.4.

2. Stability inequality and existence of minimizers

Let Ω be a bounded convex set and define

$$L(u) = \int_{\partial\Omega} u \, d\sigma - \int_{\Omega} u \, dA$$

for all convex functions $u : \overline{\Omega} \to \mathbb{R}$ with $u \in L^1(\partial\Omega, d\sigma)$. We assume that

$$\sigma \geq \rho \text{ on } \partial\Omega \text{ and } A(x) \leq \rho^{-1} \text{ in a neighborhood of } \partial\Omega, \tag{2.1}$$

for some small $\rho > 0$, and that L is stable, i.e.,

$$L(u) > 0 \text{ for all } u \text{ convex but not linear.} \tag{2.2}$$

Assume for simplicity that 0 is the center of mass of Ω. We notice that (2.2) implies $L(l) = 0$ for any l linear since l can be approximated by both convex and concave functions. We "normalize" a convex function by subtracting its tangent plane at 0, and this does not change the value of L. First, we prove some lower semicontinuity properties of L with respect to normalized solutions.

Lemma 2.1 (Lower semicontinuity)**.** *Assume that* (2.1) *holds and* (u_k) *is a normalized sequence that satisfies*

$$\int_{\partial\Omega} u_k \, d\sigma \leq C, \qquad u_k \to u \quad \textit{uniformly on compact sets of } \Omega, \tag{2.3}$$

for some function $u : \Omega \to \mathbb{R}$. Let $\bar{u}$ be the minimal convex extension of u to $\overline{\Omega}$, i.e.,

$$\bar{u} = u \quad \textit{in } \Omega, \quad \bar{u}(x) = \lim_{t \to 1^-} u(tx) \quad \textit{if } x \in \partial\Omega.$$

Then

$$\int_{\Omega} u \, dA = \lim \int_{\Omega} u_k \, dA, \qquad \int_{\partial\Omega} \bar{u} \, d\sigma \leq \liminf \int_{\partial\Omega} u_k \, d\sigma,$$

and thus

$$L(\bar{u}) \leq \liminf L(u_k).$$

Remark: The function $\bar{u}$ has the property that its upper graph is the closure of the upper graph of u in $\mathbb{R}^{n+1}$.

Proof. Since u_k are normalized, they are increasing on each ray out of the origin. For each $\eta > 0$ small, we consider the set $\Omega_\eta := \{x \in \Omega : dist(x, \partial\Omega) < \eta\}$, and from (2.1) we obtain

$$\int_{\Omega_\eta} u_k \, dA \leq C\rho^{-1}\eta \int_{\partial\Omega} u_k \, d\sigma \leq C\eta.$$

Since this inequality holds for all small $\eta \to 0$, we easily obtain

$$\int_\Omega u \, dA = \lim \int_\Omega u_k \, dA.$$

For each $z \in \partial\Omega$, and $t < 1$ we have $u_k(tz) \leq u_k(z)$. We let $k \to \infty$ in the inequality

$$\int_{\partial\Omega} u_k(tz) \, d\sigma \leq \int_{\partial\Omega} u_k(z) \, d\sigma$$

and obtain

$$\int_{\partial\Omega} u(tz) \, d\sigma \leq \liminf \int_{\partial\Omega} u_k(z) \, d\sigma,$$

and then we let $t \to 1^-$,

$$\int_{\partial\Omega} \bar{u} \, d\sigma \leq \liminf \int_{\partial\Omega} u_k \, d\sigma.$$

□

Remark 2.2. From the proof we see that if we are given functionals L_k with measures σ_k, A_k that satisfy (2.1) uniformly in k and

$$\sigma_k \rightharpoonup \sigma, \quad A_k \rightharpoonup A,$$

and if (2.3) holds for a sequence u_k, then the statement still holds, i.e.,

$$L(\bar{u}) \leq \liminf L_k(u_k).$$

By compactness, one can obtain a quantitative version of (2.2) known as *stablity inequality*. This was done by Donaldson, see Proposition 5.2.2 in [D1]. For completeness, we sketch its proof here.

Proposition 2.3. *Assume that (2.1) and (2.2) hold. Then we can find $\mu > 0$ such that*

$$L(u) := \int_{\partial\Omega} u d\sigma - \int_\Omega u dA \geq \mu \int_{\partial\Omega} u d\sigma \tag{2.4}$$

for all convex functions u normalized at 0.

Proof. Assume the conclusion does not hold, so there is a sequence of normalized convex functions (u_k) with

$$\int_{\partial\Omega} u_k d\sigma = 1, \quad \lim L(u_k) = 0,$$

thus

$$\lim \int_\Omega u_n dA = 1.$$

Using convexity, we may assume that u_k converges uniformly on compact subsets of Ω to a limiting function $u \geq 0$. Let $\bar{u}$ be the minimal convex extension of u to $\overline{\Omega}$. Then, from Lemma 2.1, we obtain

$$L(\bar{u}) = 0, \quad \int_{\Omega} \bar{u}\, dA = 1,$$

thus $\bar{u} \geq 0$ is not linear, and we contradict (2.2). $\square$

Donaldson showed that when $n = 2$, the stability condition can be checked easily (see Proposition 5.3.1 in [D1]).

Proposition 2.4. *Assume $n = 2$, (2.1) holds and for all linear functions l we have*

$$L(l) = 0 \qquad \text{and} \quad L(l^+) > 0 \quad \text{if} \quad l^+ \not\equiv 0 \quad \text{in } \Omega, \tag{2.5}$$

where $l^+ = \max(l, 0)$. Then L is stable, i.e., condition (2.2) is satisfied.

Proof. For completeness, we sketch the proof. Assume by contradiction that $L(u) \leq 0$ for some convex function u which is not linear in Ω. Let u^* be the convex envelope generated by the boundary values of $\bar{u}$ - the minimal convex extension of u to $\overline{\Omega}$. Notice that $u^* = \bar{u}$ on $\partial\Omega$. Since $L(u^*) \leq L(\bar{u}) \leq L(u)$ we find $L(u^*) \leq 0$. Notice that u^* is not linear since otherwise $0 = L(u^*) < L(\bar{u}) \leq 0$ (we used that $\bar{u}$ is not linear). After subtracting a linear function we may assume that u^* is normalized and u^* is not identically 0.

We obtain a contradiction by showing that u^* satisfies the stability inequality. By our hypotheses there exists $\mu > 0$ small such that

$$L(l^+) \geq \mu \int_{\partial\Omega} l^+\, d\sigma,$$

for any l^+. Indeed, by (2.1) this inequality is valid if the "crease" $\{l = 0\}$ is near $\partial\Omega$ and for all other l's, it follows by compactness from (2.5). We approximate from below u^* by u_k^* which is defined as the maximum of the tangent planes of u^* at some points $y_i \in \Omega$, $i = 1, .., k$. Since u^* is a convex envelope in 2D, u_k^* is a discrete sum of l^+'s hence it satisfies the stability inequality. Now we let $k \to \infty$; since $u_k^* \leq u^*$, using Lemma 2.1, we obtain that u^* also satisfies the stability inequality. $\square$

Proposition 2.5. *Assume that (2.1) and (2.2) hold. Then there exists a unique (up to linear functions) minimizer u of L subject to the constraint*

$$u \in \mathcal{A} := \{v : \overline{\Omega} \to \mathbb{R} |\, v \text{ convex}, \quad \det D^2 v \geq f\},$$

where $\rho \leq f \leq \rho^{-1}$ for some $\rho > 0$. Moreover, $\det D^2 u = f$.

Proof. Let (u_k) be a sequence of normalized solutions such that $L(u_k) \to \inf_{\mathcal{A}} L$. By the stability inequality, we see that $\int_{\partial\Omega} u_k\, d\sigma$ are uniformly bounded, and after passing to a subsequence, we may assume that u_k converges uniformly on compact subsets of Ω to a function u. Then $u \in \mathcal{A}$ and from the lower semicontinuity, we see that $L(u) = \inf_{\mathcal{A}} L$, i.e., u is a minimizer. Notice that $\det D^2 u = f$. Indeed, if a quadratic polynomial P with $\det D^2 P > f$ touches u strictly by below at some point $x_0 \in \Omega$, in a neighborhood of

x_0, then we can replace u in this neighborhood by $\max\{P+\epsilon, u\} \in \mathcal{A}$, and the energy decreases.

Next we assume w is another minimizer. We use the strict concavity of $M \mapsto \log(\det D^2 M)$ in the space of positive symmetric matrices M, and obtain that for a.e. x where u, w are twice differentiable

$$\log\det D^2(\frac{u+w}{2})(x) \geq \frac{1}{2}\log\det D^2u(x) + \frac{1}{2}\log\det D^2 w(x) \geq \log f(x).$$

This implies $(u+w)/2 \in \mathcal{A}$ is also a minimizer and $D^2u = D^2w$ a.e in Ω. Since f is bounded above and below we know that $u, w \in W^{2,1}_{loc}$ (see [DF]) in the open set Ω' where both u, w are strictly convex. This gives that $u-w$ is linear on each connected component of Ω'. If $n=2$, then $\Omega' = \Omega$ hence $u-w$ is linear. If $n \geq 3$, Labutin showed in [L] that the closed set $\Omega \setminus \Omega'$ has Hausdorff dimension $n-2+2/n < n-1$, hence Ω' is connected, and we obtain the same conclusion that $u-w$ is linear in Ω. $\square$

Remark: The arguments above show that the stability condition is also necessary for the existence of a minimizer. Indeed, if u is a minimizer and $L(u_0) = 0$ for some convex function u_0 that is not linear, then $u + u_0$ is also a minimizer and we contradict the uniqueness.

Proof of Theorem 1.1. We assume that the data $(f_k, d\sigma_k, dA_k, \Omega)$ satisfies (2.1), (2.4) uniformly in k and $\rho \leq f_k \leq \rho^{-1}$. For each k, let w_k be the convex solution to $\det D^2 w_k = f_k$ in Ω with $w_k = 0$ on $\partial\Omega$. Since f_k are bounded from above we find $w_k \geq -C$, and so by the minimality of u_k

$$L_k(u_k) \leq L_k(w_k) \leq C.$$

It follows from the stability inequality that

$$\int_{\partial\Omega} u_k \, d\sigma_k \leq C,$$

and we may assume, after passing to a subsequence, that $u_k \to u$ uniformly on compact sets.

We need to show that u is a minimizer for L with data $(f, d\sigma, dA, \Omega)$. For this it suffices to prove that for any continuous $v : \overline{\Omega} \to \mathbb{R}$ which solves $\det D^2 v = f$ in Ω, we have $L(u) \leq L(v)$.

Let v_k be the solution to $\det D^2 v_k = f_k$ with boundary data $v_k = v$ on $\partial\Omega$. Using appropriate barriers it is standard to check that $f_k \rightharpoonup f$, $f_k \leq \rho^{-1}$ implies $v_k \to v$ uniformly in $\overline{\Omega}$. Then, we let $k \to \infty$ in $L_k(u_k) \leq L_k(v_k)$, use Remark 2.2 and obtain

$$L(u) \leq \liminf L_k(u_k) \leq \lim L_k(v_k) = L(v),$$

which finishes the proof. $\square$

3. Preliminaries and the Euler-Lagrange equation

We rewrite our main hypotheses in a quantitative way. We assume that for some small $\rho > 0$ we have

H1) the curvatures of $\partial\Omega$ are bounded from below by ρ and from above by ρ^{-1};

H2) $\rho \leq f \leq \rho^{-1}$;

H3) $d\sigma = \sigma(x)\, d\mathcal{H}^{n-1}\lfloor\partial\Omega$, with $\rho \leq \sigma(x) \leq \rho^{-1}$;

H4) $dA = A(x)\, dx$ in a small neighborhood $\Omega_\rho := \{x \in \Omega| \quad dist\,(x, \partial\Omega) < \rho\}$ of $\partial\Omega$ with $A(x) \leq \rho^{-1}$.

H5) for any convex function u normalized at the center of mass of Ω, we have

$$L(u) := \int_{\partial\Omega} u\, d\sigma - \int_\Omega u\, dA \geq \rho \int_{\partial\Omega} u\, d\sigma.$$

We denote by c, C positive constants depending on ρ, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants.*

Our main theorem, Theorem 1.2, follows from the next proposition which deals with less regular data.

Proposition 3.1. *Assume that $n = 2$ and the conditions H1-H5 hold.*
(i) Then the minimizer u obtained in Proposition 2.5 satisfies $u \in C^{1,\beta}(\overline{\Omega}) \cap C^{1,1}(\partial\Omega)$ for some universal $\beta \in (0,1)$ and u separates quadratically from its tangent planes on $\partial\Omega$, i.e.,

$$C^{-1}|x-y|^2 \leq u(y) - u(x) - \nabla u(x)(y-x) \leq C|x-y|^2, \qquad \forall x, y \in \partial\Omega,$$

for some $C > 0$ universal.
(ii) If in addition $\sigma \in C^\alpha(\partial\Omega)$, then $u\mid_{\partial\Omega} \in C^{2,\gamma}(\partial\Omega)$, with $\gamma := \min\{\alpha, \beta\}$ and

$$\|u\|_{C^{2,\gamma}(\partial\Omega)} \leq C\|\sigma\|_{C^\gamma(\partial\Omega)}.$$

It is interesting to remark that in part (ii), we obtain $u \in C^{2,\gamma}(\partial\Omega)$ even though f and A are assumed to be only L^∞.

Proposition 3.1 implies Theorem 1.2. Theorem 7.3 in [S2] states that a solution to the Monge-Ampère equation which separates quadratically from its tangent planes on the boundary satisfies the classical C^α-Schauder estimates. Thus, if the assumptions of Proposition 3.1 ii) are satisfied and $f \in C^\alpha(\overline{\Omega})$ then $u \in C^{2,\gamma}(\overline{\Omega})$ with its $C^{2,\gamma}$ norm bounded by a constant C depending on ρ, α, $\|\sigma\|_{C^\alpha(\partial\Omega)}$, $\|\partial\Omega\|_{C^{2,\alpha}}$, and $\|f\|_{C^\alpha(\overline{\Omega})}$. This implies that the system (1.2) holds in the classical sense. If $\alpha \leq \beta$ then we are done. If $\alpha > \beta$ then we use $v \in C^{2,\beta}(\overline{\Omega})$ in the last equation of the system and obtain $u \in C^{2,\alpha}(\partial\Omega)$ which gives $u \in C^{2,\alpha}(\overline{\Omega})$. □

We prove Proposition 3.1 in the next two sections. Part (ii) follows from part (i) and the boundary Harnack inequality for the linearized Monge-Ampère equation which was obtained in [LS] (see Theorem 2.4). This theorem states that if a solution to the Monge-Ampère equation with bounded right hand side separates quadratically from its tangent planes on the boundary, then the classical boundary estimate of Krylov holds for solutions of the associated linearized equation.

In order to simplify the ideas we prove the proposition in the case when the hypotheses H1, H2, H4 are replaced by

H1') $\Omega = B_1$;

H2') $f \in C^\infty(\overline{\Omega})$, $\rho \le f \le \rho^{-1}$;

H4') $dA = A(x)\,dx$ with $\rho \le A(x) \le \rho^{-1}$ in Ω and $A \in C^\infty(\Omega)$.

We use H1' only for simplicity of notation. We will see from the proofs that the same arguments carry to the general case. We use H2' so that D^2u is continuous in Ω and the linearized Monge-Ampere equation is well defined. Our estimates do not depend on the smoothness of f, thus the general case follows by approximation from Theorem 1.1. Later in section 5 we show that H4' can be replaced by H4, i.e the bounds for A from below and above are not needed.

First, we establish a result on uniform modulus of convexity for minimizers of L in 2D.

Proposition 3.2. *Let u be a minimizer of L that satisfies the hypotheses above. Then, for any $\delta < 1$, there exist $c(\delta) > 0$ depending on ρ, δ such that*

$$x \in B_{1-\delta} \Rightarrow \quad S_h(x) \subset\subset B_1 \quad \textit{if} \quad h \le c(\delta).$$

In the above proposition, we denoted by $S_h(x)$ the section of u centered at x at height h:

$$S_h(x) = \{y \in \bar{B}_1 : u(y) < u(x) + \nabla u(x)(y - x) + h\}.$$

This result is well-known (see, e.g., Remark 3.2 in [TW3]). For completeness, we include its proof here.

Proof. Without loss of generality assume u is normalized in B_1, that is $u \ge 0$, $u(0) = 0$. From the stability inequality (2.4), we obtain

$$\int_{\partial B_1} u dx \le C.$$

This integral bound and the convexity of u imply

$$|u|,\ |Du| \le C(\delta) \text{ in } B_{1-\delta/2},$$

for any $\delta < 1$. We show that our statement follows from these bounds. Assume by contradiction that the conclusion is not true. Then, we can find a sequence of convex functions u_k satisfying the bounds above such that

$$u_k(y_k) \le u_k(x_k) + \nabla u_k(x_k)(y_k - x_k) + h_k \tag{3.1}$$

for sequences $x_k \in B_{1-\delta}$, $y_k \in \partial B_{1-\delta/2}$ and $h_k \to 0$. Because Du_k is uniformly bounded, after passing to a subsequence if necessary, we may assume

$$u_k \to u_* \quad \text{uniformly on } \overline{B}_{1-\delta/2}, \quad x_k \to x_*, \quad y_k \to y_*.$$

Moreover u_* satisfies $\rho \le \det D^2 u_* \le \rho^{-1}$, and

$$u_*(y_*) = u_*(x_*) + \nabla u_*(x_*)(y_* - x_*),$$

i.e., the graph of u_* contains a straight-line in the interior. However, any subsolution v to $\det D^2 v \ge \rho$ in $2D$ does not have this property and we reached a contradiction. $\square$

Since $f \in C^\alpha$ we obtain that $u \in C^{2,\alpha}(B_1)$ thus the linearized Monge-Ampère equation is well defined in B_1. Next lemma deals with general linear elliptic equations in B_1 which may become degenerate as we approach ∂B_1.

Lemma 3.3. *Let $\mathcal{L}v := a^{ij}(x)v_{ij}$ be a linear elliptic operator with continuous coefficients $a^{ij} \in C^\alpha(B_1)$ that satisfy the ellipticity condition $(a^{ij}(x))_{ij} > 0$ in B_1. Given a continuous boundary data φ, there exists a unique solution $v \in C(\overline{B}_1) \cap C^2(\Omega)$ to the Dirichlet problem*

$$\mathcal{L}v = 0 \quad \text{in } B_1, \quad v = \varphi \quad \text{on } \partial B_1.$$

Proof. For each small δ, we consider the standard Dirichlet problem for uniformly elliptic equations $\mathcal{L}v_\delta = 0$ in $B_{1-\delta}$, $v_\delta = \varphi$ on $\partial B_{1-\delta}$. Since v_δ satisfies the comparison principle with linear functions, it follows that the modulus of continuity of v_δ at points on the boundary $\partial B_{1-\delta}$ depends only on the modulus of continuity of φ. Thus, from maximum principle, we see that v_δ converges uniformly to a solution v of the Dirichlet problem above. The uniqueness of v follows from the standard comparison principle. $\square$

Remark 3.4. The modulus of continuity of v at points on ∂B_1 depends only on the modulus of continuity of φ.

Remark 3.5. If $\mathcal{L}_m$ is a sequence of operators satisfying the hypotheses of Lemma 3.3 with $a^{ij}_m \to a^{ij}$ uniformly on compact subsets of B_1 and $\mathcal{L}_m v_m = 0$ in B_1, $v_m = \varphi$ on ∂B_1, then $v_m \to v$ uniformly in $\overline{B}_1$.

Indeed, since v_m have a uniform modulus of continuity on ∂B_1 and, for all large m, a uniform modulus of continuity in any ball $B_{1-\delta}$, we see that we can always extract a uniform convergent subsequence in $\overline{B}_1$. Now it is straightforward to check that the limiting function v satisfies $\mathcal{L}v = 0$ in the viscosity sense.

Next, we establish an integral form of the Euler-Lagrange equations for the minimizers of L.

Proposition 3.6. *Assume that u is the normalized minimizer of L in the class $\mathcal{A}$. If $\varphi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to the linearized Monge-Ampère equation*

$$U^{ij}\varphi_{ij} = 0 \qquad \text{in } \Omega,$$

then

$$L(\varphi) := \int_{\partial\Omega} \varphi d\sigma - \int_\Omega \varphi dA = 0.$$

Proof. Consider the solution $u_\epsilon = u + \epsilon\varphi_\epsilon$ to

$$\begin{cases} \det D^2 u_\epsilon \ = f & \text{in } B_1, \\ \qquad u_\epsilon = u + \epsilon\varphi & \text{on } \partial B_1. \end{cases}$$

Since φ_ϵ satisfies comparison principle and comparison with planes, its existence follows as in Lemma 3.3 by solving the Dirichlet problems in $B_{1-\delta}$ and then letting $\delta \to 0$.

In B_1, φ_ϵ satisfies

$$0 = \frac{1}{\epsilon}(\det D^2 u_\epsilon - \det D^2 u) = \frac{1}{\epsilon}\int_0^1 \frac{d}{dt}\det D^2(u + t\epsilon\varphi_\epsilon)dt = a_\epsilon^{ij}\partial_{ij}\varphi_\epsilon$$

where

$$(a_\epsilon^{ij})_{ij} = \int_0^1 Cof\ (D^2(u + t\epsilon\varphi_\epsilon))dt.$$

Because u is strictly convex in 2D and $u_\epsilon \to u$ uniformly on $\overline{B}_1$, $D^2u_\epsilon \to D^2 u$ uniformly on compact sets of B_1. Thus, as $\epsilon \to 0$, $a_\epsilon^{ij} \to U^{ij}$ uniformly on compact sets of B_1 and by Remark 3.5, we find $\varphi_\epsilon \to \varphi$ uniformly in $\overline{B}_1$. By the minimality of u, we find

$$0 \le \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}(L(u_\epsilon) - L(u)) = \int_{\partial B_1} \varphi\, d\sigma - \int_{B_1} \varphi\, dA.$$

By replacing φ with $-\varphi$ we obtain the opposite inequality. □

4. Proof of Proposition 3.1

In this section, we prove Proposition 3.1 where H1', H2' and H4' are satisfied. Given a convex function $u \in C^\infty(B_1)$ (not necessarily a minimizer of L) with $\rho \le \det D^2 u \le \rho^{-1}$, we let v be the solution to the following Dirichlet problem

$$U^{ij}v_{ij} = -A \quad \text{in } B_1, \qquad v = 0 \quad \text{on } \partial B_1. \tag{4.1}$$

Notice that $\Psi := C(1 - |x|^2)$ is an upper barrier for v if C is large enough, since

$$U^{ij}\Psi_{ij} \le -C\, tr\, U \le -C(\det D^2 U)^{1/n} = -C(\det D^2 u)^{\frac{n-1}{n}} \le -C\rho^{\frac{n-1}{n}} \le -A,$$

hence

$$0 \le v(x) \le C(1 - |x|^2) \ \sim\ dist(x, \partial B_1). \tag{4.2}$$

As in Lemma 3.3, the function v is the uniform limit of the corresponding v_δ that solve the Dirichlet problem in $B_{1-\delta}$. Indeed, since v_δ also satisfies (4.2), we see that

$$|v_{\delta_1} - v_{\delta_2}|_{L^\infty} \le C\max\{\delta_1, \delta_2\}.$$

Let φ be the solution of the homogenous problem

$$U^{ij}\varphi_{ij} = 0 \quad \text{in } B_1, \qquad \varphi = l^+ \quad \text{on } \partial B_1,$$

where $l^+ = \max\{0, l\}$ for some linear function $l = b + \nu\cdot x$ of slope $|\nu| = 1$. Denote by $\mathcal{S} := \overline{B}_1 \cap \{l = 0\}$ the segment of intersection of the crease of l with $\overline{B}_1$. Then

Lemma 4.1.
$$\int_{B_1} \varphi \, dA = \int_{B_1} l^+ \, dA + \int_{\mathcal{S}} u_{\tau\tau} v \, d\mathcal{H}^1,$$
where τ is the unit vector in the direction of $\mathcal{S}$, hence $\tau \perp \nu$.

Proof. It suffices to show the equality in the case when $u \in C^\infty(\overline{B}_1)$. The general case follows by writing the identity in $B_{1-\delta}$ with v_δ (which increases as δ decreases), and then letting $\delta \to 0$.

Let $\tilde{l}_\epsilon$ be a smooth approximation of l^+ with
$$D^2 l_\epsilon \rightharpoonup \nu \otimes \nu \; dH^1 \lfloor \mathcal{S} \quad \text{as } \epsilon \to 0,$$
and let φ_ϵ solve the corresponding Dirichlet problem with boundary $\tilde{l}_\epsilon$. Then, we integrate by parts and use $\partial_i U^{ij} = 0$,
$$\begin{aligned}\int_{B_1} (\varphi_\epsilon - \tilde{l}_\epsilon) \, dA &= -\int_{B_1} (\varphi_\epsilon - \tilde{l}_\epsilon) U^{ij} v_{ij} \, dx \\ &= \int_{B_1} \partial_i(\varphi_\epsilon - \tilde{l}_\epsilon) U^{ij} v_j \, dx \\ &= -\int_{B_1} \partial_{ij}(\varphi_\epsilon - \tilde{l}_\epsilon) U^{ij} v \, dx \\ &= \int_{B_1} U^{ij} \partial_{ij} \tilde{l}_\epsilon \; v \, dx.\end{aligned}$$
We let $\epsilon \to 0$ and obtain
$$\int_{B_1} (\varphi - l^+) \, dA = \int_{\mathcal{S}} U^{\nu\nu} v \, d\mathcal{H}^1,$$
which is the desired conclusion since $U^{\nu\nu} = u_{\tau\tau}$. □

From Lemma 4.1 and Proposition 3.6, we obtain

Corollary 4.2. *If u is a minimizer of L in the class $\mathcal{A}$ then*
$$\int_{\mathcal{S}} u_{\tau\tau} v \, d\mathcal{H}^1 = \int_{\partial B_1} l^+ \, d\sigma - \int_{B_1} l^+ \, dA.$$
The hypotheses on σ and A imply that if the segment $\mathcal{S}$ has length $2h$ with $h \leq h_0$ small, universal then
$$ch^3 \leq \int_{\mathcal{S}} u_{\tau\tau} v \, d\mathcal{H}^1 \leq Ch^3,$$
for some c, C universal.

Lemma 4.3. *Let X_1 and X_2 be the endpoints of the segment $\mathcal{S}$ defined as above. Then*
$$\int_{\mathcal{S}} u_{\tau\tau}(1 - |x|^2) \, d\mathcal{H}^1 = 4h \left(\frac{u(X_1) + u(X_2)}{2} - \fint_{\mathcal{S}} u d\mathcal{H}^1 \right), \tag{4.3}$$
where $2h$ denotes the length of $\mathcal{S}$.

Proof. Again we may assume that $u \in C^2(\overline{B}_1)$ since the general case follows by approximating B_1 by $B_{1-\delta}$. Assume for simplicity that $\tau = e_1$. Then

$$\int_{\mathcal{S}} u_{\tau\tau}(1-|x|^2)d\mathcal{H}^1 = \int_{-h}^{h} \partial_t^2 u(t,a)(h^2-t^2)\,dt$$

for some fixed a and integrating by parts twice, we obtain (4.3). □

We remark that the right hand side in (4.3) represents twice the area between the segment with end points $(X_1, u(X_1))$, $(X_2, u(X_2))$ and the graph of u above $\mathcal{S}$.

Definition 4.4. We say that u admits a tangent plane at a point $z \in \partial B_1$, if there exists a linear function l_z such that

$$x_{n+1} = l_z(x)$$

is a supporting hyperplane for the graph of u at $(z, u(z))$ but for any $\epsilon > 0$,

$$x_{n+1} = l_z(x) - \epsilon z \cdot (x-z)$$

is not a supporing hyperplane. We call l_z a tangent plane for u at z.

Remark 4.5. Notice that if $\det D^2u \le C$ then the set of points where u admits a tangent plane is dense in ∂B_1. Indeed, using standard barriers it is not difficult to check that any point on ∂B_1 where the boundary data $u|_{\partial B_1}$ admits a quadratic polynomial from below satisfies the definition above. In the definition above we assumed $u = \bar{u}$ on ∂B_1 with $\bar{u}$ defined as in the Lemma 2.1, therefore $u|_{\partial B_1}$ is lower semicontinuous.

Assume that u admits a tangent plane at z, and denote by

$$\tilde{u} = u - l_z.$$

Lemma 4.6. *There exists $\eta > 0$ small universal such that the section*

$$\tilde{S}_z := \{x \in \overline{B}_1 | \quad \tilde{u} < \eta(x-z)\cdot(-z)\},$$

satisfies

$$\tilde{S}_z \subset B_1 \setminus B_{1-\rho}, \quad |\tilde{S}_z| \ge c,$$

for some small c universal.

Proof. We notice that (4.3) is invariant under additions with linear functions. We apply it to $\tilde{u}$ with $X_1 = z$, $X_2 = x$ and use $\tilde{u} \ge 0$, $\tilde{u}(z) = 0$ together with (4.2) and Corollary 4.2 and obtain

$$\tilde{u}(x) \ge c|x-z|^2 \quad x \in \partial B_1 \cap B_{h_0}(z).$$

From the uniform strict convexity of $\tilde{u}$, which was obtained in Proposition 3.2, we find that the inequality above holds for all $x \in \partial B_1$ for possibly a different value of c. Thus, by choosing η sufficiently small, we obtain

$$\tilde{S}_z \subset B_1, \qquad \tilde{S}_z \cap B_{1-\rho} = \emptyset,$$

where the second statement follows also from Proposition 3.2.

Next we show that $|\tilde{S}_z|$ cannot be arbitrarily small. Otherwise, by the uniform strict convexity of $\tilde{u}$, we obtain that $\tilde{S}_z \subset B_{\epsilon^4}(z)$ for some small $\epsilon > 0$. Assume for simplicity of notation that $z = -e_2$. Then the function

$$w := \eta(x_2+1) + \frac{\epsilon}{2}x_1^2 + \frac{1}{2\rho\epsilon}(x_2+1)^2 - 2\epsilon(x_2+1),$$

is a lower barrier for $\tilde{u}$ in $B_1 \cap B_{\epsilon^4}(z)$. Indeed, notice that if ϵ is sufficiently small then

$$w \leq \eta(x_2+1) \leq \tilde{u} \quad \text{on } \partial(B_1 \cap B_{\epsilon^4}(z)), \qquad \det D^2 w = \rho^{-1} \geq \det D^2 \tilde{u}.$$

In conclusion, $\tilde{u} \geq w \geq (\eta/2)(x_2+1)$ and we contradict that 0 is a tangent plane for $\tilde{u}$ at z. ◻

Lemma 4.7. *Let u be the normalized minimizer of L. Then $\|u\|_{C^{0,1}(\overline{B}_1)} \leq C$, and u admits tangent planes at all points of ∂B_1. Also, u separates at least quadratically from its tangent planes i.e*

$$u(x) \geq l_z(x) + c|x-z|^2, \quad \forall x, z \in \partial B_1.$$

Proof. Let z be a point on ∂B_1 where u admits a tangent plane l_z. From the previous lemma we know that u satisfies the quadratic separation inequality at z and also that $\tilde{u} = u - l_z$ is bounded from above and below in $\tilde{S}_z$, i.e.,

$$|u - l_z| \leq C \quad \text{in } \tilde{S}_z.$$

We obtain

$$\int_{\tilde{S}_z} |l_z|\, dx - C \leq \int_{\tilde{S}_z} u\, dx \leq \int_{B_1} u\, dx \leq C \int_{\partial B_1} u\, d\sigma \leq C,$$

and since $\tilde{S}_z \subset B_1$ has measure bounded from below we find

$$l_z(z), |\nabla l_z| \leq C.$$

By Remark 4.5, this holds for a.e. $z \in \partial B_1$ and, by approximation, we find that any point in ∂B_1 admits a tangent plane that satisfies the bounds above. This also shows that u is Lipschitz and the lemma is proved. ◻

Lemma 4.8. *The function v satisfies the lower bound*

$$v(x) \geq c\, dist(x, \partial B_1),$$

for some small c universal.

Proof. Let $z \in \partial B_1$ and let l be a linear functional with

$$l(x) = l_z(x) - b\, z \cdot (x - z), \quad \text{for some } 0 \leq b \leq \eta.$$

where l_z denotes a tangent plane at z. We consider all sections

$$S = \{x \in \overline{B}_1 | \quad u < l\}$$

which satisfy

$$\inf_S (u - l) \leq -c_0,$$

for some appropriate c_0 small, universal. We denote the collection of such sections $\mathcal{M}_z$. From Lemma 4.6, we see that $\mathcal{M}_z \neq \emptyset$ since $\tilde{S}_z$ (or $b = \eta$) satisfies the property above. Notice also that $S \subset \tilde{S}_z \subset B_1$ and $z \in \partial S$. For any section $S \in \mathcal{M}_z$ we consider its center of mass z^S, and from the property above we see that $z^S \in B_{1-c}$ for some small $c > 0$ universal.

First, we show that the lower bound for v holds on the segment $[z, z^S]$. Indeed, since

$$U^{ij}[c(l-u)]_{ij} = -2c \det D^2 u \geq -2c\rho^{-1} \geq -A = U^{ij} v_{ij},$$

and $c(l-u) \leq 0 = v$ on ∂B_1 we conclude that

$$c(l-u)^+ \leq v \quad \text{in } B_1. \tag{4.4}$$

Now, we use the convexity of u and the fact that the property of S implies $(u-l)(z^S) < -c$, and conclude that

$$v(x) \geq c(l-u)(x) \geq c|x-z| \geq c \; dist(x, \partial B_1) \qquad \forall x \in [z, z^S].$$

Now, it remains to prove that the collection of segments $[z, z^S]$, $z \in \partial B_1$, $S \in \mathcal{M}_z$ cover a fixed neighborhood of ∂B_1. To this aim we show that the multivalued map

$$z \in \partial B_1 \longmapsto F(z) := \{z^S | \quad S \in \mathcal{M}_z\}$$

has the following properties

1) the map F is *closed* in the sense that

$$z_n \to z_* \quad \text{and} \quad z_n^{S_n} \to y_* \quad \Rightarrow y^* \in F(z_*)$$

2) $F(z)$ is a connected set for any z.

The first property follows easily from the following facts: z^S varies continuously with the linear map l that defines $S = \{u < l\}$; and if $l_{z_n} \to l_*$ then $l_* \leq l_{z_*}$ for some tangent plane l_{z_*}.

To prove the second property we notice that if we increase continuously the value of the parameter b (which defines l) up to η then all the corresponding sections belong also to $\mathcal{M}_z$. This means that in $F(z)$ we can connect continuously z^S with $z^{\tilde{S}_z}$ for some section $\tilde{S}_z$. On the other hand the set of all possible $z^{\tilde{S}_z}$ is connected since the set l_z of all tangent planes at z is connected in the space of linear functions.

Since $F(z) \subset B_{1-c}$, it follows that for all $\delta < c$ the intersection map

$$z \longmapsto G_\delta(z) = \{[z, y] \cap \partial B_{1-\delta} | \quad y \in F(z)\}$$

has also the properties 1 and 2 above. Now it is easy to check that the image of G_δ covers the whole $\partial B_{1-\delta}$, hence the collection of segments $[z, z^S]$ covers $B_1 \setminus B_{1-c}$ and the lemma is proved.

□

Now, we are ready to prove the first part of Proposition 3.1.

Proof of Proposition 3.1 (i). In Lemma 4.6, we obtained the quadratic separation from below for $\tilde{u} = u - l_z$. Next we show that $\tilde{u}$ separates at most quadratically on ∂B_1 in a neighborhood of z.

Assume for simplicity of notation that $z = -e_2$. We apply (4.3) to $\tilde{u}$ with $X_1 = (-h, a)$, $X_2 = (h, a)$, then use Corollary 4.2 and Lemma 4.8 and obtain

$$\frac{\tilde{u}(X_1) + \tilde{u}(X_2)}{2} - \fint_S \tilde{u} \le Ch^2.$$

On the other hand, for small h, the segment $[z, z^{\tilde{S}_z}]$ intersects $[X_1, X_2]$ at a point $y = (t, a)$ with $|t| \le Ch^2 \le h/2$. Moreover, since $y \in \tilde{S}_z$ we have $\tilde{u}(y) \le \eta(a+1) \le Ch^2$. On the segment $[X_1, X_2]$, $\tilde{u}$ satisfies the conditions of Lemma 4.9 which we prove below, hence

$$\tilde{u}(X_1), \tilde{u}(X_2) \le Ch^2.$$

In conclusion, u separates quadratically on ∂B_1 from its tangent planes and therefore satisfies the hypotheses of the Localization Theorem in [S2], [LS]. From Theorem 2.4 and Proposition 2.6 in [LS], we conclude that

$$\|u\|_{C^{1,\beta}(\overline{B}_1)},\ \|v\|_{C^{\beta}(\overline{B}_1)},\ \|v_\nu\|_{C^{\beta}(\partial B_1)} \le C, \tag{4.5}$$

for some $\beta < 1$, C universal. $\square$

Lemma 4.9. *Let $f : [-h, h] \to \mathbb{R}^+$ be a nonnegative convex function such that*

$$\frac{f(-h) + f(h)}{2} - \frac{1}{2h}\int_{-h}^{h} f(x)dx \le Mh^2, \qquad f(t) \le Mh^2,$$

for some $t \in [-h/2, h/2]$. Then

$$f(\pm h) \le Ch^2,$$

for some C depending on M.

Proof. The inequality above states that the area between the line segment with end points $(-h, f(-h))$, $(h, f(h))$ and the graph of f is bounded by $2Mh^3$. By convexity, this area is greater than the area of the triangle with vertices $(-h, f(-h))$, $(t, f(t))$, $(h, f(h))$. Now the inequality of the heights $f(\pm h)$ follows from elementary euclidean geometry. $\square$

Finally, we are ready to prove the second part of Proposition 3.1.

Proof of Proposition 3.1 (ii). Let φ be such that

$$U^{ij}\varphi_{ij} = 0 \quad \text{ in } \Omega, \qquad \varphi \in C^{1,1}(\partial B_1) \cap C^0(\overline{B}_1).$$

Since u satisfies the quadratic separation assumption and f is smooth up to the boundary, we obtain from Theorem 2.5 and Proposition 2.6 in [LS]

$$\|v\|_{C^{1,\beta}(\overline{B}_1)},\ \|\varphi\|_{C^{1,\beta}(\overline{B}_1)} \le K, \qquad \text{and} \quad |U^{ij}| \le K|\log\delta|^2 \quad \text{on } B_{1-\delta},$$

for some constant K depending on ρ, $\|f\|_{C^{\beta}(\overline{B}_1)}$, and $\|\varphi\|_{C^{1,1}(\partial B_1)}$.

We will use the following identity in 2D:

$$U^{ij}v_j\nu_i = U^{\tau\nu}v_\tau + U^{\nu\nu}v_\nu.$$

Integrating by parts twice, we obtain as in (1.1)

$$\begin{aligned}\int_{B_{1-\delta}} \varphi\, dA &= -\int_{B_{1-\delta}} \varphi\, U^{ij} v_{ij}\, dx \\ &= \int_{\partial B_{1-\delta}} \varphi_i U^{ij} v \nu_j - \int_{\partial B_{1-\delta}} \varphi U^{ij} v_j \nu_i \\ &= -\int_{\partial B_{1-\delta}} \varphi U^{\nu\nu} v_\nu + o(\delta)\end{aligned}$$

where in the last equality we used the estimates

$$|v| \le C\delta, \quad |v_\tau| \le K\delta^\beta, \quad |\varphi|, |\nabla \varphi| \le K, \quad U^{ij} \le K|\log \delta|^2 \quad \text{on } \partial B_{1-\delta}.$$

Since on ∂B_r

$$U^{\nu\nu} = u_{\tau\tau} = r^{-2} u_{\theta\theta} + r^{-1} u_\nu,$$

$u \in C^{1,\beta}(\overline{B}_1)$ and $u(re^{i\theta})$ converges uniformly as $r \to 1$, and $u_{\theta\theta}$ is uniformly bounded from below, we obtain

$$U^{\nu\nu}\, d\mathcal{H}^1 \lfloor_{\partial B_r} \rightharpoonup (u_{\theta\theta} + u_\nu)\, d\mathcal{H}^1 \lfloor_{\partial B_1} \quad \text{as } r \to 1.$$

We let $\delta \to 0$ in the equality above and find

$$\int_{B_1} \varphi\, dA = -\int_{\partial B_1} \varphi\, (u_{\theta\theta} + u_\nu) v_\nu\, d\mathcal{H}^1.$$

Now the Euler-Lagrange equation, Lemma 3.6, gives

$$(u_{\theta\theta} + u_\nu) v_\nu = -\sigma \quad \text{on } \partial B_1.$$

We use that $\|v_\nu\|_{C^\beta(\partial B_1)} \le C$ and, from Lemma 4.8, $v_\nu \le -c$ on ∂B_1 and obtain

$$\|u\|_{C^{2,\gamma}(\partial B_1)} \le C\|\sigma\|_{C^\gamma(\partial B_1)}.$$

□

5. The general case for A

In this section, we remove the assumptions that A is bounded from below by ρ in B_1 and also we assume that A is bounded from above only in a neighborhood of the boundary. Precisely, we assume that $A \ge 0$ in B_1 and $A \le \rho^{-1}$ in $B_1 \setminus \overline{B}_{1-\rho}$. We may also assume A is smooth in B_1 since the general case follows by approximation. Notice that $\int_{B_1} A\, dx$ is bounded from above and below since it equals $\int_{\partial B_1} d\sigma$.

Let v be the solution of the Dirichlet problem

$$U^{ij} v_{ij} = -A, \qquad v = 0 \quad \text{on } \partial B_1. \tag{5.1}$$

In Section 4, we used that A is bounded from above when we obtained $v \le C(1 - |x|^2)$, and we used that A is bounded from below in Lemma 4.8 (see (4.4)). We need to show that these bounds for v also hold in a neighborhood of ∂B_1 under the weaker hypotheses above. First, we show

Lemma 5.1.

$$v \le C \quad on\ \partial B_{1-\rho/2}, \qquad v \ge c(\delta) \quad on\ B_{1-\delta},$$

with C universal, and $c(\delta) > 0$ depending also on δ.

Proof. As before, we may assume that $u \in C^\infty(\overline{B}_1)$ since the general case follows by approximating B_1 by $B_{1-\epsilon}$.

We multiply the equation in (5.1) by $(1-|x|^2)$, integrate by parts twice and obtain

$$\int_{B_1} 2v\ tr\, U\, dx = \int_{B_1} A(x)(1-|x|^2)\, dx \le C,$$

and since $tr\, U \ge c$ we obtain

$$\int_{B_1} v\, dx \le C.$$

We know

1) $v \ge 0$ solves a linearized Monge-Ampère equation with bounded right hand side in $B_1 \setminus B_{1-\rho}$,

2) u has a uniform modulus of convexity on compact sets of B_1.

Now we use the Harnack inequality of Caffarelli-Gutierrez [CG] and conclude that

$$\sup_{\mathcal{V}} v \le C(\inf_{\mathcal{V}} v + 1), \qquad \mathcal{V} := B_{1-\rho/4} \setminus \overline{B}_{1-3\rho/4},$$

and the integral inequality above gives $\sup_{\mathcal{V}} v \le C$.

Next, we prove the lower bound. We multiply the equation in (5.1) by $\varphi \in C_0^\infty(B_1)$ with

$$\varphi = 0 \quad \text{if } |x| \ge 1-\delta/2, \quad \varphi = 1 \quad \text{in } B_{1-\delta}, \quad \|D^2\varphi\| \le C/\delta^2,$$

integrate by parts twice and obtain

$$C(\delta)\int_{\mathcal{U}} v\ tr\, U \ge -\int_{B_1} v\, U^{ij}\varphi_{ij} = \int_{B_1} A\varphi \ge c, \qquad \mathcal{U} := B_{1-\delta/2} \setminus \overline{B}_{1-\delta},$$

where the last inequality holds provided that δ is sufficiently small. Since u is normalized we obtain (see Proposition 3.2) , $|\nabla u| \le C(\delta)$ in $\mathcal{U}$ thus

$$\int_{\mathcal{U}} tr\, U = \int_{\mathcal{U}} \triangle u = \int_{\partial\mathcal{U}} u_\nu \le C(\delta).$$

The last two inequalities imply $\sup_{\mathcal{U}} v \ge c(\delta)$, hence there exists $x_0 \in \mathcal{U}$ such that $v(x_0) \ge c(\delta)$. We use 1), 2) above and Harnack inequality and find $v \ge c(\delta)$ in $B_{\bar\delta}(x_0)$ for some small $\bar\delta$ depending on ρ and δ. Since v is a supersolution, i.e $U^{ij}v_{ij} \le 0$, we can apply the weak Harnack inequality of Caffarelli-Gutierrez, Theorem 4 in [CG]. From property 2) above, we see that we can extend the lower bound of v from $B_{\bar\delta}(x_0)$ all the way to $\mathcal{U}$, and by the maximum principle this bound holds also in $B_{1-\delta/2}$.

□

The upper bound in Lemma 5.1 gives as in (4.2) the upper bound for v in a neighborhood of ∂B_1, i.e

$$v(x) \le C(1-|x|^2) \quad \text{on} \quad B_1 \setminus B_{1-\rho/2}.$$

This implies as in Section 4 that Lemma 4.7 holds i.e., u separates at least quadratically from its tangent planes on ∂B_1. It remains to show that also Lemma 4.8 holds. Since A is not strictly positive, $c(l-u)$ is no longer a subsolution for the equation (5.1) and we cannot bound v below as we did in (4.4). In the next lemma, we construct another barrier which alows us to bound v from below on the segment $[z, z^S]$.

Lemma 5.2. *Let $\tilde{u} : B_1 \to \mathbb{R}$ be a convex function with $\tilde{u} \in C(\overline{B}_1) \cap C^2(B_1)$, and*

$$\rho \le \det D^2\tilde{u} \le \rho^{-1}.$$

Assume that the section $S := \{\tilde{u} < 0\}$ is included in B_1 and is tangent to ∂B_1 at a point $z \in \partial B_1$, and also that

$$\inf_S \tilde{u} \le -\mu,$$

for some $\mu > 0$. If

$$\tilde{U}^{ij} v_{ij} \le 0 \quad \text{in } B_1, \quad v \ge 0 \quad \text{on } \partial B_1,$$

then

$$v(x) \ge c(\mu,\rho)|x-z| \inf_{S'} v \quad \forall x \in [z, z^S], \qquad S' := \{\tilde{u} \le \frac{1}{2} \inf_S \tilde{u}\},$$

where z^S denotes the center of mass of S, and $c(\mu,\rho)$ is a positive constant depending on μ and ρ.

The functions $\tilde{u} = u - l$ and v in the proof of Lemma 4.8 satisfy the lemma above, if η in Lemma 4.6 is small, universal. Using also the lower bound on v from Lemma 5.1, we find

$$v \ge c|x-z| \quad \text{on } [z, z^S],$$

for some c universal, and the rest of the proof of Lemma 4.8 follows as before. This shows that Proposition 3.1 holds also with our assumptions on the measure A.

Proof of Lemma 5.2. We construct a lower barrier for v of the type

$$w := e^{k\bar{w}} - 1, \quad \bar{w} := -\tilde{u} + \frac{\epsilon}{2}(|x|^2 - 1),$$

for appropriate constants k large and $\epsilon \ll \mu$ small. Notice that $w \le 0$ on ∂B_1 since $\bar{w} \le 0$ on ∂B_1. Also

$$\bar{w} \ge c\,|x-z| \quad \text{on } [z, z^S],$$

since, by convexity, $-\tilde{u} \ge c|x-z|$ on $[z, z^S]$ for some c depending on μ and ρ. It suffices to check that

$$\tilde{U}^{ij} w_{ij} \ge 0 \quad \text{on } B_1 \setminus S',$$

since then we obtain $v \ge (\inf_{S'} v)\, c\, w$ in $B_1 \setminus S'$ which easily implies the conclusion. In $B_1 \setminus S'$ we have $|\nabla \bar{w}| \ge c(\mu) > 0$ provided that ϵ is sufficiently small, thus

$$\tilde{U}^{ij} \bar{w}_i \bar{w}_j = (\det D^2\tilde{u})(\nabla \bar{w})^T\,(D^2\tilde{u})^{-1}\,\nabla \bar{w} \ge c\Lambda^{-1},$$

where Λ is the largest eigenvalue of $D^2\tilde{u}$. Then, we use that $tr\,\tilde{U} \geq c\lambda^{-1} \geq c\Lambda^{\frac{1}{n-1}}$ where λ is the smallest eigenvalue of $D^2\tilde{u}$, and obtain

$$\begin{aligned}\tilde{U}^{ij}w_{ij} &= ke^{k\bar{w}}\left(\tilde{U}^{ij}\bar{w}_{ij} + k\tilde{U}^{ij}\bar{w}_i\bar{w}_j\right)\\ &\geq ke^{k\bar{w}}\left(-n + \epsilon\, tr\,\tilde{U} + kc\Lambda^{-1}\right)\\ &\geq ke^{k\bar{w}}\left(-n + c(\epsilon\,\Lambda^{\frac{1}{n-1}} + k\Lambda^{-1})\right)\\ &\geq 0,\end{aligned}$$

if k is chosen large depending on ϵ, ρ, μ and n. $\square$

6. Singular minimizers in dimension $n \geq 3$.

Let

$$u(x) := |x'|^{2-\frac{2}{n}}h(x_n),$$

be the singular solution to $\det D^2u = 1$ constructed by Pogorelov, with h a smooth even function, defined in a neighborhood of 0 and $h(0) = 1$, satisfying an ODE

$$\left((1-\frac{2}{n})hh'' - (2-\frac{2}{n})h'^2\right)h^{n-2} = c.$$

We let

$$v(x) := |x'|^{2-\frac{2}{n}}q(x_n)$$

be obtained as the infinitesimal difference between u and a rescaling of u,

$$v(x', x_n) := \lim_{\epsilon\to 0}\frac{1}{\epsilon}[u(x', x_n) - (1+\epsilon)^{-\gamma}u(x', (1+\epsilon)x_n)],$$

for some small $\gamma < 2/n$. Notice that

$$q(t) = \gamma h(t) - h'(t)t$$

and $q > 0$ in a small interval $(-a, a)$ and q vanishes at its end points. Also,

$$U^{ij}v_{ij} = n\gamma - 2 < 0 \quad \text{in} \quad \Omega := \mathbb{R}^{n-1}\times[-a, a],$$

$$v = 0, \quad U^{\nu\nu}v_\nu = U^{nn}v_n = -\sigma_0 \quad \text{on} \quad \partial\Omega,$$

for some constant $\sigma_0 > 0$. The last equality follows since U^{nn} is homogenous of degree $-(n-1)(2/n)$ in $|x'|$ and v_n is homogenous of degree $2-2/n$ in $|x'|$.

Notice that if u, v are solutions of the system (1.2) in the infinite cylinder Ω for uniform measures A and σ. In order to obtain a solution in a finite domain Ω_0 we modify v outside a neighborhood of the line $|x'| = 0$ by subtracting a smooth convex function ψ which vanishes in B_1 and increases rapidly outside B_1. Precisely we let

$$\tilde{v} := v - \psi, \quad \Omega_0 := \{\tilde{v} > 0\}$$

and then we notice that u, $\tilde{v}$, solve the system (1.2) in the smooth bounded domain Ω_0 for smooth measures A and σ.

Since

$$|U^{ij}| \le C r^{\frac{2}{n}-2}, \quad \text{if } |x'| \ge r,$$

we integrate by parts in the domain $\Omega_0 \setminus \{|x'| \le \epsilon\}$ and then let $\epsilon \to 0$ and find

$$\int_{\Omega_0} \varphi \, dA = -\int_{\Omega_0} U^{ij} \varphi_{ij} v + \int_{\partial\Omega_0} \varphi \, d\sigma, \qquad \forall \varphi \in C^2(\overline{\Omega}_0),$$

or

$$L(\varphi) = \int_{\Omega_0} U^{ij} \varphi_{ij} \, v.$$

This implies that L is stable, i.e $L(\varphi) > 0$ for any convex φ which is not linear. Also, if $w \in C^2(\overline{\Omega}_0)$ satisfies $\det D^2 w = 1$, then $U^{ij}(w-u)_{ij} \ge 0$, and we obtain

$$L(w) - L(u) = \int_{\Omega_0} U^{ij} (w-u)_{ij} \, v \ge 0,$$

i.e u is a minimizer of L.

We remark that the domain Ω_0 has flat boundary in a neighborhood of the line $\{|x'| = 0\}$ and therefore is not uniformly convex. However this is not essential in our example. One can construct for example a function $\bar{v}$ in a uniformly convex domain by modifying v as

$$\bar{v} := |x'|^{2-\frac{2}{n}} q(x_n(1 + \delta |x'|^2)),$$

for some small $\delta > 0$.

7. Proof of Theorem 1.4

We assume for simplicity that $\Omega = B_1$. The existence of a minimizer u for the convex functional E follows as in Section 2. First, we show that

$$t_1 \le \det D^2 u \le t_0 \tag{7.1}$$

for some t_1 depending on F and ρ. The upper bound follows easily. If $\det D^2 u > t_0$ in a set of positive measure then the function w defined as

$$\det D^2 w = \min\{\, t_0, \, \det D^2 u\}, \qquad w = u \text{ on } \partial B_1,$$

satisfies $E(w) < E(u)$ since $F(\det D^2 w) = F(\det D^2 u)$ and $L(w) < L(u)$.

In order to obtain the lower bound in (7.1) we need the following lemma.

Lemma 7.1. *Let w be a convex functions in B_1 with*

$$(\det D^2 w)^{\frac{1}{n}} = g \in L^n(B_1).$$

Let $w + \varphi$ be another convex function in B_1 with the same boundary values as w such that

$$(\det D^2 (w + \varphi))^{\frac{1}{n}} = g - h, \quad \textit{for some } h \ge 0.$$

Then

$$\int_{B_1} \varphi \, g^{n-1} \le C(n) \int_{B_1} h \, g^{n-1}.$$

Proof. By approximation, we may assume that w, φ are smooth in $\overline{B}_1$. Using the concavity of the map $M \mapsto (\det M)^{\frac{1}{n}}$ in the space of symmetric matrices $M \geq 0$, we obtain

$$(\det D^2(w+\varphi))^{\frac{1}{n}} \leq (\det D^2 w)^{\frac{1}{n}} + \frac{1}{n}(\det D^2 w)^{\frac{1}{n}-1} W^{ij}\varphi_{ij},$$

hence

$$-n\, h\, g^{n-1} \leq W^{ij}\varphi_{ij}.$$

We multiply both sides with $\Phi := \frac{1}{2}(1-|x|^2)$ and integrate. Since both φ and Φ vanish on ∂B_1 we integrate by parts twice and obtain

$$-C(n)\int_{B_1} h\, g^{n-1} \leq \int_{B_1} W^{ij}\Phi_{ij}\ \varphi = -\int_{B_1}(tr\ W)\varphi.$$

Using

$$tr\, W \geq c(n)(\det W)^{\frac{1}{n}} = c(n)(\det D^2 w)^{\frac{n-1}{n}} = c(n) g^{n-1}$$

we obtain the desired conclusion. $\square$

Now we prove the lower bound in (7.1). Define w such that $w = u$ on ∂B_1 and

$$\det D^2 w = \max\{\ t_1,\, \det D^2 u\},$$

for some small t_1. Since $G(t) = F(t^n)$ is convex and $\det D^2 w \geq t_1$, we have

$$G((\det D^2 w)^{1/n}) \leq G((\det D^2 u)^{1/n}) + G'(t_1^{1/n})((\det D^2 w)^{1/n} - (\det D^2 u)^{1/n}).$$

We denote

$$u - w = \varphi, \quad (\det D^2 w)^{1/n} = g, \quad (\det D^2 u)^{1/n} = g - h,$$

and we rewrite the inequality above as

$$F(\det D^2 w) \leq F(\det D^2 u) + G'(t_1^{1/n})\, h.$$

From Lemma 7.1, we obtain

$$\int_{B_1} h\, g^{n-1} \geq c(n)\int_{B_1}\varphi\, g^{n-1}$$

and since h is supported on the set where the value of $g = t_1^{1/n}$ is minimal, we find that

$$\int_{B_1} h \geq c(n)\int_{B_1}\varphi.$$

This gives

$$\int_{B_1} F(\det D^2 w) - F(\det D^2 u) \leq c(n) G'(t_1^{1/n})\int_{B_1}\varphi,$$

thus, using the minimality of u and $G'(0^+) = -\infty$,

$$0 \leq E(w) - E(u) \leq \int_{B_1}\varphi dA + c(n) G'(t_1^{1/n})\int_{B_1}\varphi \leq 0,$$

if t_1 is small enough. In conclusion, $\varphi = 0$ and $u = w$ and (7.1) is proved.

We denote

$$\det D^2 u = f, \quad t_1 \le f \le t_0.$$

Any minimizer for L in the class of functions whose determinant equals f is a minimizer for E as well. In order to apply Theorem 1.2 we need f to be Holder continuous. However, we can approximate f by smooth functions f_n and find smooth minimizers u_n for approximate linear functionals L_n with the constraint $\det D^2 u_n = f_n$. By Proposition 3.1 (see (4.5)),

$$\|u_n\|_{C^{1,\beta}(\overline{B}_1)}, \ \|v_n\|_{C^{\beta}(\overline{B}_1)} \le C,$$

hence we may assume (see Theorem 1.1) that, after passing to a subsequence, $u_n \to u$ and $v_n \to v$ uniformly for some function $v \in C^\beta(\overline{B}_1)$. We show that

$$v = -F'(f). \tag{7.2}$$

Then by the hypotheses on F we obtain $\det D^2 u = f \in C^\beta(\overline{B}_1)$ and from Theorem 1.2 we easily obtain

$$\|u\|_{C^{2,\alpha}(\overline{B}_1)}, \ \|v\|_{C^{2,\alpha}(\overline{B}_1)} \le C,$$

for some C depending on ρ, α, $\|\sigma\|_{C^\alpha(\overline{B}_1)}$, $\|A\|_{C^\alpha(\overline{B}_1)}$ and F.

In order to prove (7.2) we need a uniform integral bound (in 2D) between solutions to the Monge-Ampère equation and solutions of the corresponding linearized equation.

Lemma 7.2. *Assume $n = 2$ and let w be a smooth convex function in $\overline{B}_1$ with*

$$\lambda \le \det D^2 w := g \le \Lambda,$$

for some positive constants λ, Λ. Let $w + \epsilon\varphi$ be a convex function with

$$\det D^2(w + \epsilon\varphi) = g + \epsilon h, \quad \varphi = 0 \quad \text{on } \partial B_1$$

for some smooth function h with $\|h\|_{L^\infty} \le 1$. If $\epsilon \le \epsilon_0$ then

$$\int_{B_1} |h - W^{ij}\varphi_{ij}| \le C\epsilon.$$

for some C, ϵ_0 depending only on λ, Λ.

We postpone the proof of the lemma untill the end of the section.

Now let h be a smooth function, $\|h\|_{L^\infty} \le 1$, and we solve the equations

$$\det D^2(u_n + \epsilon\varphi_n) = f_n + \epsilon h, \quad \varphi_n = 0 \quad \text{on } \partial B_1,$$

with u_n, f_n as above. From (1.1) we see that

$$L_n(\varphi_n) = \int_{B_1} (U_n^{ij}\partial_{ij}\varphi_n)\, v_n,$$

hence, by the lemma above

$$|L_n(\varphi_n) - \int_{B_1} h\, v_n| \le C\epsilon$$

with C universal. We let $n \to \infty$ and obtain

$$|L(\varphi) - \int_{B_1} h\, v| \le C\epsilon.$$

with φ the solution of

$$\det D^2(u+\epsilon\varphi) = f+\epsilon h, \quad \varphi = 0 \quad \text{on } \partial B_1.$$

The inequality $E(u+\epsilon\varphi) \geq E(u)$ implies

$$\int_{B_1} (F(f+\epsilon h) - F(f) + \epsilon h\, v) \geq -C\epsilon^2,$$

hence, as $\epsilon \to 0$,

$$\int_{B_1} (F'(f) + v)\, h \geq 0 \quad \text{for any smooth } h,$$

which gives (7.2). □

Proof of Lemma 7.2. Using the concavity of $(\det D^2 w)^{1/n}$ we obtain

$$(g+\epsilon h)^{1/n} \leq g^{1/n} + \frac{\epsilon}{n} g^{1/n-1} W^{ij}\varphi_{ij},$$

thus, for $\epsilon \leq \epsilon_0$

$$h - C\epsilon \leq W^{ij}\varphi_{ij}. \tag{7.3}$$

Since $n = 2$ we have

$$\det D^2(w+\epsilon\varphi) = \det D^2 w + \epsilon W^{ij}\varphi_{ij} + \epsilon^2 \det D^2\varphi,$$

hence

$$h - W^{ij}\varphi_{ij} = \epsilon \det D^2\varphi.$$

From the pointwise inequality (7.3), we see that in order to prove the lemma it suffices to show that

$$\int_{B_1} \det D^2\varphi \geq -C.$$

Integrating by parts and using $\varphi = 0$ on ∂B_1 we find

$$\int_{B_1} 2\det D^2\varphi = \int_{B_1} \Phi^{ij}\varphi_{ij} = \int_{\partial B_1} \Phi^{ij}\varphi_i\nu_j = \int_{\partial B_1} \Phi^{\nu\nu}\varphi_\nu = \int_{\partial B_1} \varphi_\nu^2 \geq 0$$

where we used that $\Phi^{\nu\nu} = \varphi_{\tau\tau} = \varphi_\nu$. □

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