BOUNDARY REGULARITY FOR SOLUTIONS TO THE LINEARIZED MONGE-AMPÈRE EQUATIONS

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Abstract. We obtain boundary Hölder gradient estimates and regularity for solutions to the linearized Monge-Ampère equations under natural assumptions on the domain, Monge-Ampère measures and boundary data. Our results are affine invariant analogues of the boundary Hölder gradient estimates of Krylov.

1. Introduction

This paper is concerned with boundary regularity for solutions to the linearized Monge-Ampère equations. The equations we are interested in are of the form

$$L_u v = g,$$

with

$$L_u v := \sum_{i,j=1}^{n} U^{ij} v_{ij},$$

where u is a locally uniformly convex function and U^{ij} is the cofactor of the Hessian D^2u. The operator L_u appears in several contexts including affine differential geometry [TW, TW1, TW2, TW3], complex geometry [D2], and fluid mechanics [B, CNP, Loe]. As $U = (U^{ij})$ is divergence-free, we can write

$$L_u v = \sum_{i,j}^{n} \partial_i (U^{ij} D_j v) = \sum_{i,j=1}^{n} \partial_i \partial_j (U^{ij} v).$$

Because the matrix of cofactors U is positive semi-definite, L_u is a linear elliptic partial differential operator, possibly degenerate.

In [CG], Caffarelli and Gutiérrez developed a Harnack inequality theory for solutions of the homogeneous equations $L_u v = 0$ in terms of the pinching of the Hessian determinant

$$\lambda \leq \det D^2 u \leq \Lambda.$$

This theory is an affine invariant version of the classical Harnack inequality for uniformly elliptic equations with measurable coefficients.

In this paper, we establish boundary Hölder gradient estimates and regularity for solutions to the linearized Monge-Ampère equations $L_u v = g$ under natural assumptions on the domain, Monge-Ampère measures and boundary data; see Theorems 2.1, 2.4 and 2.5. These theorems are affine invariant analogues of the boundary Hölder gradient estimates of Krylov [K].

The motivation for our estimates comes from the study of convex minimizers u for convex energies E of the type

$$E(u) = \int_\Omega F(\det D^2 u)\, dx + \int_{\partial\Omega} u d\sigma - \int_\Omega u dA,$$

which we considered in [LS2]. Such energies appear in the work of Donaldson [D1]-[D4] in the context of existence of Kähler metrics of constant scalar curvature for toric varieties. Minimizers of E satisfy a system of the form

$$\begin{cases} -F'(\det D^2 u) &= v \quad \text{in } \Omega, \\ U^{ij} v_{ij} &= -dA \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \\ U^{\nu\nu} v_\nu &= -\sigma \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $U^{\nu\nu} = \det D^2_{x'} u$ with $x' \perp \nu$ denoting the tangential directions along $\partial\Omega$. The minimizer u solves a fourth order elliptic equation with two nonstandard boundary conditions involving the second and third order derivatives of u. In [LS2] we apply the boundary Hölder gradient estimates established in this paper and show that $u \in C^{2,\alpha}(\overline{\Omega})$ in dimensions $n = 2$ under suitable conditions on the function F and the measures dA and $d\sigma$.

Our boundary Hölder gradient estimates depend only on the bounds on the Hessian determinant $\det D^2 u$, the quadratic separations of u from its tangent planes on the boundary $\partial\Omega$ and the geometry of Ω. Under these assumptions, the linearized Monge-Ampère operator L_u is in general not uniformly elliptic, i.e., the eigenvalues of $U = (U^{ij})$ are not necessarily bounded away from 0 and ∞. Moreover, L_u can be possibly singular near the boundary; even if $\det D^2 u$ is constant in $\overline{\Omega}$, U can blow up logarithmically at the boundary, see Proposition 2.6. The degeneracy and singularity of L_u are the main difficulties in establishing our boundary regularity results. We handle the degeneracy of L_u by working as in [CG] with sections of solutions to the Monge-Ampère equations. These sections have the same role as euclidean balls have in the classical theory. To overcome the singularity of L_u near the boundary, we use a Localization Theorem at the boundary for solutions to the Monge-Ampère equations which was obtained in [S, S2].

The rest of the paper is organized as follows. We state our main results in Section 2. In Section 3, we discuss the Localization Theorem and weak Harnack inequality, which are the main tools used in the proof of our local boundary regularity result, Theorem 2.1. In Sections 4 and 5, we study boundary behavior and the main properties of the rescaled functions u_h obtained from the Localization Theorem. The proofs of Theorems 2.1 and 2.5 will be given in Section 6 and Section 7.

2. Statement of the main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_{\frac{1}{\rho}}, \tag{2.1}$$

for some small $\rho > 0$. Assume that

(2.2) $\quad \Omega$ contains an interior ball of radius ρ tangent to $\partial\Omega$ at each point on $\partial\Omega \cap B_\rho$.

Let $u : \overline{\Omega} \to \mathbb{R}$, $u \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying

$$\det D^2 u = f, \qquad 0 < \lambda \leq f \leq \Lambda \quad \text{in } \Omega. \tag{2.3}$$

Throughout, we denote by $U = (U^{ij})$ the matrix of cofactors of the Hessian matrix $D^2 u$, i.e.,

$$U = (\det D^2 u)(D^2 u)^{-1}.$$

We assume that on $\partial\Omega \cap B_\rho$, u separates quadratically from its tangent planes on $\partial\Omega$. Precisely we assume that if $x_0 \in \partial\Omega \cap B_\rho$ then

$$\rho\, |x - x_0|^2 \leq u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \leq \rho^{-1} |x - x_0|^2, \tag{2.4}$$

for all $x \in \partial\Omega$.

When $x_0 \in \partial\Omega$, the term $\nabla u(x_0)$ is understood in the sense that

$$x_{n+1} = u(x_0) + \nabla u(x_0) \cdot (x - x_0)$$

is a supporting hyperplane for the graph of u but for any $\varepsilon > 0$,

$$x_{n+1} = u(x_0) + (\nabla u(x_0) - \varepsilon \nu_{x_0}) \cdot (x - x_0)$$

is not a supporting hyperplane, where ν_{x_0} denotes the exterior unit normal to $\partial\Omega$ at x_0. In fact we will show in Proposition 4.1 that our hypotheses imply that u is always differentiable at x_0 and then $\nabla u(x_0)$ is defined also in the classical sense.

We are ready to state our main theorem.

Theorem 2.1. *Assume u and Ω satisfy the assumptions (2.1)-(2.4) above. Let $v : B_\rho \cap \overline{\Omega} \to \mathbb{R}$ be a continuous solution to*

$$\begin{cases} U^{ij} v_{ij} = g & \text{in } B_\rho \cap \Omega, \\ v = 0 & \text{on } \partial\Omega \cap B_\rho, \end{cases}$$

Then

$$\|v_\nu\|_{C^{0,\alpha}(\partial\Omega \cap B_{\rho/2})} \leq C \left(\|v\|_{L^\infty(\Omega \cap B_\rho)} + \|g/\, tr\, U\|_{L^\infty(\Omega \cap B_\rho)} \right),$$

and, for $r \leq \rho/2$, we have the estimate

$$\max_{B_r \cap \overline{\Omega}} |v + v_\nu(0) x_n| \leq C r^{1+\alpha} \left(\|v\|_{L^\infty(\Omega \cap B_\rho)} + \|g/\, tr\, U\|_{L^\infty(\Omega \cap B_\rho)} \right),$$

where $\alpha \in (0,1)$ and C are constants depending only on $n, \rho, \lambda, \Lambda$.

We remark that our estimates do not depend on the $C^{0,1}(\overline{\Omega})$ norm of u or the smoothness of u.

Remark 2.2. The theorem is still valid if we consider the equation

$$tr\,(A D^2 v) = g, \qquad \text{with} \quad 0 < \tilde{\lambda} U \leq A \leq \tilde{\Lambda} U$$

and then the constants α, C depend also on $\tilde{\lambda}$, $\tilde{\Lambda}$.

Theorem 2.1 is concerned with boundary regularity in the case when the potential u is nondegenerate along $\partial\Omega$. It is an affine invariant analogue of the boundary Hölder gradient estimate of Krylov [K].

Theorem 2.3 (Krylov)**.** *Let $w \in C(\overline{B_1^+}) \cap C^2(B_1^+)$ satisfy*

$$Lw = f \quad \text{in } B_1^+, \qquad w = 0 \quad \text{on } \{x_n = 0\},$$

where $L = a^{ij}\partial_{ij}$ is a uniformly elliptic operator with bounded measurable coefficients with ellipticity constants λ, Λ. Then there are constants $0 < \alpha < 1$ and $C > 0$ depending on λ, Λ, n such that

$$\|w_n\|_{C^\alpha(B_{1/2}\cap\{x_n=0\})} \leq C(\|w\|_{L^\infty(B_1^+)} + \|f\|_{L^\infty(B_1^+)}).$$

We also obtain global boundary regularity estimates under global conditions on the domain Ω and the potential function u.

Theorem 2.4. *Assume that $\Omega \subset B_{1/\rho}$ contains an interior ball of radius ρ tangent to $\partial\Omega$ at each point on $\partial\Omega$. Assume further that*

$$\det D^2 u = f \qquad \text{with} \quad \lambda \leq f \leq \Lambda,$$

and on $\partial\Omega$, u separates quadratically from its tangent planes, namely

$$\rho\,|x - x_0|^2 \leq u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \leq \rho^{-1}\,|x - x_0|^2\,, \ \forall x, x_0 \in \partial\Omega.$$

Let $v : \overline{\Omega} \to \mathbb{R}$ be a continuous function that solves

$$\begin{cases} U^{ij} v_{ij} &= g \qquad \text{in } \Omega, \\ \qquad v &= \varphi \qquad \text{on } \partial\Omega, \end{cases}$$

where φ is a $C^{1,1}$ function defined on $\partial\Omega$. Then

$$\|v_\nu\|_{C^{0,\alpha}(\partial\Omega)} \leq C\left(\|\varphi\|_{C^{1,1}(\partial\Omega)} + \|g/\,tr\,U\|_{L^\infty(\Omega)}\right),$$

and for all $x_0 \in \partial\Omega$

$$\max_{B_r(x_0)\cap\overline{\Omega}} |v - v(x_0) - \nabla v(x_0)(x - x_0)| \leq C\left(\|\varphi\|_{C^{1,1}(\partial\Omega)} + \|g/\,tr\,U\|_{L^\infty(\Omega)}\right) r^{1+\alpha},$$

where $\alpha \in (0,1)$ and C are constants depending on $n, \rho, \lambda, \Lambda$.

Theorem 2.4 follows easily from Theorem 2.1. Indeed, first we notice that v is bounded by the use of barriers

$$\pm C(|x|^2 - 2/\rho^2),$$

for appropriate C, and then we apply Theorem 2.1 on $\partial\Omega$ for $\tilde{v} := v - \varphi$, where φ is a $C^{1,1}$ extension of φ to $\overline{\Omega}$.

If, in addition, we assume that $\det D^2 u$ is globally Hölder continuous, then the solutions to the linearized Monge-Ampère equations have global $C^{1,\alpha}$ estimates as stated in the next theorem.

Theorem 2.5. *Assume the hypotheses of Theorem 2.4 hold and $f \in C^\beta(\overline{\Omega})$ for some $\beta > 0$. Then*

$$\|v\|_{C^{1,\alpha}(\overline{\Omega})} \leq K(\|\varphi\|_{C^{1,1}(\partial\Omega)} + \|g/\,tr\, U\|_{L^\infty(\Omega)}),$$

with K a constants depending on $n, \beta, \rho, \lambda, \Lambda$ and $\|f\|_{C^\beta(\overline{\Omega})}$.

Finally we mention also the regularity properties of the potentials u that satisfy our hypotheses.

Proposition 2.6. *If u satisfies the hypotheses of Theorem 2.4 then*

$$[\nabla u]_{C^\alpha(\overline{\Omega})} \leq C.$$

If in addition $f \in C^\beta(\overline{\Omega})$ then

$$\|D^2 u\| \leq K|\log\varepsilon|^2 \quad on \quad \Omega_\varepsilon = \{x \in \Omega, dist(x, \partial\Omega) > \varepsilon\},$$

where K is a constant depending on $n, \beta, \rho, \lambda, \Lambda$ and $\|f\|_{C^\beta(\overline{\Omega})}$.

The proof of Theorem 2.1 follows the same lines as the proof of the standard boundary estimate of Krylov. Our main tools are a localization theorem at the boundary for solutions to the Monge-Ampère equation which was obtained in [S], and the interior Harnack estimates for solutions to the linearized Monge-Ampère equations which were established in [CG] (see Section 3).

3. The Localization Theorem and Weak Harnack Inequality

In this section, we state the main tools used in the proof of Theorem 2.1, the localization theorem and the weak Harnack inequality.

We start with the localization theorem. Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous convex function and assume that

$$u(0) = 0, \quad \nabla u(0) = 0. \tag{3.1}$$

Let $S_h(u)$ be the section of u at 0 with level h:

$$S_h := \{x \in \overline{\Omega} : \quad u(x) < h\}.$$

If the boundary data has quadratic growth near $\{x_n = 0\}$ then, as $h \to 0$, S_h is equivalent to a half-ellipsoid centered at 0. This is the content of the Localization Theorem proved in [S, S2]. Precisely, this theorem reads as follows.

Theorem 3.1 (Localization Theorem [S, S2])**.** *Assume that Ω satisfies* (2.1) *and u satisfies* (2.3), (3.1) *above and,*

$$\rho|x|^2 \leq u(x) \leq \rho^{-1}|x|^2 \quad on\ \partial\Omega \cap \{x_n \leq \rho\}. \tag{3.2}$$

Then, for each $h < k$ there exists an ellipsoid E_h of volume $\omega_n h^{n/2}$ such that

$$kE_h \cap \overline{\Omega} \subset S_h \subset k^{-1}E_h \cap \overline{\Omega}.$$

Moreover, the ellipsoid E_h is obtained from the ball of radius $h^{1/2}$ by a linear transformation A_h^{-1} (sliding along the $x_n = 0$ plane)

$$A_h E_h = h^{1/2} B_1$$
$$A_h(x) = x - \tau_h x_n, \quad \tau_h = (\tau_1, \tau_2, \ldots, \tau_{n-1}, 0),$$

with

$$|\tau_h| \le k^{-1} |\log h|.$$

The constant k above depends only on $\rho, \lambda, \Lambda, n$.

The ellipsoid E_h, or equivalently the linear map A_h, provides useful information about the behavior of u near the origin. From Theorem 3.1 we also control the shape of sections that are tangent to $\partial\Omega$ at the origin. Before we state this result we introduce the notation for the section of u centered at $x \in \overline{\Omega}$ at height h:

$$S_{x,h}(u) := \{y \in \overline{\Omega} : u(y) < u(x) + \nabla u(x)(y - x) + h\}.$$

Proposition 3.2. *Let u and Ω satisfy the hypotheses of the Localization Theorem 3.1 at the origin. Assume that for some $y \in \Omega$ the section $S_{y,h} \subset \Omega$ is tangent to $\partial\Omega$ at 0 for some $h \le c$. Then there exists a small constant $k_0 > 0$ depending on λ, Λ, ρ and n such that*

$$\nabla u(y) = a e_n \quad \text{for some} \quad a \in [k_0 h^{1/2}, k_0^{-1} h^{1/2}],$$
$$k_0 E_h \subset S_{y,h} - y \subset k_0^{-1} E_h, \qquad k_0 h^{1/2} \le dist(y, \partial\Omega) \le k_0^{-1} h^{-1/2},$$

with E_h the ellipsoid defined in the Localization Theorem 3.1.

Proposition 3.2 is a consequence of Theorem 3.1 and was proved [S3]. For completeness we sketch its proof at the end of the paper.

Next, we state the weak Harnack inequality. Caffarelli and Gutiérrez [CG] proved Hölder estimates and Harnack inequalities for solutions of the homogeneous equation $L_u v = 0$. Their approach is based on the Krylov and Safonov's Hölder estimates for linear elliptic equations in general form, with the sections of u having the same role as euclidean balls have in the classical theory. We state the weak Harnack inequality in this setting (see also [TW3]).

Theorem 3.3. *(Theorem 4* [CG]*) Let $u \in C^2(\Omega)$ be a locally strictly convex function satisfying*

$$0 < \lambda \le \det D^2 u \le \Lambda,$$

and let $v \ge 0$ be a nonnegative supersolution defined in a section $S_{x,h}(u) \subset\subset \Omega$,

$$L_u v := U^{ij} v_{ij} \le 0.$$

If

$$|\{v \ge 1\} \cap S_{x,h}(u)| \ge \mu |S_{x,h}(u)|$$

then

$$\inf_{S_{x,h/2}(u)} v \ge c,$$

with $c > 0$ a constant depending only on n, λ, Λ and μ.

4. Boundary behavior of the rescaled functions

We denote by c, C positive constants depending on ρ, λ, Λ, n, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants.*

Sometimes, for simplicity of notation, we write $S_{x,h}$ instead of $S_{x,h}(u)$ and we drop the x subindex whenever $x = 0$, i.e., $S_h = S_{0,h}(u)$.

We denote the distance from a point x to a closed set Γ as

$$d_\Gamma(x) = dist(x, \Gamma). \tag{4.1}$$

First we obtain pointwise $C^{1,\alpha}$ estimates on the boundary in the setting of the Localization Theorem 3.1. We know that for all $h \le k$, S_h satisfies

$$kE_h \cap \bar{\Omega} \subset S_h \subset k^{-1}E_h,$$

with A_h being a linear transformation and

$$\det A_h = 1, \;\; E_h = A_h^{-1} B_{h^{1/2}}, \qquad A_h x = x - \tau_h x_n$$

$$\tau_h \cdot e_n = 0, \quad \|A_h^{-1}\|, \, \|A_h\| \le k^{-1} |\log h|.$$

This gives

$$\overline{\Omega} \cap B^+_{ch^{1/2}/|\log h|} \subset S_h \subset B^+_{Ch^{1/2}|\log h|}, \tag{4.2}$$

or

$$|u| \le h \quad \text{in} \quad \overline{\Omega} \cap B^+_{ch^{1/2}/|\log h|}.$$

Then for all x close to the origin

$$|u(x)| \le C|x|^2 |\log x|^2,$$

which shows that u is differentiable at 0. We remark that the other inclusion of (4.2) gives a lower bound for u near the origin

$$u(x) \ge c|x|^2 |\log x|^{-2} \ge |x|^3. \tag{4.3}$$

We summarize the differentiability of u in the next lemma.

Lemma 4.1. *Assume u and Ω satisfy the hypotheses of the Localization Theorem 3.1 at a point $x_0 \in \partial\Omega$. If $x \in \overline{\Omega} \cap B_r(x_0)$, $r \le 1/2$, then*

$$|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| \le Cr^2 |\log r|^2. \tag{4.4}$$

Moreover, if u, Ω satisfy the hypotheses of the Localization Theorem 3.1 also at a point $x_1 \in \partial\Omega \cap B_r(x_0)$ then

$$|\nabla u(x_1) - \nabla u(x_0)| \le Cr |\log r|^2.$$

Clearly, the second statement follows from writing (4.4) for x_0 and x_1 at all points x in a ball $B_{cr}(y) \subset \Omega$.

Next we discuss the scaling for our linearized Monge-Ampère equation. Under the linear transformations

$$\tilde{u}(x) = \frac{1}{a}u(Tx), \qquad \tilde{v}(x) = \frac{1}{b}v(Tx),$$
$$\tilde{g}(x) = \frac{1}{a^{n-1}b}(\det T)^2 g(Tx),$$

we find that

$$\tilde{U}^{ij}\tilde{v}_{ij} = \tilde{g}. \tag{4.5}$$

Indeed, we note that

$$D^2\tilde{u} = \frac{1}{a}T^t D^2 u T, \quad D^2\tilde{v} = \frac{1}{b}T^t D^2 v T,$$

and

$$\begin{aligned}\tilde{U} &= (\det D^2\tilde{u})(D^2\tilde{u})^{-1}\\ &= \frac{1}{a^{n-1}}(\det T)^2(\det D^2 u)\, T^{-1}(D^2u)^{-1}(T^{-1})^t\\ &= \frac{1}{a^{n-1}}(\det T)^2 T^{-1} U (T^{-1})^t\end{aligned}$$

and (4.5) easily follows.
We use the rescaling above with

$$a = h, \quad b = h^{1/2}, \; T = h^{1/2}A_h^{-1}$$

where A_h is the matrix in the Localization theorem. We denote the rescaled functions by

$$u_h(x) := \frac{u(h^{1/2}A_h^{-1}x)}{h}, \qquad v_h(x) := \frac{v(h^{1/2}A_h^{-1}x)}{h^{1/2}}.$$
$$g_h(x) := h^{1/2}g(h^{1/2}A_h^{-1}x),$$

and they satisfy

$$U_h^{ij}D_{ij}v_h = g_h. \tag{4.6}$$

The function u_h is continuous and is defined in $\overline{\Omega}_h$ with

$$\Omega_h := h^{-1/2}A_h\Omega,$$

and solves the Monge-Ampère equation

$$\det D^2 u_h = f_h(x), \qquad \lambda \le f_h \le \Lambda,$$

with

$$f_h(x) := f(h^{1/2}A_h^{-1}x).$$

The section at height 1 for u_h centered at the origin satisfies

$$S_1(u_h) = h^{-1/2}A_h S_h,$$

and by the localization theorem we obtain

$$B_k \cap \overline{\Omega}_h \subset S_1(u_h) \subset B^+_{k^{-1}}.$$

We remark that since

$$tr\, U = h^{-1}\, tr(TU_hT^t) \leq h^{-1}\|T\|^2\, tr\, U_h,$$

we obtain

$$\|g_h/\, tr\, U_h\|_{L^\infty} \leq Ch^{1/2}|\log h|^2 \;\; \|g/\, tr\, U\|_{L^\infty}. \tag{4.7}$$

In the next lemma we investigate the properties of the rescaled function u_h. We recall that if $x_0 \in \partial\Omega \cap B_\rho$ then Ω has an interior tangent ball of radius ρ at x_0, and u satisfies

$$\rho\, |x - x_0|^2 \leq u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \leq \rho^{-1}\, |x - x_0|^2\,, \quad \forall x \in \partial\Omega. \tag{4.8}$$

Lemma 4.2. *If $h \leq c$, then*

a) $\partial\Omega_h \cap B_{2/k}$ is a graph in the e_n direction whose $C^{1,1}$ norm is bounded by $Ch^{1/2}$;

b) for any $x, x_0 \in \partial\Omega_h \cap B_{2/k}$ we have

$$\frac{\rho}{4}\, |x - x_0|^2 \leq u_h(x) - u_h(x_0) - \nabla u_h(x_0)(x - x_0) \leq 4\rho^{-1}\, |x - x_0|^2\,, \tag{4.9}$$

c) if $r \leq c$ small, we have

$$|\nabla u_h| \leq Cr|\log r|^2 \quad in \quad \overline{\Omega}_h \cap B_r.$$

Proof. For $x, x_0 \in \partial\Omega_h \cap B_{2/k}$ we denote

$$X = Tx, \quad X_0 = Tx_0, \quad T := h^{1/2}A_h^{-1},$$

hence

$$X, X_0 \in \partial\Omega \cap B_{Ch^{1/2}|\log h|}.$$

First we show that

$$\frac{|x - x_0|}{2} \leq \frac{|X - X_0|}{h^{1/2}} \leq 2\, |x - x_0|\,, \tag{4.10}$$

which is equivalent to

$$1/2 \leq |A_hZ|/|Z| \leq 2, \qquad Z := X - X_0.$$

Since $\partial\Omega$ is $C^{1,1}$ in a neighborhood of the origin we find

$$|Z_n| \leq Ch^{1/2}|\log h||Z'|$$

hence, if h is small

$$|A_hZ - Z| = |\tau_h Z_n| \leq Ch^{1/2}|\log h|^2|Z'| \leq |Z|/2,$$

and (4.10) is proved.

Part b) follows now from (4.8) and the equality

$$u_h(x) - u_h(x_0) - \nabla u_h(x_0)(x - x_0) = \frac{1}{h}(u(X) - u(X_0) - \nabla u(X_0)(X - X_0).$$

Next we show that $\partial\Omega_h$ has small $C^{1,1}$ norm. Since $\partial\Omega$ has an interior tangent ball at X_0 we see that

$$|(X-X_0)\cdot\nu_0|\le C|X-X_0|^2,$$

where ν_0 is the exterior normal to Ω at X_0. This implies, in view of (4.10)

$$|(x-x_0)\cdot T^t\nu_0|\le Ch|x-x_0|^2,$$

or

$$|(x-x_0)\cdot\tilde{\nu}_0|\le C\frac{h}{|T^t\nu_0|}|x-x_0|^2,$$

where

$$\tilde{\nu}_0:=T^t\nu_0/|T^t\nu_0|.$$

From the formula for A_h we see that

$$e_n\cdot((A_h^{-1})^Te_n)=(A_h^{-1}e_n)\cdot e_n=1,$$

hence

$$|(A_h^{-1})^Te_n|\ge 1.$$

Since

$$|\nu_0+e_n|\le Ch^{1/2}|\log h|$$

we obtain

$$|(A_h^{-1})^T\nu_0|\ge 1-Ch^{1/2}\,|\log h|\,\|A_h^{-1}\|\ge 1/2,$$

thus

$$|T^t\nu_0|=h^{1/2}|(A_h^{-1})^T\nu_0|\ge h^{1/2}/2.$$

In conclusion

$$|(x-x_0)\cdot\tilde{\nu}_0|\le Ch^{1/2}|x-x_0|^2,$$

which easily implies our claim about the $C^{1,1}$ norm of $\partial\Omega_h$.

Next we prove property c). From a), b) above we see that u_h satisfies in $S_1(u_h)$ the hypotheses of the Localization Theorem 3.1 at 0 for a small $\tilde{\rho}$ depending on the given constants. We consider a point $x_0\in\partial\Omega_h\cap B_r$, and by Lemma 4.1, it remains to show that u_h, $S_1(u_h)$ satisfy the hypotheses of the Localization Theorem 3.1 also at x_0. From (4.4) we have

$$|u_h|\le Cr^2|\log r|^2 \quad \text{in} \quad \overline{\Omega}_h\cap B_{2r}, \tag{4.11}$$

which, by convexity of u_h gives

$$\partial_n u_h(x_0)\le Cr|\log r|^2.$$

On the other hand, we use part b) at x_0 and 0 (see (4.9)) and obtain

$$|u_h(x_0)+\nabla u_h(x_0)\,(x-x_0)|\le Cr^2 \quad \text{on} \quad \partial\Omega\cap B_r,$$

thus

$$|\nabla u_h(x_0)\cdot x|\le Cr^2 \quad \text{on} \quad \partial\Omega\cap B_r.$$

Since $x_n\ge 0$ on $\partial\Omega$, we see that if $\partial_n u_h(x_0)\ge 0$ then,

$$\nabla_{x'}u_h(x_0)\cdot x'\le Cr^2 \quad \text{if} \quad |x'|\le r/2,$$

which gives

$$|\nabla_{x'} u_h(x_0)| \leq Cr.$$

We obtain the same conclusion similarly if $\partial_n u_h(x_0) \leq 0$. The upper bounds on $\partial_n u_h(x_0)$ and $|\nabla_{x'} u_h(x_0)|$ imply that if $x \in S_1(u_h) \subset B_{1/k}$ we have

$$u_h(x_0) + \nabla u_h(x_0) \cdot (x - x_0) + 1/2 \leq Cr^2 + Cr|\log r|^2 + 1/2 < 1,$$

provided that r is small. This shows that

$$S_{x_0, \frac{1}{2}}(u_h) \subset S_1(u_h) \subset B_{1/k}.$$

Moreover, since

$$|S_{x_0, \frac{1}{2}}(u_h)| = h^{-n/2} |S_{X_0, \frac{h}{2}}(u)| \sim 1$$

we obtain a bound for $|\nabla u(x_0)|$. Now we can easily conclude from parts a) and b) and the inclusion above that u_h satisfies in $S_1(u_h)$ the hypotheses of the Localization Theorem at x_0 for a small $\tilde{\rho}$.

□

In the next proposition we compare the distance functions under the following transformations of point and domain:

$$x \to X := Tx, \ \Omega_h \to \Omega = T\Omega_h, \qquad T = h^{1/2} A_h^{-1}.$$

Proposition 4.3. *For $x \in \Omega_h \cap B^+_{k^{-1}}$, let $X = Tx \in \Omega$. Then (see notation* (4.1)*)*

$$1 - Ch^{1/2} |\log h|^2 \leq \frac{h^{-1/2} d_{\partial\Omega}(X)}{d_{\partial\Omega_h}(x)} \leq 1 + Ch^{1/2} |\log h|^2 .$$

Proof. Denote by ξ_x, ξ_X the unit vectors at x, X which give the perpendicular direction to $\partial\Omega_h$ respectively $\partial\Omega$, and which point inside the domain. Since $\partial\Omega$ is $C^{1,1}$ at the origin and $|X| \leq Ch^{1/2}|\log h|$ we find

$$|\xi_X - e_n| \leq Ch^{1/2}|\log h|.$$

Moreover, the $C^{1,1}$ bound of $\partial\Omega_h$ from Lemma 4.2 shows that

$$|\xi_x - e_n| \leq Ch^{1/2}.$$

We compare $h^{-1/2} d_{\partial\Omega}(X)$ with $d_{\partial\Omega_h}(x)$ by computing the directional derivative of $h^{-1/2} d_{\partial\Omega}(X)$ along ξ_x. We have

$$\begin{aligned} \nabla_x (h^{-1/2} d_{\partial\Omega}(X)) \cdot \xi_x &= h^{-1/2} \nabla_X d_{\partial\Omega}(X) \ T \, \xi_x \\ &= h^{-1/2} \xi_X \cdot (T\xi_x) \\ &= \xi_X \cdot (A_h^{-1} \xi_x) \end{aligned}$$

From the inequalities above on ξ_x, ξ_X we find

$$|\xi_X \cdot (A_h^{-1}\xi_x) - e_n \cdot (A_h^{-1} e_n)| \leq Ch^{1/2}|\log h| \|A_h^{-1}\| \leq Ch^{1/2}|\log h|^2.$$

Using

$$e_n \cdot (A_h^{-1} e_n) = 1$$

we obtain

$$|\nabla_x (h^{-1/2} d_{\partial\Omega}(X)) \cdot \xi_x - 1| \leq C h^{1/2} |\log h|^2,$$

which implies our result.

□

5. The class $\mathcal{D}_\sigma$ and its main properties

In this section we introduce the class $\mathcal{D}_\sigma$ that captures the properties of the rescaled functions u_h in $S_1(u_h)$. By abuse of notation we use u and Ω when we define $\mathcal{D}_\sigma$.

Fix ρ, λ, Λ. We introduce the class $\mathcal{D}_\sigma$ consisting of pairs of function u and domain Ω satisfying the following conditions:

(i) $0 \in \partial\Omega, \quad \Omega \subset B^+_{1/k}, \quad |\Omega| \geq c_0,$

(ii) $u : \overline{\Omega} \to \mathbb{R}$ is convex, continuous satisfying

$$u(0) = 0, \quad \nabla u(0) = 0, \quad \lambda \leq \det D^2 u \leq \Lambda;$$

(iii)

$$\partial\Omega \cap \{u < 1\} \subset G \subset \{x_n \leq \sigma\}$$

where G is a graph in the e_n direction which is defined in $B_{2/k}$, and its $C^{1,1}$ norm is bounded by σ.

(iv)

$$\frac{\rho}{4}|x - x_0|^2 \leq u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \leq \frac{4}{\rho}|x - x_0|^2 \qquad \forall x, x_0 \in G \cap \partial\Omega;$$

(v) If $r \leq c_0$,

$$|\nabla u| \leq C_0 r |\log r|^2 \qquad \text{in} \quad \overline{\Omega} \cap B_r.$$

The constants k, c_0, C_0 above depend explicitly on ρ, λ, Λ, and n.

We remark that the properties above imply that if $x_0 \in \partial\Omega$ is close to the origin then

$$S_{x_0, \frac{1}{2}}(u) \subset \{u < 1\},$$

and u satisfies in $S_{x_0,\frac{1}{2}}(u)$ the hypotheses of the Localization Theorem at x_0 for some $\tilde{\rho}$ depending on the given constants.

Lemma 4.2 can be restated in the following way.

Lemma 5.1. *Let (u, Ω) be as in Theorem 2.1. Then, if $h \leq c$,*

$$(u_h, S_1(u_h)) \in \mathcal{D}_\sigma \quad \textit{with } \sigma = C h^{1/2}.$$

We first construct a useful subsolution.

Lemma 5.2 (Subsolution)**.** *Suppose $(u, \Omega) \in \mathcal{D}_\delta$. If $\delta \leq c$ then the function*

$$\underline{w} := x_n - u + \delta^{\frac{1}{n-1}} |x'|^2 + \frac{\Lambda^n}{\lambda^{n-1}\delta} x_n^2$$

satisfies

$$L_u(\underline{w}) := U^{ij}\underline{w}_{ij} \geq \delta^{\frac{1}{n-1}} \; tr\, U,$$

and on the boundary of the domain $D := \{x_n \leq 2\delta\} \cap \Omega$ *we have*

$$\underline{w} \leq 0 \;\; on \;\partial D \backslash F_\delta, \quad \underline{w} \leq 1 \;\; on \; F_\delta,$$

where

$$F_\delta := \{x_n = 2\delta, \quad |x'| \leq \delta^{\frac{1}{6(n-1)}}\}. \tag{5.1}$$

Proof. Let

$$p(x) = \frac{1}{2}\left(\delta^{\frac{1}{n-1}}|x'|^2 + \frac{\Lambda^n}{\lambda^{n-1}\delta}x_n^2\right).$$

Then

$$\det D^2 p(x) = \frac{\Lambda^n}{\lambda^{n-1}}.$$

Using the matrix inequality

$$tr(AB) \geq n(\det A \det B)^{1/n} \text{ for } A, B \text{ symmetric } \geq 0,$$

we get

$$L_u p = U^{ij}p_{ij} \geq n(\det(U)\det D^2 p)^{1/n} = n((\det D^2 u)^{n-1}\frac{\Lambda^n}{\lambda^{n-1}})^{1/n} \geq n\Lambda.$$

Since δ is small

$$D^2 p \geq \delta^{\frac{1}{n-1}} I,$$

hence

$$L_u p = U^{ij}p_{ij} \geq \delta^{\frac{1}{n-1}} \; tr\, U.$$

Using $L_u x_n = 0$ and

$$L_u u = U^{ij}u_{ij} = n \det D^2 u \leq n\Lambda$$

we find

$$L_u \underline{w} = L_u(x_n - u + 2p) \geq \delta^{\frac{1}{n-1}} \; tr\, U.$$

Next we check the behavior of $\underline{w}$ on ∂D. We decompose $\partial D \subset G \cup E_\delta \cup F_\delta$ where

$$E_\delta := \partial D \cap \{|x| \geq \delta^{\frac{1}{6(n-1)}}\}.$$

On $G \cap \partial\Omega$, we use the properties of $\mathcal{D}_\delta$ and obtain

$$u \geq (\rho/4)|x|^2, \qquad x_n \leq \delta|x'|^2$$

which follows from the $C^{1,1}$ bound on the graph G. Then

$$\underline{w} \leq (\delta + \delta^{\frac{1}{n-1}} + C\delta)|x'|^2 - (\rho/4)|x|^2 \leq 0,$$

provided that δ is small.

On E_δ we use (4.3) and find

$$u \geq (\delta^{\frac{1}{6(n-1)}})^3 = \delta^{\frac{1}{2(n-1)}}$$

hence, for small δ,

$$w \leq C\delta - \delta^{\frac{1}{2(n-1)}} + c\delta^{\frac{1}{n-1}} \leq -\delta^{\frac{1}{2(n-1)}}/2 < 0.$$

On F_δ, the positive terms in $\underline{w}$ are bounded by $1/3$ for small δ and we obtain $w \leq 1$. □

Remark 5.3. For any point $x_0 \in \partial\Omega$ close to the origin we can construct the corresponding subsolution

$$\underline{w}_{x_0} = z_n - u_{x_0} + 2p(z)$$

where

$$u_{x_0} := u - u(x_0) - \nabla u(x_0) \cdot (x - x_0)$$

and with z denoting the coordinates of the point x in a system of coordinates centered at x_0 with the z_n-axis perpendicular to $\partial\Omega$. From the proof above we see that $\underline{w}_{x_0}$ satisfies the same conclusion of Lemma 5.2 if $|x_0| \ll \delta$.

Next we show that u has uniform modulus of convexity on the set F_δ introduced above (see (5.1)).

Lemma 5.4. *Let $(u, \Omega) \in \mathcal{D}_\delta$. If $\delta \leq c$ then for any $y \in F_\delta$ we have*

$$S_{y,c\delta^2}(u) \subset \Omega.$$

Remark 5.5. From now on we fix the value of δ to be small, universal so that it satisfies the hypotheses of Lemma 5.2 and Lemma 5.4.

Remark 5.6. Since the section $S_{y,c\delta^2/2}$ is contained in $\Omega \subset B_{1/k}$ and has volume bounded from below we can conclude that it contains a ball $B_{\bar{\delta}}(y)$ for some $\bar{\delta} \ll \delta$ small, universal.

We sketch the proof of Lemma 5.4 below.

Proof. Let h_0 be the maximal value of h for which $S_{y,h} \subset \Omega$, and let

$$x_0 \in \partial S_{y,h_0} \cap \partial\Omega.$$

Since S_{y,h_0} is balanced around y and u grows quadratically away from 0 on G we see that the point x_0 lies also in a neighborhood of the origin. Now we can apply Proposition 3.2 at x_0 and obtain

$$h_0 \geq c d_{\partial\Omega}(y)^2 \geq c\delta^2.$$

□

A consequence of Lemma 5.2 is the following proposition.

Proposition 5.7. *Assume $(u, \Omega) \in \mathcal{D}_\delta$, and let $v \geq 0$ be a nonnegative function satisfying*

$$L_u v \leq \delta^{\frac{1}{n-1}} \, tr\, U \text{ in } \Omega, \qquad v \geq 1 \text{ in } F_\delta.$$

Then,

$$v \geq \frac{1}{2} d_G \text{ in } S_\theta.$$

for some small θ universal.

Proof. Lemma 5.2 and the maximum principle for the operator L_u imply $v \geq \underline{w}$ in D, which gives

$$v(0, x_n) \geq \frac{1}{2} x_n \quad \text{for } x_n \in [0, c].$$

The same argument can be repeated at points $x_0 \in \partial\Omega$ if $|x_0|$ is sufficiently small, by comparing u with the corresponding subsolution $\underline{w}_{x_0}$. We obtain

$$v \geq \frac{1}{2} d_G \quad \text{in} \quad \overline{\Omega} \cap B_c,$$

and the lemma follows by choosing θ sufficiently small. □

Proposition 5.8. *Let $(u, \Omega) \in \mathcal{D}_\sigma$, $(\sigma \leq \delta)$ and suppose v satisfies in Ω*

$$L_u v = g, \qquad a\, d_G \leq v \leq b\, d_G,$$

for some $a, b \in [-1, 1]$. There exists c_1 small, universal such that if

$$\max\{\sigma, \|g/tr\, U\|_{L^\infty}\} \leq c_1(b - a),$$

then

$$a' d_G \leq v \leq b' d_G \quad in \quad S_\theta,$$

for some a', b' that satisfy

$$a \leq a' \leq b' \leq b, \qquad b' - a' \leq \eta(b - a),$$

with $\eta \in (0, 1)$, universal, close to 1.

Proof. We define the functions

$$v_1 = \frac{v - a\, d_G}{b - a}, \ v_2 = \frac{b\, d_G - v}{b - a}$$

which are nonnegative. Since

$$v_1 + v_2 = d_G,$$

we might assume (see Remark 5.6) that the function v_1 satisfies

$$|\{v_1 \geq \frac{\delta}{2}\} \cap B_{\bar{\delta}}(2\delta e_n)| \geq \frac{1}{2}|B_{\bar{\delta}}(2\delta e_n)|.$$

Next we apply Theorem 3.3, for the function

$$\tilde{v}_1 := v_1 + c_1(k^{-2} - |x|^2).$$

Notice that $\tilde{v}_1 \geq v_1 \geq 0$ in Ω and

$$L_u \tilde{v}_1 \leq (g + \sigma\ tr\, U)(b - a)^{-1} - 2c_1\, trU \leq 0.$$

Using Lemma 5.4 we can apply weak Harnack inequality Theorem 3.3 a finite number of times and obtain

$$\tilde{v}_1 \geq 2c_2 > 0 \quad \text{on} \quad F_\delta,$$

for some universal c_2. By choosing c_1 sufficiently small we find

$$v_1 \geq c_2 \quad \text{on} \quad F_\delta.$$

Now we can apply Proposition 5.7 to v_1/c_2 since

$$L_u(v_1/c_2) \leq 2(c_1/c_2)\, trU \leq \delta^{\frac{1}{n-1}}\, tr\, U,$$

provided that c_1 is small. We obtain

$$v_1 \geq (c_2/2)\, d_G \quad \text{in } S_\theta,$$

hence

$$v \geq a' d_G, \quad a' = a + c_2(b-a)/2.$$

□

6. Proof of Theorem 2.1

Throughout this section we assume that u, v satisfy the hypotheses of Theorem 2.1 and we also assume for simplicity that

$$u(0) = 0, \quad \nabla u(0) = 0.$$

Our boundary gradient estimate states as follows.

Proposition 6.1. *Let v be as in Theorem 2.1. Then, in $\Omega \cap B_{\rho/2}$, we have*

$$|v(x)| \leq C(\|v\|_{L^\infty(\Omega\cap B_\rho)} + \|g/\, tr\, U\|_{L^\infty(\Omega \cap B_\rho)}) d_{\partial\Omega}(x).$$

The proposition follows easily from the construction of a suitable supersolution.

Lemma 6.2 (Supersolution). *There exists universal constants M large, and $\tilde{\delta}$ small such that the function*

$$\overline{w} := Mx_n + u - \tilde{\delta}|x'|^2 - \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}} x_n^2$$

satisfies

$$L_u(\overline{w}) \leq -\tilde{\delta}\; tr\, U,$$

and

$$\overline{w} \geq 0 \text{ on } \partial(\Omega \cap B_\rho), \quad \overline{w} \geq \tilde{\delta} \text{ on } \partial(\Omega\cap B_\rho) \setminus B_{\rho/2}.$$

Proof. We first choose $\tilde{\delta} \leq \rho$ small such that

$$u - \tilde{\delta}|x'|^2 \geq \tilde{\delta} \quad \text{on } \overline{\Omega} \setminus B_{\rho/2}.$$

The existence of $\tilde{\delta}$ follows for example from (4.3). We choose M such that

$$Mx_n - \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}} x_n^2 \geq 0 \qquad \text{on } \overline{\Omega}.$$

Then on $\partial\Omega$,

$$\overline{w} \geq u - \tilde{\delta}|x'|^2 \geq 0,$$

and we obtain the desired inequalities for $\overline{w}$ on $\partial\Omega$.

If we denote

$$q(x) := \frac{1}{2}\left(\tilde{\delta}|x'|^2 + \frac{\Lambda^n}{(\lambda\tilde{\delta})^{n-1}}x_n^2\right),$$

then

$$\det D^2 q = \frac{\Lambda^n}{\lambda^{n-1}}, \quad D^2 q \geq \tilde{\delta} I$$

and we obtain as in Lemma 5.2

$$L_u \overline{w} \leq -\tilde{\delta}\ tr\, U.$$

□

Proof of Proposition 6.1. By dividing the equation by a suitable constant we may suppose that

$$\|v\|_{L^\infty} \leq \tilde{\delta}, \qquad \|g/\,tr\, U\|_{L^\infty} \leq \tilde{\delta},$$

and we need to show that

$$|v| \leq C d_{\partial\Omega} \quad \text{in } \partial\Omega \cap B_{\rho/2}.$$

Since $v \leq \overline{w}$ on $\partial(\Omega \cap B_\rho)$ and $L_u v \geq L_u \overline{w}$ we obtain $v \leq \overline{w}$ in $\Omega \cap B_\rho$ hence

$$v(0, x_n) \leq C x_n, \qquad \text{if } x_n \in [0, \rho/2].$$

The same argument applies at all points $x_0 \in \partial\Omega \cap B_{\rho/2}$ and we obtain the upper bound for v. The lower bound follows similarly and the proposition is proved.

□

Proof of Theorem 2.1. By dividing by a suitable constant we may suppose that

$$\|v\|_{L^\infty} + \|g/\,tr\, U\|_{L^\infty}$$

is sufficiently small such that, by Proposition 6.1,

$$|v| \leq \frac{1}{2} d_{\partial\Omega} \quad \text{in } \overline{\Omega} \cap B_{\rho/2}.$$

We focus our attention on the sections at the origin and we show that we can improve these bounds in the form

$$a_h d_{\partial\Omega} \leq v \leq b_h d_{\partial\Omega}, \text{ in } S_h, \tag{6.1}$$

for appropriate constants a_h, b_h. First we fix h_0 small universal and let

$$a_{h_0} = -1/2, \quad b_{h_0} = 1/2.$$

Then we show by induction that for all

$$h = h_0 \theta^k, \qquad k \geq 0,$$

we can find a_h increasing and b_h decreasing with k such that (6.1) holds and

$$b_h - a_h = \left(\frac{1+\eta}{2}\right)^k \geq C_1 h^{1/2} |\log h|^2. \tag{6.2}$$

for some large universal constant C_1. We notice that this statement holds for $k = 0$ if h_0 is chosen sufficiently small.

Assume the statement holds for k. Proposition 4.3 implies that

$$\overline{a}_h d_{\partial\Omega_h} \leq v_h \leq \overline{b}_h d_{\partial\Omega_h} \text{ in } S_1(u_h)$$

with

$$|\overline{a}_h - a_h| \leq Ch^{1/2} \left|\log h\right|^2, \ |\overline{b}_h - b| \leq Ch^{1/2} \left|\log h\right|^2.$$

Since

$$(u_h, S_1(u_h)) \in \mathcal{D}_\sigma, \quad \text{for } \sigma = Ch^{1/2},$$

and (see (4.6), (4.7))

$$L_{u_h} v_h = g_h,$$

$$\begin{aligned}\|g_h/trU_h\|_{L^\infty} &\leq Ch^{1/2}|\log h|^2 \|g/trU\|_{L^\infty}\\ &\leq Ch^{1/2}|\log h|^2\\ &\leq c_1(\overline{b}_h - \overline{a}_h),\end{aligned}$$

we can apply Proposition 5.8 and conclude

$$\overline{a}_{\theta h} d_{\partial\Omega_h} \leq v_h(x) \leq \overline{b}_{\theta h} d_{\partial\Omega_h}, \text{ in } S_\theta(u_h).$$

with

$$\overline{b}_{\theta h} - \overline{a}_{\theta h} \leq \eta(\overline{b}_h - \overline{a}_h).$$

Rescaling back to $S_{\theta h}$, and using Proposition 4.3 again, we obtain

$$a_{\theta h} d_{\partial\Omega} \leq v \leq b_{\theta h} d_{\partial\Omega}, \text{ in } S_{\theta h}(u) \tag{6.3}$$

where

$$b_{\theta h} - a_{\theta h} \leq \eta(b_h - a_h) + Ch^{1/2}|\log h| \leq (1+\eta)/2(b_h - a_h).$$

By possibly modifying their values we may take $a_{\theta h}$, $b_{\theta h}$ such that

$$a_h \leq a_{\theta h} \leq b_{\theta h} \leq b, \qquad b_{\theta h} - a_{\theta h} = \frac{1+\eta}{2}(b_h - a_h).$$

From (6.1), (6.2) we find

$$osc_{S_h} v \leq Ch^{1/2+\alpha},$$

for some small α universal. Using (4.2) we obtain

$$osc_{B_r} v \leq Cr^{1+\alpha} \quad \text{if } r \leq c,$$

and the theorem is proved. □

7. Proof of Theorem 2.5

In this last section we prove Propositions 2.6 and 3.2 and Theorem 2.5.

Proof of Proposition 2.6. Let $y \in \Omega$ with

$$r := d_{\partial\Omega}(y) \le c,$$

and consider the maximal section $S_{\bar h, y}$ centered at y, i.e.,

$$\bar h = max\{h \,|\quad S_{y,h} \subset \Omega\}.$$

By Proposition 3.2 applied at the point

$$x_0 \in \partial S_{y,\bar h} \cap \partial\Omega,$$

we have

$$\bar h^{1/2} \sim r, \quad |\nabla u(y) - \nabla u(x_0)| \le C \bar h^{1/2},$$

and $S_{\bar h, y}$ is equivalent to an ellipsoid E i.e

$$c\tilde E \subset S_{\bar h, y} - y \subset C\tilde E,$$

where

$$E := \bar h^{1/2} A_{\bar h}^{-1} B_1, \quad \text{with} \quad \|A_{\bar h}\|, \|A_{\bar h}^{-1}\| \le C|\log \bar h|.$$

We denote

$$u_y := u - u(y) - \nabla u(y)(x-y).$$

The rescaling $\tilde u : \tilde S_1 \to \mathbb{R}$ of u

$$\tilde u(\tilde x) := \frac{1}{\bar h} u_y(T\tilde x) \qquad x = T\tilde x := y + \bar h^{1/2} A_{\bar h}^{-1} \tilde x,$$

satisfies

$$\det D^2 \tilde u(\tilde x) = \tilde f(\tilde x) := f(T\tilde x),$$

and

$$B_c \subset \tilde S_1 \subset B_C, \qquad \tilde S_1 = \bar h^{-1/2} A_{\bar h}(S_{\bar h, y} - y),$$

where $\tilde S_1$ represents the section of $\tilde u$ at the origin at height 1.

The interior $C^{1,\gamma}$ estimate for solutions of the Monge-Ampere equation (see [C1]) gives

$$|\nabla \tilde u(\tilde z_1) - \nabla \tilde u(\tilde z_2)| \le C|\tilde z_1 - \tilde z_2|^\gamma \quad \forall \tilde z_1, \tilde z_2 \in \tilde S_{1/2}$$

for some $\gamma \in (0,1)$, C universal. Rescaling back and using

$$\nabla \tilde u(\tilde z_1) - \nabla \tilde u(\tilde z_2) = \bar h^{-1/2} (A_{\bar h}^{-1})^T (\nabla u(z_1) - \nabla u(z_2)), \qquad \tilde z_1 - \tilde z_2 = \bar h^{-1/2} A_{\bar h}(z_1 - z_2)$$

we find

$$|\nabla u(z_1) - \nabla u(z_2)| \le |z_1 - z_2|^\gamma \quad \forall z_1, z_2 \in S_{\bar h/2, y}.$$

Notice that this inequality holds also in the Euclidean ball $B_{r^2}(y) \subset S_{\bar h/2, y}$. Also, if $y_0 \in \partial\Omega$ denotes the closest point to y on $\partial\Omega$ i.e $|y - y_0| = r$, by Lemma 4.1, we find

$$|\nabla u(y) - \nabla u(y_0)| \le |\nabla u(y) - \nabla u(x_0)| + |\nabla u(x_0) - \nabla u(y_0)| \le r^{1/2}.$$

These oscillation properties for ∇u and Lemma 4.1 easily imply that

$$[\nabla u]_{C^\alpha(\bar\Omega)} \le C,$$

for some $\alpha \in (0,1)$, C universal.

If we assume that $f \in C^\beta(\overline{\Omega})$ then

$$\|\tilde f\|_{C^\beta(\tilde S_1)} \le \|f\|_{C^\beta(\overline{\Omega})},$$

and the interior $C^{2,\beta}$ estimates for $\tilde u$ in $\tilde S_1$ (see [C2]) give

$$\|D^2 \tilde u\|_{C^\beta(\tilde S_{1/2})} \le K. \tag{7.1}$$

In particular

$$\|D^2 u(y)\| = \|A_{\bar h}^T D^2 \tilde u(0) A_{\bar h}\| \le K|\log h|^2 \le K|\log r|^2,$$

where by K we denote various constants depending on β, $\|f\|_{C^\beta(\overline{\Omega})}$ and the universal constants.

□

Proof of Theorem 2.5. We use the same notations as in the proof of Proposition 2.6.

After multiplying v by a suitable constant we may assume that

$$\|\varphi\|_{C^{1,1}} + \|g/\, tr\, U\|_{L^\infty} = 1.$$

We define also the rescaling $\tilde v$ for v

$$\tilde v(x) := \bar h^{-1/2} v(Tx).$$

From Theorem 2.4 we obtain

$$\max_{S_{\bar h, y}} |v - v(x_0) - \nabla v(x_0)(x - x_0)| \le C\, r^{1+\alpha'}$$

for some universal $\alpha' \in (0,1)$ and C, hence

$$\max_{\tilde S_1} |\tilde v(\tilde x) - \tilde v(\tilde x_0) - \nabla \tilde v(\tilde x_0)(\tilde x - \tilde x_0)| \le C r^{1+\alpha'} \bar h^{-1/2} \le C r^{\alpha'}.$$

Using the computations in Section 4 we see that $\tilde v$ solves

$$\tilde U^{ij} \tilde v_{ij} = \tilde g(x) := \bar h^{1/2} g(Tx),$$

with

$$\|\tilde g(x)/tr\, \tilde U\|_{L^\infty(\tilde S_{1/2})} \le C h^{1/2} |\log h|^2 \|g/\, tr\, U\|_{L^\infty} \le C r^{\alpha'}.$$

Since (7.1) holds, we can apply Schauder estimates and find that for any $\tilde z_1, \tilde z_2 \in \tilde S_{1/4}$

$$|\nabla \tilde v(0) - \nabla \tilde v(\tilde x_0)| \le K r^{\alpha'}, \qquad |\nabla \tilde v(\tilde z_1) - \nabla \tilde v(\tilde z_2)| \le K r^{\alpha'} |\tilde z_1 - \tilde z_2|^{\alpha'/2}.$$

Using that $\nabla \tilde v(\tilde z_i) = (A_{\bar h}^{-1})^T \nabla v(z_i)$ we obtain

$$|\nabla v(y) - \nabla v(x_0)| \le K r^{\alpha'/2}, \quad |\nabla v(z_1) - \nabla v(z_2)| \le K |z_1 - z_2|^{\alpha'/2}$$

for any $z_1, z_2 \in B_{r^2}(y)$. These inequalities and Theorem 2.4 give as in the proof of Proposition 2.6 above the desired $C^{1,\alpha}$ bound for v.

□

Remark 7.1. Theorem 2.5 still holds if we only assume that $f \in C(\overline{\Omega})$. In this case one needs to apply the interior $C^{1,\alpha}$ estimates for the linearized Monge-Ampere equation obtained by Gutiérrez and Nguyen [GN].

We conclude the paper with a sketch of the proof of Proposition 3.2.

Proof of Proposition 3.2. Assume that the hypotheses of the Localization Theorem 3.1 hold at the origin. For $a \geq 0$ we denote

$$S'_a := \{x \in \overline{\Omega} | \quad u(x) < a x_n\},$$

and clearly $S'_{a_1} \subset S'_{a_2}$ if $a_1 \leq a_2$. The proposition easily follows once we show that $S'_{ch^{1/2}}$ has the shape of the ellipsoid E_h for all small h.

From Theorem 3.1 we know

$$S_h := \{u < h\} \subset k^{-1} E_h \subset \{x_n \leq k^{-1} h^{1/2}\}$$

and since $u(0) = 0$ we use the convexity of u and obtain

$$S'_{kh^{1/2}} \subset S_h \cap \Omega. \tag{7.2}$$

This inclusion shows that in order to prove that $S'_{kh^{1/2}}$ is equivalent to E_h it suffices to bound its volume by below

$$|S'_{kh^{1/2}}| \geq c|E_h|.$$

From Theorem 3.1, there exists $y \in \partial S_{\theta h}$ such that $y_n \geq k(\theta h)^{1/2}$. We evaluate

$$\tilde{u} := u - k h^{1/2} x_n,$$

at y and find

$$\tilde{u}(y) \leq \theta h - k h^{1/2} k (\theta h)^{1/2} \leq -\delta h,$$

for some $\delta > 0$ provided that we choose θ small depending on k. Since $\tilde{u} = 0$ on $\partial S'_{kh^{1/2}}$ and

$$\det D^2 \tilde{u} \geq \lambda$$

we have

$$|\inf \tilde{u}| \leq C(\lambda) |S'_{kh^{1/2}}|^{2/n},$$

hence

$$c h^{n/2} \leq |S'_{kh^{1/2}}|.$$

□

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