POINTWISE $C^{2,\alpha}$ ESTIMATES AT THE BOUNDARY FOR THE MONGE-AMPERE EQUATION

O. SAVIN

ABSTRACT. We prove a localization property of boundary sections for solutions to the Monge-Ampere equation. As a consequence we obtain pointwise $C^{2,\alpha}$ estimates at boundary points under appropriate local conditions on the right hand side and boundary data.

1. INTRODUCTION

Boundary estimates for the second derivatives of the solution to the Dirichlet problem for the Monge-Ampere equation

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

were first obtained by Ivockina [I] in 1980. A few years later independently Krylov [K] and Caffarelli-Nirenberg-Spruck [CNS] obtained the global $C^{2,\alpha}$ estimates in the case when $\partial\Omega$, φ and f are sufficiently smooth and this led to the solvability of the classical Dirichlet problem for the Monge-Ampere equation. When the right hand side f is less regular, i.e $f \in C^\alpha$, the global $C^{2,\alpha}$ estimates were obtained recently by Trudinger and Wang in [TW] for φ, $\partial\Omega \in C^3$.

In this paper we discuss pointwise $C^{2,\alpha}$ estimates at boundary points under appropriate local conditions on the right hand side and boundary data. Our main result can be viewed as a boundary Schauder estimate for the Monge-Ampere equation which extends up to the boundary the pointwise interior $C^{2,\alpha}$ estimate of Caffarelli [C2] (see also [JW]). These sharp estimates play an important role for example when dealing with fourth order Monge-Ampere type equations arising in geometry, (see [TW], [LS]) or when the right hand side f depends also on the second derivatives.

We start with the following definition (see [CC]).

Definition: Let $0 < \alpha \leq 1$. We say that a function u is *pointwise* $C^{2,\alpha}$ at x_0 and write

$$u \in C^{2,\alpha}(x_0)$$

if there exists a quadratic polynomial P_{x_0} such that

$$u(x) = P_{x_0}(x) + O(|x - x_0|^{2+\alpha}).$$

We say that $u \in C^2(x_0)$ if

$$u(x) = P_{x_0}(x) + o(|x - x_0|^2).$$

Similarly one can define the notion for a function to be C^k and $C^{k,\alpha}$ at a point for any integer $k \geq 0$.

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It is easy to check that if u is *pointwise* $C^{2,\alpha}$ at all points of a Lipschitz domain $\bar\Omega$ and the equality in the definition above is uniform in x_0 then $u \in C^{2,\alpha}(\bar\Omega)$ in the classical sense. Precisely, if there exist M and δ such that for all points $x_0 \in \bar\Omega$

$$|u(x) - P_{x_0}(x)| \le M|x-x_0|^{2+\alpha} \quad \text{if } |x-x_0| \le \delta, \;\; x \in \bar\Omega$$

then

$$[D^2 u]_{C^\alpha(\bar\Omega)} \le C(\delta, \Omega) M.$$

Caffarelli showed in [C2] that if u is a strictly convex solution of

$$\det D^2 u = f$$

and $f \in C^\alpha(x_0)$, $f(x_0) > 0$ at some interior point $x_0 \in \Omega$, then $u \in C^{2,\alpha}(x_0)$. Our main theorem deals with the case when $x_0 \in \partial\Omega$.

Theorem 1.1. *Let Ω be a convex domain and let $u : \bar\Omega \to \mathbb{R}$ convex, continuous, solve the Dirichlet problem for the Monge-Ampere equation*

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with positive, bounded right hand side i.e

$$0 < \lambda \le f \le \Lambda,$$

for some constants λ, Λ.

Assume that for some point $x_0 \in \partial\Omega$ we have

$$f \in C^\alpha(x_0), \quad \varphi, \partial\Omega \in C^{2,\alpha}(x_0),$$

for some $\alpha \in (0,1)$. If φ separates quadratically on $\partial\Omega$ from the tangent plane of u at x_0, then

$$u \in C^{2,\alpha}(x_0).$$

The way φ separates locally from the tangent plane at x_0 is given by the tangential second derivatives of u at x_0. Thus the assumption that this separation is quadratic is in fact necessary for the $C^{2,\alpha}$ estimate to hold. Heuristically, Theorem 1.1 states that if the tangential pure second derivatives of u are bounded below then the boundary Schauder estimates hold for the Monge-Ampere equation.

A more precise, quantitative version of Theorem 1.1 is given in section 7 (see Theorem 7.1).

Given the boundary data, it is not always easy to check the quadratic separation since it involves some information about the slope of the tangent plane at x_0. However, this can be done in several cases (see Proposition 3.2). One example is when $\partial\Omega$ is uniformly convex and φ, $\partial\Omega \in C^3(x_0)$. The C^3 condition of the data is optimal as it was shown by Wang in [W]. Other examples are when $\partial\Omega$ is uniformly convex and φ is linear, or when $\partial\Omega$ is tangent of second order to a plane at x_0 and φ has quadratic growth near x_0.

As a consequence of Theorem 1.1 we obtain a pointwise $C^{2,\alpha}$ estimate in the case when the boundary data and the domain are pointwise C^3. As mentioned above, the global version was obtained by Trudinger and Wang in [TW].

Theorem 1.2. *Let Ω be uniformly convex and let u solve* (1.1). *Assume that*

$$f \in C^\alpha(x_0), \quad \varphi, \partial\Omega \in C^3(x_0),$$

for some point $x_0 \in \partial\Omega$, and some $\alpha \in (0,1)$. Then $u \in C^{2,\alpha}(x_0)$.

We also obtain the $C^{2,\alpha}$ estimate in the simple situation when $\partial\Omega \in C^{2,\alpha}$ and φ is constant.

Theorem 1.3. *Let Ω be a uniformly convex domain and assume u solves* (1.1) *with $\varphi \equiv 0$. If $f \in C^{\alpha}(\bar\Omega)$, $\partial\Omega \in C^{2,\alpha}$, for some $\alpha \in (0,1)$ then $u \in C^{2,\alpha}(\bar\Omega)$.*

The key step in the proof of Theorem 1.1 is a localization theorem for boundary points (see also [S]). It states that under natural local assumptions on the domain and boundary data, the sections

$$S_h(x_0) = \{x \in \overline{\Omega} \mid \quad u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h\},$$

with $x_0 \in \partial\Omega$ are "equivalent" to ellipsoids centered at x_0.

Theorem 1.4. *Let Ω be convex and u satisfy* (1.1)*, and assume*

$$\partial\Omega, \varphi \in C^{1,1}(x_0).$$

If φ separates quadratically from the tangent plane of u at x_0, then for each small $h > 0$ there exists an ellipsoid E_h of volume $h^{n/2}$ such that

$$cE_h \cap \overline{\Omega} \subset \, S_h(x_0) - x_0 \subset \, CE_h \cap \overline{\Omega},$$

with c, C constants independent of h.

Theorem 1.4 provides useful information about the geometry of the level sets under rather mild assumptions and it extends up to the boundary the localization theorem at interior points due to Caffarelli in [C1].

The paper is organized as follows. In section 2 we discuss briefly the compactness of solutions to the Monge-Ampere equation which we use later in the paper (see Theorem 2.7). For this we need to consider also solutions with possible discontinuities at the boundary. In section 3 we give a quantitative version of the Localization Theorem (see Theorem 3.1). In sections 4 and 5 we provide the proof of Theorem 3.1. In section 6 we obtain a version of the classical Pogorelov estimate in half-domain (Theorem 6.4). Finally, in section 7 we use the previous results together with a standard approximation method and prove our main theorem.

2. Solutions with discontinuities on the boundary

Let $u : \Omega \to \mathbb{R}$ be a convex function with $\Omega \subset \mathbb{R}^n$ bounded and convex. Denote by

$$U := \{(x, x_{n+1}) \in \Omega \times \mathbb{R} | \quad x_{n+1} \geq u(x)\}$$

the upper graph of u.

Definition 2.1. We define the values of u on $\partial\Omega$ to be equal to φ i.e

$$u|_{\partial\Omega} = \varphi,$$

if the upper graph of $\varphi : \partial\Omega \to \mathbb{R} \cup \{\infty\}$

$$\Phi := \{(x, x_{n+1}) \in \partial\Omega \times \mathbb{R} | \quad x_{n+1} \geq \varphi(x)\}$$

is given by the closure of U restricted to $\partial\Omega \times \mathbb{R}$,

$$\Phi := \bar U \cap (\partial\Omega \times \mathbb{R}).$$

From the definition we see that φ is *lower semicontinuous*.

If $u : \Omega \to \mathbb{R}$ is a viscosity solution to

$$\det D^2 u = f(x),$$

with $f \geq 0$ continuous and bounded on Ω, then there exists an increasing sequence of subsolutions, continuous up to the boundary,

$$u_n : \bar{\Omega} \to \mathbb{R}, \qquad \det D^2 u_n \geq f(x)$$

with

$$\lim u_n = u \quad \text{in } \bar{\Omega},$$

where the values of u on $\partial\Omega$ are defined as above.

Indeed, let us assume for simplicity that $0 \in \Omega$, $u(0) = 0$, $u \geq 0$. Then, on each ray from the origin u is increasing, hence $v_\varepsilon : \bar{\Omega} \to \mathbb{R}$,

$$v_\varepsilon(x) = u((1-\varepsilon)x)$$

is an increasing family of continuous functions as $\varepsilon \to 0$, with

$$\lim v_\varepsilon = u \quad \text{in } \bar{\Omega}.$$

In order to obtain a sequence of subsolutions we modify v_ε as

$$u_\varepsilon(x) := v_\varepsilon(x) + w_\varepsilon(x),$$

with $w_\varepsilon \leq -\varepsilon$, convex, so that

$$\det D^2 w_\varepsilon \geq |f(x) - (1-\varepsilon)^{2n} f((1-\varepsilon)x)|,$$

thus

$$\det D^2 u_\varepsilon(x) = \det(D^2 v_\varepsilon + D^2 w_\varepsilon) \geq \det D^2 v_\varepsilon + \det D^2 w_\varepsilon \geq f(x).$$

The claim is proved since as $\varepsilon \to 0$ we can choose w_ε to converge uniformly to 0.

Proposition 2.2 (Comparison principle)**.** *Let u, v be defined on Ω with*

$$\det D^2 u \geq f(x) \geq \det D^2 v$$

in the viscosity sense and

$$u|_{\partial\Omega} \leq v|_{\partial\Omega}.$$

Then

$$u \leq v \quad \textit{in } \Omega.$$

Proof. Since u can be approximated by a sequence of continuous functions on $\bar{\Omega}$ it suffices to prove the result in the case when u is continuous on $\bar{\Omega}$ and $u < v$ on $\partial\Omega$. Then, $u < v$ in a small neighborhood of $\partial\Omega$ and the inequality follows from the standard comparison principle.

□

A consequence of the comparison principle is that a solution $\det D^2 u = f$ is determined uniquely by its boundary values $u|_{\partial\Omega}$.

Next we define the notion of convergence for functions which are defined on different domains.

Definition 2.3. a) Let $u_k : \Omega_k \to \mathbb{R}$ be a sequence of convex functions with Ω_k convex. We say that u_k converges to $u : \Omega \to \mathbb{R}$ i.e

$$u_k \to u$$

if the upper graphs converge

$$\bar{U}_k \to \bar{U} \quad \text{in the Haudorff distance.}$$

In particular it follows that $\bar{\Omega}_k \to \bar{\Omega}$ in the Hausdorff distance.

b) Let $\varphi_k : \partial\Omega_k \to \mathbb{R} \cup \{\infty\}$ be a sequence of lower semicontinuous functions. We say that φ_k converges to $\varphi : \partial\Omega \to \mathbb{R} \cup \{\infty\}$ i.e

$$\varphi_k \to \varphi$$

if the upper graphs converge

$$\Phi_k \to \Phi \quad \text{in the Haudorff distance.}$$

c) We say that $f_k : \Omega_k \to \mathbb{R}$ converge to $f : \Omega \to \mathbb{R}$ if f_k are uniformly bounded and

$$f_k \to f$$

uniformly on compact sets of Ω.

Remark: When we restrict the Hausdorff distance to the nonempty closed sets of a compact set we obtain a compact metric space. Thus, if Ω_k, u_k are uniformly bounded then we can always extract a convergent subsequence $u_{k_m} \to u$. Similarly, if Ω_k, φ_k are uniformly bounded we can extract a convergent subsequence $\varphi_{k_m} \to \varphi$.

Proposition 2.4. *Let $u_k : \Omega_k \to \mathbb{R}$ be convex and*

$$\det D^2 u_k = f_k, \quad u_k|_{\partial\Omega_k} = \varphi_k.$$

If

$$u_k \to u, \quad \varphi_k \to \varphi, \quad f_k \to f,$$

then

$$\det D^2 u = f, \quad u = \varphi^* \quad \text{on } \partial\Omega, \tag{2.1}$$

where φ^* is the convex envelope of φ on $\partial\Omega$ i.e Φ^* is the restriction to $\partial\Omega \times \mathbb{R}$ of the convex hull generated by Φ.

Remark: If Ω is strictly convex then $\varphi^* = \varphi$.

Proof. Since

$$\bar{U}_k \to \bar{U}, \quad \Phi_k \to \Phi, \quad \Phi_k \subset \bar{U}_k,$$

we see that $\Phi \subset \bar{U}$. Thus, if K denotes the convex hull generated by Φ, then $\Phi^* \subset K \subset \bar{U}$. It remains to show that $\bar{U} \cap (\partial\Omega \times \mathbb{R}) \subset K$.

Indeed consider a hyperplane

$$x_{n+1} = l(x)$$

which lies strictly below K. Then for all large k

$$\{u_k - l \le 0\} \subset \Omega_k$$

and by Alexandrov estimate we have that

$$u_k - l \ge -C d_k^{1/n}$$

where d_k represents the distance to $\partial\Omega_k$. By taking $k \to \infty$ we see that

$$u - l \geq -Cd^{1/n}$$

thus no point on $\partial\Omega \times \mathbb{R}$ below the hyperplane belongs to $\bar{U}$.

$\square$

Proposition 2.4 says that given any φ bounded and lower semicontinuous, and $f \geq 0$ bounded and continuous we can always solve uniquely the Dirichlet problem

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

by approximation. Indeed, we can find sequences φ_k, f_k of continuous, uniformly bounded functions defined on strictly convex domains Ω_k such that $\varphi_k \to \varphi$ and $f_k \to f$. Then the corresponding solutions u_k are uniformly bounded and continuous up to the boundary. Using compactness and the proposition above we see that u_k must converge to the unique solution u in (2.1).

In view of Proposition 2.4 we extend the Definition 2.1 in order to allow boundary data that is not necessarily convex.

Definition 2.5. Let $\varphi : \partial\Omega \to \mathbb{R}$ be a lower semicontinuous function. When we write that a convex function u satisfies

$$u = \varphi \quad \text{on } \partial\Omega$$

we understand

$$u|_{\partial\Omega} = \varphi^*$$

where φ^* is the convex envelope of φ on $\partial\Omega$.

Whenever φ^* and φ do not coincide we can think of the graph of u as having a vertical part on $\partial\Omega$ between φ^* and φ.

It follows easily from the definition above that the boundary values of u when we restrict to the domain

$$\Omega_h := \{u < h\}$$

are given by

$$\varphi_h = \varphi \quad \text{on} \quad \partial\Omega \cap \{\varphi \leq h\} \subset \partial\Omega_h$$

and $\varphi_h = h$ on the remaining part of $\partial\Omega_h$.

By Proposition 2.2, the comparison principle still holds. Precisely, if

$$u = \varphi, \quad v = \psi, \quad \varphi \leq \psi \qquad \text{on } \partial\Omega,$$
$$\det D^2 u \geq f \geq \det D^2 v \quad \text{in } \Omega,$$

then

$$u \leq v \quad \text{in } \Omega.$$

The advantage of introducing the notation of Definition 2.5 is that the boundary data is preserved under limits.

Proposition 2.6. *Assume*

$$\det D^2 u_k = f_k, \quad u_k = \varphi_k \quad \text{on } \partial\Omega_k,$$

with Ω_k, φ_k uniformly bounded and

$$\varphi_k \to \varphi, \quad f_k \to f.$$

Then

$$u_k \to u$$

and u satisfies

$$\det D^2 u = f, \quad u = \varphi \quad \text{on } \partial\Omega.$$

Proof. Using compactness we may assume also that $\varphi_k^* \to \psi$. Since $\varphi_k \to \varphi$ we find

$$\varphi \ge \psi \ge \varphi^*.$$

and the conclusion follows from Proposition 2.4.

□

Finally, we state a version of the last proposition for solutions with bounded right-hand side i.e

$$\lambda \le \det D^2 u \le \Lambda,$$

where the two inequalities are understood in the viscosity sense.

Theorem 2.7. *Assume*

$$\lambda \le \det D^2 u_k \le \Lambda, \quad u_k = \varphi_k \quad \text{on } \partial\Omega_k,$$

and Ω_k, φ_k uniformly bounded.

Then there exists a subsequence k_m such that

$$u_{k_m} \to u, \quad \varphi_{k_m} \to \varphi$$

with

$$\lambda \le \det D^2 u \le \Lambda, \quad u = \varphi \quad \text{on } \partial\Omega.$$

3. The Localization Theorem

In this section we state the quantitative version of the localization theorem at boundary points (Theorem 3.1).

Let Ω be a bounded convex set in $\mathbb{R}^n$. We assume that

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \ge 0\} \cap B_{\frac{1}{\rho}}, \tag{3.1}$$

for some small $\rho > 0$, that is $\Omega \subset (\mathbb{R}^n)^+$ and Ω contains an interior ball tangent to $\partial\Omega$ at 0.

Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying

$$\det D^2 u = f, \qquad 0 < \lambda \le f \le \Lambda \quad \text{in } \Omega. \tag{3.2}$$

We extend u to be ∞ outside $\overline{\Omega}$.

After subtracting a linear function we assume that

$$x_{n+1} = 0 \text{ is the tangent plane to } u \text{ at } 0, \tag{3.3}$$

in the sense that

$$u \ge 0, \quad u(0) = 0,$$

and any hyperplane $x_{n+1} = \varepsilon x_n$, $\varepsilon > 0$, is not a supporting plane for u.

We investigate the geometry of the sections of u at 0 that we denote for simplicity of notation

$$S_h := \{x \in \overline{\Omega} : \quad u(x) < h\}.$$

We show that if the boundary data has quadratic growth near $\{x_n = 0\}$ then, as $h \to 0$, S_h is equivalent to a half-ellipsoid centered at 0.

Precisely, our theorem reads as follows.

Theorem 3.1 (Localization Theorem)**.** *Assume that* Ω*,* u *satisfy* (3.1)-(3.3) *above and for some* $\mu > 0$*,*

$$\mu |x|^2 \le u(x) \le \mu^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \le \rho\}. \tag{3.4}$$

Then, for each $h < c(\rho)$ *there exists an ellipsoid* E_h *of volume* $h^{n/2}$ *such that*

$$kE_h \cap \overline{\Omega} \, \subset \, S_h \, \subset \, k^{-1}E_h \cap \overline{\Omega}.$$

Moreover, the ellipsoid E_h *is obtained from the ball of radius* $h^{1/2}$ *by a linear transformation* A_h^{-1} *(sliding along the* $x_n = 0$ *plane)*

$$A_h E_h = h^{1/2} B_1$$

$$A_h(x) = x - \nu x_n, \quad \nu = (\nu_1, \nu_2, \dots, \nu_{n-1}, 0),$$

with

$$|\nu| \le k^{-1} |\log h|.$$

The constant k *above depends on* μ, λ, Λ, n *and* $c(\rho)$ *depends also on* ρ*.*

The ellipsoid E_h, or equivalently the linear map A_h, provides information about the behavior of the second derivatives near the origin. Heuristically, the theorem states that in S_h the tangential second derivatives are bounded from above and below and the mixed second derivatives are bounded by $|\log h|$.

The hypothesis that u is continuous up to the boundary is not necessary, we just need to require that (3.4) holds in the sense of Definition 2.5.

Given only the boundary data φ of u on $\partial\Omega$, it is not always easy to check the main assumption (3.4) i.e that φ separates quadratically on $\partial\Omega$ (in a neighborhood of $\{x_n = 0\}$) from the tangent plane at 0. Proposition 3.2 provides some examples when this is satisfied depending on the local behavior of $\partial\Omega$ and φ (see also the remarks below).

Proposition 3.2. *Assume* (3.1),(3.2) *hold. Then* (3.4) *is satisfied if any of the following holds:*

1) φ *is linear in a neighborhood of* 0 *and* Ω *is uniformly convex at the origin.*

2) $\partial\Omega$ *is tangent of order 2 to* $\{x_n = 0\}$ *and* φ *has quadratic growth in a neighborhood of* $\{x_n = 0\}$*.*

3) φ*,* $\partial\Omega \in C^3(0)$*, and* Ω *is uniformly convex at the origin.*

Proposition 3.2 is standard (see [CNS], [W]). We sketch its proof below.

Proof. 1) Assume $\varphi = 0$ in a neighborhood of 0. By the use of standard barriers, the assumptions on Ω imply that the tangent plane at the origin is given by

$$x_{n+1} = -\mu x_n$$

for some bounded $\mu > 0$. Then (3.4) clearly holds.

2) After subtracting a linear function we may assume that

$$\mu |x'|^2 \le \varphi \le \mu^{-1} |x'|^2$$

on $\partial\Omega$ in a neighborhood of $\{x_n = 0\}$. Using a barrier we obtain that l_0, the tangent plane at the origin, has bounded slope. But $\partial\Omega$ is tangent of order 2 to $\{x_n = 0\}$, thus l_0 grows less than quadratic on $\partial\Omega$ in a neighborhood of $\{x_n = 0\}$ and (3.4) is again satisfied.

3) Since Ω is uniformly convex at the origin, we can use barriers and obtain that l_0 has bounded slope. After subtracting this linear function we may assume $l_0 = 0$. Since φ, $\partial\Omega \in C^3(0)$ we find that

$$\varphi = Q_0(x') + o(|x'|^3)$$

with Q_0 a cubic polynomial. Now $\varphi \geq 0$, hence Q_0 has no linear part and its quadratic part is given by, say

$$\sum_{i<n} \frac{\mu_i}{2} x_i^2, \quad \text{with} \quad \mu_i \geq 0.$$

We need to show that $\mu_i > 0$.

If $\mu_1 = 0$, then the coefficient of x_1^3 is 0 in Q_0. Thus, if we restrict to $\partial\Omega$ in a small neighborhood near the origin, then for all small h the set $\{\varphi < h\}$ contains

$$\{|x_1| \leq r(h)h^{1/3}\} \cap \{|x'| \leq ch^{1/2}\}$$

for some $c > 0$ and with

$$r(h) \to \infty \quad \text{as } h \to 0.$$

Now S_h contains the convex set generated by $\{\varphi < h\}$ thus, since Ω is uniformly convex,

$$|S_h| \geq c'(r(h)h^{1/3})^3 h^{(n-2)/2} \geq c' r(h)^3 h^{n/2}.$$

On the other hand, since u satisfies (3.2) and

$$0 \leq u \leq h \quad \text{in } S_h$$

we obtain (see (4.4))

$$|S_h| \leq Ch^{n/2},$$

for some C depending on λ and n, and we contradict the inequality above as $h \to 0$. $\square$

Remark 3.3. The proof easily implies that if $\partial\Omega$, $\varphi \in C^3(\Omega)$ and Ω is uniformly convex, then we can find a constant μ which satisfies (3.4) for all $x \in \partial\Omega$.

Remark 3.4. From above we see that we can often verify (3.4) in the case when φ, $\partial\Omega \in C^{1,1}(0)$ and Ω is uniformly convex at 0. Indeed, if l_φ represents the tangent plane at 0 to $\varphi : \partial\Omega \to \mathbb{R}$ (in the sense of (3.3)), then (3.4) holds if either φ separates from l_φ quadratically near 0, or if φ is tangent to l_φ of order 3 in some tangential direction.

Remark 3.5. Given φ, $\partial\Omega \in C^{1,1}(0)$ and Ω uniformly convex at 0, then (3.4) holds if λ is sufficiently large.

4. Proof of Theorem 3.1 (I)

We prove Theorem 3.1 in the next two sections. In this section we obtain some preliminary estimates and reduce the theorem to a statement about the rescalings of u. This statement is proved in section 5 using compactness.

Next proposition was proved by Trudinger and Wang in [TW]. It states that the volume of S_h is proportional to $h^{n/2}$ and after an affine transformation (of controlled norm) we may assume that the center of mass of S_h lies on the x_n axis. Since our setting is slightly different we provide its proof.

Proposition 4.1. *Under the assumptions of Theorem 3.1, for all $h \le c(\rho)$, there exists a linear transformation (sliding along $x_n = 0$)*

$$A_h(x) = x - \nu x_n,$$

with

$$\nu_n = 0, \quad |\nu| \le C(\rho) h^{-\frac{n}{2(n+1)}}$$

such that the rescaled function

$$\tilde{u}(A_h x) = u(x),$$

satisfies in

$$\tilde{S}_h := A_h S_h = \{\tilde{u} < h\}$$

the following:

(i) *the center of mass of $\tilde{S}_h$ lies on the x_n-axis;*

(ii)

$$k_0 h^{n/2} \le |\tilde{S}_h| = |S_h| \le k_0^{-1} h^{n/2};$$

(iii) *the part of $\partial \tilde{S}_h$ where $\{\tilde{u} < h\}$ is a graph, denoted by*

$$\tilde{G}_h = \partial \tilde{S}_h \cap \{\tilde{u} < h\} = \{(x', g_h(x'))\}$$

that satisfies

$$g_h \le C(\rho)|x'|^2$$

and

$$\frac{\mu}{2}|x'|^2 \le \tilde{u} \le 2\mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h.$$

The constant k_0 above depends on μ, λ, Λ, n and the constants $C(\rho), c(\rho)$ depend also on ρ.

In this section we denote by c, C positive constants that depend on n, μ, λ, Λ. For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion. Constants that depend also on ρ are denote by $c(\rho)$, $C(\rho)$.

Proof. The function

$$v := \mu|x'|^2 + \frac{\Lambda}{\mu^{n-1}} x_n^2 - C(\rho) x_n$$

is a lower barrier for u in $\Omega \cap \{x_n \le \rho\}$ if $C(\rho)$ is chosen large.

Indeed, then

$$v \le u \quad \text{on } \partial\Omega \cap \{x_n \le \rho\},$$

$$v \le 0 \le u \quad \text{on } \Omega \cap \{x_n = \rho\},$$

and

$$\det D^2 v > \Lambda.$$

In conclusion,

$$v \le u \quad \text{in } \Omega \cap \{x_n \le \rho\},$$

hence

$$S_h \cap \{x_n \le \rho\} \subset \{v < h\} \subset \{x_n > c(\rho)(\mu|x'|^2 - h)\}. \tag{4.1}$$

Let x_h^* be the center of mass of S_h. We claim that

$$x_h^* \cdot e_n \geq c_0(\rho) h^\alpha, \quad \alpha = \frac{n}{n+1}, \tag{4.2}$$

for some small $c_0(\rho) > 0$.

Otherwise, from (4.1) and John's lemma we obtain

$$S_h \subset \{x_n \leq C(n) c_0 h^\alpha \leq h^\alpha\} \cap \{|x'| \leq C_1 h^{\alpha/2}\},$$

for some large $C_1 = C_1(\rho)$. Then the function

$$w = \varepsilon x_n + \frac{h}{2}\left(\frac{|x'|}{C_1 h^{\alpha/2}}\right)^2 + \Lambda C_1^{2(n-1)} h \left(\frac{x_n}{h^\alpha}\right)^2$$

is a lower barrier for u in S_h if c_0 is sufficiently small.

Indeed,

$$w \leq \frac{h}{4} + \frac{h}{2} + \Lambda C_1^{2(n-1)} (C(n) c_0)^2 h < h \quad \text{in } S_h,$$

and for all small h,

$$w \leq \varepsilon x_n + \frac{h^{1-\alpha}}{C_1^2} |x'|^2 + C(\rho) h c_0 \frac{x_n}{h^\alpha} \leq \mu |x'|^2 \leq u \quad \text{on } \partial\Omega,$$

and

$$\det D^2 w = 2\Lambda.$$

Hence

$$w \leq u \quad \text{in } S_h,$$

and we contradict that 0 is the tangent plane at 0. Thus claim (4.2) is proved.

Now, define

$$A_h x = x - \nu x_n, \quad \nu = \frac{x_h^{*'}}{x_h^* \cdot e_n},$$

and

$$\tilde{u}(A_h x) = u(x).$$

The center of mass of $\tilde{S}_h = A_h S_h$ is

$$\tilde{x}_h^* = A_h x_h^*$$

and lies on the x_n-axis from the definition of A_h. Moreover, since $x_h^* \in S_h$, we see from (4.1)-(4.2) that

$$|\nu| \leq C(\rho) \frac{(x_h^* \cdot e_n)^{1/2}}{(x_h^* \cdot e_n)} \leq C(\rho) h^{-\alpha/2},$$

and this proves (i).

If we restrict the map A_h on the set on $\partial\Omega$ where $\{u < h\}$, i.e. on

$$\partial S_h \cap \partial\Omega \subset \{x_n \leq \frac{|x'|^2}{\rho}\} \cap \{|x'| < C h^{1/2}\}$$

we have

$$|A_h x - x| = |\nu| x_n \leq C(\rho) h^{-\alpha/2} |x'|^2 \leq C(\rho) h^{\frac{1-\alpha}{2}} |x'|,$$

and part (iii) easily follows.

Next we prove (ii). From John's lemma, we know that after relabeling the x' coordinates if necessary,

$$D_h B_1 \subset \tilde{S}_h - \tilde{x}_h^* \subset C(n) D_h B_1 \tag{4.3}$$

where

$$D_h = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

Since

$$\tilde{u} \le 2\mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h = \{(x', g_h(x'))\},$$

we see that the domain of definition of g_h contains a ball of radius $(\mu h/2)^{1/2}$. This implies that

$$d_i \ge c_1 h^{1/2}, \qquad i = 1, \cdots, n-1,$$

for some c_1 depending only on n and μ. Also from (4.2) we see that

$$\tilde{x}_h^* \cdot e_n = x_h^* \cdot e_n \ge c_0(\rho) h^\alpha$$

which gives

$$d_n \ge c(n) \tilde{x}_h^* \cdot e_n \ge c(\rho) h^\alpha.$$

We claim that for all small h,

$$\prod_{i=1}^n d_i \ge k_0 h^{n/2},$$

with k_0 small depending only on μ, n, Λ, which gives the left inequality in (ii).

To this aim we consider the barrier,

$$w = \varepsilon x_n + \sum_{i=1}^n ch \left(\frac{x_i}{d_i}\right)^2.$$

We choose c sufficiently small depending on μ, n, Λ so that for all $h < c(\rho)$,

$$w \le h \quad \text{on } \partial \tilde{S}_h,$$

and on the part of the boundary $\tilde{G}_h$, we have $w \le \tilde{u}$ since

$$\begin{aligned} w &\le \varepsilon x_n + \frac{c}{c_1^2}|x'|^2 + ch\left(\frac{x_n}{d_n}\right)^2 \\ &\le \frac{\mu}{4}|x'|^2 + chC(n)\frac{x_n}{d_n} \\ &\le \frac{\mu}{4}|x'|^2 + ch^{1-\alpha}C(\rho)|x'|^2 \\ &\le \frac{\mu}{2}|x'|^2. \end{aligned}$$

Moreover, if our claim does not hold, then

$$\det D^2 w = (2ch)^n (\prod d_i)^{-2n} > \Lambda,$$

thus $w \le \tilde{u}$ in $\tilde{S}_h$. By definition, $\tilde{u}$ is obtained from u by a sliding along $x_n = 0$, hence 0 is still the tangent plane of $\tilde{u}$ at 0. We reach again a contradiction since $\tilde{u} \ge w \ge \varepsilon x_n$ and the claim is proved.

Finally we show that

$$|\tilde{S}_h| \le Ch^{n/2} \tag{4.4}$$

for some C depending only on λ, n. Indeed, if

$$v = h \quad \text{on } \partial \tilde{S}_h,$$

and

$$\det D^2 v = \lambda$$

then

$$v \geq u \geq 0 \quad \text{in } \tilde{S}_h.$$

Since

$$h \geq h - \min_{\tilde{S}_h} v \geq c(n, \lambda)|\tilde{S}_h|^{2/n}$$

we obtain the desired conclusion.

□

In the proof above we showed that for all $h \leq c(\rho)$, the entries of the diagonal matrix D_h from (4.3) satisfy

$$d_i \geq c h^{1/2}, \quad i = 1, \dots n-1$$

$$d_n \geq c(\rho) h^{\alpha}, \quad \alpha = \frac{n}{n+1}$$

$$c h^{n/2} \leq \prod d_i \leq C h^{n/2}.$$

The main step in the proof of Theorem 3.1 is the following lemma that will be completed in Section 5.

Lemma 4.2. *There exist constants c, $c(\rho)$ such that*

$$d_n \geq c h^{1/2}, \tag{4.5}$$

for all $h \leq c(\rho)$.

Using Lemma 4.2 we can easily finish the proof of our theorem.

Proof of Theorem 3.1. Since all d_i are bounded below by $ch^{1/2}$ and their product is bounded above by $Ch^{n/2}$ we see that

$$C h^{1/2} \geq d_i \geq c h^{1/2} \qquad i = 1, \cdots, n$$

for all $h \leq c(\rho)$. Using (4.3) we obtain

$$\tilde{S}_h \subset C h^{1/2} B_1.$$

Moreover, since

$$\tilde{x}_h^* \cdot e_n \geq d_n \geq c h^{1/2}, \qquad (\tilde{x}_h^*)' = 0,$$

and the part $\tilde{G}_h$ of the boundary $\partial \tilde{S}_h$ contains the graph of $\tilde{g}_h$ above $|x'| \leq ch^{1/2}$, we find that

$$c h^{1/2} B_1 \cap \tilde{\Omega} \subset \tilde{S}_h,$$

with $\tilde{\Omega} = A_h \Omega$, $\tilde{S}_h = A_h S_h$. In conclusion

$$c h^{1/2} B_1 \cap \tilde{\Omega} \subset A_h S_h \subset C h^{1/2} B_1.$$

We define the ellipsoid E_h as

$$E_h := A_h^{-1}(h^{1/2} B_1),$$

hence

$$c E_h \cap \overline{\Omega} \subset S_h \subset C E_h.$$

Comparing the sections at levels h and $h/2$ we find

$$cE_{h/2} \cap \overline{\Omega} \subset CE_h$$

and we easily obtain the inclusion

$$A_h A_{h/2}^{-1} B_1 \subset CB_1.$$

If we denote

$$A_h x = x - \nu_h x_n$$

then the inclusion above implies

$$|\nu_h - \nu_{h/2}| \leq C,$$

which gives the desired bound

$$|\nu_h| \leq C|\log h|$$

for all small h.

□

In order to prove Lemma 4.2 we introduce a new quantity $b(h)$ which is proportional to $d_n h^{-1/2}$ and is appropriate when dealing with affine transformations.

Notation. Given a convex function u we define

$$b_u(h) = h^{-1/2} \sup_{S_h} x_n.$$

Whenever there is no possibility of confusion we drop the subindex u and use the notation $b(h)$.

Below we list some basic properties of $b(h)$.

1) If $h_1 \leq h_2$ then

$$\left(\frac{h_1}{h_2}\right)^{\frac{1}{2}} \leq \frac{b(h_1)}{b(h_2)} \leq \left(\frac{h_2}{h_1}\right)^{\frac{1}{2}}.$$

2) A rescaling

$$\tilde{u}(Ax) = u(x)$$

given by a linear transformation A which leaves the x_n coordinate invariant does not change the value of b, i.e

$$b_{\tilde{u}}(h) = b_u(h).$$

3) If A is a linear transformation which leaves the plane $\{x_n = 0\}$ invariant the values of b get multiplied by a constant. However the quotients $b(h_1)/b(h_2)$ do not change values i.e

$$\frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$

4) If we multiply u by a constant, i.e.

$$\tilde{u}(x) = \beta u(x)$$

then

$$b_{\tilde{u}}(\beta h) = \beta^{-1/2} b_u(h),$$

and

$$\frac{b_{\tilde{u}}(\beta h_1)}{b_{\tilde{u}}(\beta h_2)} = \frac{b_u(h_1)}{b_u(h_2)}.$$

From (4.3) and property 2 above,

$$c(n)d_n \leq b(h)h^{1/2} \leq C(n)d_n,$$

hence Lemma 4.2 will follow if we show that $b(h)$ is bounded below. We achieve this by proving the following lemma.

Lemma 4.3. *There exist c_0, $c(\rho)$ such that if $h \leq c(\rho)$ and $b(h) \leq c_0$ then*

$$\frac{b(th)}{b(h)} > 2, \tag{4.6}$$

for some $t \in [c_0, 1]$.

This lemma states that if the value of $b(h)$ on a certain section is less than a critical value c_0, then we can find a lower section at height still comparable to h where the value of b doubled. Clearly Lemma 4.3 and property 1 above imply that $b(h)$ remains bounded for all h small enough.

The quotient in (4.6) is the same for $\tilde{u}$ which is defined in Proposition 4.1. We normalize the domain $\tilde{S}_h$ and $\tilde{u}$ by considering the rescaling

$$v(x) = \frac{1}{h}\tilde{u}(h^{1/2}Ax)$$

where A is a multiple of D_h (see (4.3)), $A = \gamma D_h$ such that

$$\det A = 1.$$

Then

$$ch^{-1/2} \leq \gamma \leq Ch^{-1/2},$$

and the diagonal entries of A satisfy

$$a_i \geq c, \qquad i = 1, 2, \cdots, n-1,$$

$$cb_u(h) \leq a_n \leq Cb_u(h).$$

The function v satisfies

$$\lambda \leq \det D^2 v \leq \Lambda,$$

$$v \geq 0, \quad v(0) = 0,$$

is continuous and it is defined in $\bar{\Omega}_v$ with

$$\Omega_v := \{v < 1\} = h^{-1/2}A^{-1}\tilde{S}_h.$$

Then

$$x^* + cB_1 \subset \Omega_v \subset CB_1^+,$$

for some x^*, and

$$ct^{n/2} \leq |S_t(v)| \leq Ct^{n/2}, \quad \forall t \leq 1,$$

where $S_t(v)$ denotes the section of v. Since

$$\tilde{u} = h \quad \text{in} \quad \partial\tilde{S}_h \cap \{x_n \geq C(\rho)h\},$$

then

$$v = 1 \quad \text{on } \partial\Omega_v \cap \{x_n \geq \sigma\}, \quad \sigma := C(\rho)h^{1-\alpha}.$$

Also, from Proposition 4.1 on the part G of the boundary of $\partial\Omega_v$ where $\{v < 1\}$ we have

$$\frac{1}{2}\mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq v \leq 2\mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2. \tag{4.7}$$

In order to prove Lemma 4.3 we need to show that if σ, a_n are sufficiently small depending on n, μ, λ, Λ then the function v above satisfies

$$b_v(t) \geq 2b_v(1) \tag{4.8}$$

for some $1 > t \geq c_0$.

Since $\alpha < 1$, the smallness condition on σ is satisfied by taking $h < c(\rho)$ sufficiently small. Also a_n being small is equivalent to one of the a_i, $1 \leq i \leq n-1$ being large since their product is 1 and a_i are bounded below.

In the next section we prove property (4.8) above by compactness, by letting $\sigma \to 0$, $a_i \to \infty$ for some i (see Proposition 5.1).

5. Proof of Theorem 3.1 (II)

In this section we consider the class of solutions v that satisfy the properties above. After relabeling the constants μ and a_i, and by abuse of notation writing u instead of v, we may assume we are in the following situation.

Fix μ small and λ, Λ. For an increasing sequence

$$a_1 \leq a_2 \leq \ldots \leq a_{n-1}$$

with

$$a_1 \geq \mu,$$

we consider the family of solutions

$$u \in \mathcal{D}_\sigma^\mu(a_1, a_2, \ldots, a_{n-1})$$

of convex functions $u : \Omega \to \mathbb{R}$ that satisfy

$$\lambda \leq \det D^2 u \leq \Lambda \quad \text{in } \Omega, \quad 0 \leq u \leq 1 \text{ in } \Omega; \tag{5.1}$$

$$0 \in \partial\Omega, \quad B_\mu(x_0) \subset \Omega \subset B_{1/\mu}^+ \quad \text{for some } x_0; \tag{5.2}$$

$$\mu h^{n/2} \leq |S_h| \leq \mu^{-1} h^{n/2}. \tag{5.3}$$

Moreover we assume that the boundary $\partial\Omega$ has a closed subset G

$$G \subset \{x_n \leq \sigma\} \cap \partial\Omega \tag{5.4}$$

which is a graph in the e_n direction with projection $\pi_n(G) \subset \mathbb{R}^{n-1}$ along e_n

$$\{\, \mu^{-1} \sum_1^{n-1} a_i^2 x_i^2 \leq 1 \,\} \subset \pi_n(G) \subset \{\, \mu \sum_1^{n-1} a_i^2 x_i^2 \leq 1 \,\}, \tag{5.5}$$

and (see Definition 2.5), the boundary values of $u = \varphi$ on $\partial\Omega$ satisfy

$$\varphi = 1 \quad \text{on } \partial\Omega \setminus G; \tag{5.6}$$

and

$$\mu \sum_{1}^{n-1} a_i^2 x_i^2 \le \varphi \le \min\{\, 1, \quad \mu^{-1} \sum_{1}^{n-1} a_i^2 x_i^2 \,\} \qquad \text{on } G. \tag{5.7}$$

In this section we prove

Proposition 5.1. *For any $M > 0$ there exists C_* depending on $M, \mu, \lambda, \Lambda, n$ such that if $u \in \mathcal{D}_\sigma^\mu(a_1, a_2, \dots, a_{n-1})$ with*

$$a_{n-1} \ge C_*, \quad \sigma \le C_*^{-1}$$

then

$$b(h) = (\sup_{S_h} x_n) h^{-1/2} \ge M$$

for some h with $C_*^{-1} \le h \le 1$.

Property (4.8) (hence Theorem 3.1), easily follows from this proposition. Indeed, by choosing

$$M = 2\mu^{-1} \ge 2b(1)$$

in Proposition 5.1 we prove the existence of a section S_h with $h \ge c_0$ such that

$$b(h) \ge 2b(1).$$

Clearly the function v of the previous section satisfies the hypotheses above (after renaming the constant μ) provided that σ, a_n are sufficiently small.

We prove Proposition 5.1 by compactness. We introduce the limiting solutions from the class $\mathcal{D}_\sigma^\mu(a_1, \dots, a_{n-1})$ when $a_{k+1} \to \infty$ and $\sigma \to 0$.

If $\mu \le a_1 \le \dots \le a_k$, we denote by

$$\mathcal{D}_0^\mu(a_1, \dots, a_k, \infty, \infty, \dots, \infty), \quad 0 \le k \le n-2$$

the class of functions u that satisfy properties (5.1)-(5.2)-(5.3) with,

$$G \subset \{x_i = 0, \quad i > k\} \cap \partial\Omega \tag{5.8}$$

and if we restrict to the space generated by the first k coordinates then

$$\{\, \mu^{-1} \sum_{1}^{k} a_i^2 x_i^2 \le 1 \,\} \subset G \subset \{\, \mu \sum_{1}^{k} a_i^2 x_i^2 \le 1 \,\}. \tag{5.9}$$

Also, $u = \varphi$ on $\partial\Omega$ with

$$\varphi = 1 \quad \text{on } \partial\Omega \setminus G; \tag{5.10}$$

$$\mu \sum_{1}^{k} a_i^2 x_i^2 \le \varphi \le \min\{\, 1, \quad \mu^{-1} \sum_{1}^{k} a_i^2 x_i^2 \,\} \qquad \text{on } G. \tag{5.11}$$

The compactness theorem (Theorem 2.7) implies that if

$$u_m \in D_{\sigma_m}^\mu(a_1^m, \dots, a_{n-1}^m)$$

is a sequence with

$$\sigma_m \to 0 \quad \text{ and } \quad a_{k+1}^m \to \infty$$

for some fixed $0 \le k \le n-2$, then we can extract a convergent subsequence to a function u (see Definition 2.3) with

$$u \in D_0^\mu(a_1, .., a_l, \infty, .., \infty),$$

for some $l \le k$ and $a_1 \le \dots \le a_l$.

Proposition 5.1 follows easily from the next proposition.

Proposition 5.2. *For any $M > 0$ and $0 \le k \le n-2$ there exists c_k depending on $M, \mu, \lambda, \Lambda, n, k$ such that if*

$$u \in \mathcal{D}_0^\mu(a_1, \dots, a_k, \infty, \dots, \infty) \tag{5.12}$$

then

$$b(h) = (\sup_{S_h} x_n) h^{-1/2} \ge M$$

for some h with $c_k \le h \le 1$.

Indeed, if Proposition 5.1 fails for a sequence of constants $C_* \to \infty$ then we obtain a limiting solution u as in (5.12) for which $b(h) \le M$ for all $h > 0$. This contradicts Proposition 5.2 (with M replaced by $2M$).

We prove Proposition 5.2 by induction on k. We start by introducing some notation.

Denote

$$x = (y, z, x_n), \quad y = (x_1, \dots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \dots, x_{n-1}) \in \mathbb{R}^{n-1-k}.$$

Definition 5.3. We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a *sliding along the y direction* if

$$Tx := x + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n$$

with

$$\nu_1, \nu_2, \dots, \nu_{n-k} \in span\{e_1, \dots, e_k\}$$

We see that T leaves the (z, x_n) components invariant together with the subspace $(y, 0, 0)$. Clearly, if T is a sliding along the y direction then so is T^{-1} and

$$\det T = 1.$$

The key step in the proof of Proposition 5.2 is the following lemma.

Lemma 5.4. *Assume that*

$$u \ge p(|z| - q x_n),$$

for some $p, q > 0$ and assume that for each section S_h of u, $h \in (0,1)$, there exists T_h a sliding along the y direction such that

$$T_h S_h \subset C_0 h^{1/2} B_1^+,$$

for some constant C_0. Then

$$u \notin D_0^\mu(1, \dots, 1, \infty, \dots, \infty).$$

Proof. Assume by contradiction that $u \in D_0^\mu$ and it satisfies the hypotheses with $q \le q_0$ for some q_0. We show that

$$u \ge p'(|z| - q' x_n), \qquad q' = q - \eta, \tag{5.13}$$

for some $0 < p' \ll p$, where the constant $\eta > 0$ depends only on q_0 and μ, C_0, Λ, n.

Then, since $q' \le q_0$, we can apply this result a finite number of times and obtain

$$u \ge \varepsilon(|z| + x_n),$$

for some small $\varepsilon > 0$. This gives $S_h \subset \{x_n \le \varepsilon^{-1} h\}$ hence

$$T_h S_h \subset \{x_n \le \varepsilon^{-1} h\}$$

and by the hypothesis above

$$|S_h| = |T_h S_h| = O(h^{(n+1)/2}) \quad \text{as } h \to 0,$$

and we contradict (5.3).

Now we prove (5.13). Since $u \in D_0^\mu$ as above, there exists a closed set

$$G_h \subset \partial S_h \cap \{z = 0, x_n = 0\}$$

such that on the subspace $(y, 0, 0)$

$$\{\mu^{-1}|y|^2 \le h\} \subset G_h \subset \{\mu|y|^2 \le h\},$$

and the boundary values φ_h of u on ∂S_h satisfy (see Section 2)

$$\varphi_h = h \quad \text{on } \partial S_h \setminus G_h;$$

$$\mu|y|^2 \le \varphi_h \le \min\left\{h, \mu^{-1}|y|^2\right\} \qquad \text{on } G_h.$$

Let w be a rescaling of u,

$$w(x) := \frac{1}{h} u(h^{1/2} T_h^{-1} x)$$

for some small $h \ll p$. Then

$$S_1(w) := \Omega_w = h^{-1/2} T_h S_h \subset B_{C_0}^+$$

and our hypothesis becomes

$$w \ge \frac{p}{h^{1/2}}(|z| - q x_n). \tag{5.14}$$

Moreover the boundary values φ_w of w on $\partial\Omega_w$ satisfy

$$\varphi_w = 1 \quad \text{on } \partial\Omega_w \setminus G_w$$

$$\mu|y|^2 \le \varphi_w \le \min\{1, \mu^{-1}|y|^2\} \quad \text{on} \quad G_w := h^{-1/2} G_h.$$

Next we show that $\varphi_w \ge v$ on $\partial\Omega_w$ where v is defined as

$$v := \delta|x|^2 + \frac{\Lambda}{\delta^{n-1}}(z_1 - q x_n)^2 + N(z_1 - q x_n) + \delta x_n,$$

and δ is small depending on μ and C_0, and N is chosen large such that

$$\frac{\Lambda}{\delta^{n-1}} t^2 + N t$$

is increasing in the interval $|t| \le (1 + q_0)C_0$.

From the definition of v we see that

$$\det D^2 v > \Lambda.$$

On the part of the boundary $\partial\Omega_w$ where $z_1 \le q x_n$ we use that $\Omega_w \subset B_{C_0}$ and obtain

$$v \le \delta(|x|^2 + x_n) \le \varphi_w.$$

On the part of the boundary $\partial\Omega_w$ where $z_1 > q x_n$ we use (5.14) and obtain

$$1 = \varphi_w \ge C(|z| - q x_n) \ge C(z_1 - q x_n)$$

with C arbitrarily large provided that h is small enough. We choose C such that the inequality above implies

$$\frac{\Lambda}{\delta^{n-1}}(z_1 - q x_n)^2 + N(z_1 - q x_n) < \frac{1}{2}.$$

Then

$$\varphi_w = 1 > \frac{1}{2} + \delta(|x|^2 + x_n) \geq v.$$

In conclusion $\varphi_w \geq v$ on $\partial\Omega_w$ hence the function v is a lower barrier for w in Ω_w. Then

$$w \geq N(z_1 - qx_n) + \delta x_n$$

and, since this inequality holds for all directions in the z-plane, we obtain

$$w \geq N(|z| - (q - \eta)x_n), \qquad \eta := \frac{\delta}{N}.$$

Scaling back we get

$$u \geq p'(|z| - (q - \eta)x_n) \qquad \text{in } S_h.$$

Since u is convex and $u(0) = 0$, this inequality holds globally, and (5.13) is proved.

□

Lemma 5.5. *Proposition 5.2 holds for $k = 0$.*

Proof. By compactness we need to show that there does not exist $u \in \mathcal{D}_0^\mu(\infty, \dots, \infty)$ with $b(h) \leq M$ for all h. If such u exists then $G = \{0\}$. Let

$$v := \delta(|x'| + \frac{1}{2}|x'|^2) + \frac{\Lambda}{\delta^{n-1}}x_n^2 - Nx_n$$

with δ small depending on μ, and N large so that

$$\frac{\Lambda}{\delta^{n-1}}x_n^2 - Nx_n \leq 0$$

in $B_{1/\mu}^+$. Then

$$v \leq \varphi \quad \text{on } \partial\Omega, \quad \det D^2 v > \Lambda,$$

hence

$$v \leq u \quad \text{in } \Omega.$$

This gives

$$u \geq \delta|x'| - Nx_n,$$

and we obtain

$$S_h \subset \{|x'| \leq C(x_n + h)\}.$$

Since $b(h) \leq M$ we conclude

$$S_h \subset Ch^{1/2}B_1^+,$$

and we contradict Lemma 5.4 for $k = 0$.

□

Now we prove Proposition 5.2 by induction on k.

Proof of Proposition 5.2. In this proof we denote by c, C positive constants that depend on $M, \mu, \lambda, \Lambda, n$ and k.

We assume that the proposition holds for all nonnegative integers up to $k-1$, $1 \leq k < n-2$, and we prove it for k. Let

$$u \in D_0^\mu(a_1, \dots, a_k, \infty, \dots, \infty).$$

By the induction hypotheses and compactness we see that there exists a constant

$$C_k(\mu, M, \lambda, \Lambda, n)$$

such that if $a_k \geq C_k$ then $b(h) \geq M$ for some $h \geq C_k^{-1}$. Thus, it suffices to consider only the case when $a_k < C_k$.

If no c_{k+1} exists then we can find a limiting solution that, by abuse of notation, we still denote by u such that

$$u \in \mathcal{D}_0^{\tilde{\mu}}(1, 1, \dots, 1, \infty, \dots, \infty) \tag{5.15}$$

with

$$b(h) \leq M h^{1/2}, \quad \forall h > 0 \tag{5.16}$$

where $\tilde{\mu}$ depends on μ and C_k.

We show that such a function u does not exist.

Denote as before

$$x = (y, z, x_n), \quad y = (x_1, \dots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \dots, x_{n-1}) \in \mathbb{R}^{n-1-k}.$$

On $\partial\Omega$ we have

$$\varphi(x) \geq \delta|x'|^2 + \delta|z| + \frac{\Lambda}{\delta^{n-1}} x_n^2 - N x_n$$

where δ is small depending on $\tilde{\mu}$, and N is large so that

$$\frac{\Lambda}{\delta^{n-1}} x_n^2 - N x_n \leq 0$$

in $B^+_{1/\tilde{\mu}}$. As before we obtain that the inequality above holds in Ω, hence

$$u(x) \geq \delta|z| - N x_n. \tag{5.17}$$

From (5.16)-(5.17) we see that the section S_h of u satisfies

$$S_h \subset \{|z| < \delta^{-1}(N x_n + h)\} \cap \{x_n \leq M h^{1/2}\}. \tag{5.18}$$

From John's lemma we know that S_h is equivalent to an ellipsoid E_h of the same volume i.e

$$c(n) E_h \subset S_h - x_h^* \subset C(n) E_h, \quad |E_h| = |S_h|, \tag{5.19}$$

with x_h^* the center of mass of S_h.

For any ellipsoid E_h in $\mathbb{R}^n$ of positive volume we can find T_h, a sliding along the y direction (see Definition 5.3), such that

$$T_h E_h = |E_h|^{1/n} A B_1, \tag{5.20}$$

with a matrix A that leaves the $(y, 0, 0)$ and $(0, z, x_n)$ subspaces invariant, and $\det A = 1$. By choosing an appropriate system of coordinates in the y and z variables we may assume in fact that

$$A(y, z, x_n) = (A_1 y, A_2(z, x_n))$$

with

$$A_1 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix}$$

with $0 < \beta_1 \leq \cdots \leq \beta_k$, and

$$A_2 = \begin{pmatrix} \gamma_{k+1} & 0 & \cdots & 0 & \theta_{k+1} \\ 0 & \gamma_{k+2} & \cdots & 0 & \theta_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_{n-1} & \theta_{n-1} \\ 0 & 0 & \cdots & 0 & \theta_n \end{pmatrix}$$

with γ_j, $\theta_n > 0$.

The h section $\tilde{S}_h = T_h S_h$ of the rescaling

$$\tilde{u}(x) = u(T_h^{-1}x)$$

satisfies (5.18) and since $u \in \mathcal{D}_0^\mu$, there exists $\tilde{G}_h = G_h$,

$$\tilde{G}_h \subset \{z = 0, x_n = 0\} \cap \partial \tilde{S}_h$$

such that on the subspace $(y, 0, 0)$

$$\{\mu^{-1}|y|^2 \le h\} \subset \tilde{G}_h \subset \{\mu |y|^2 \le h\},$$

and the boundary values $\tilde{\varphi}_h$ of $\tilde{u}$ on $\partial \tilde{S}_h$ satisfy

$$\tilde{\varphi}_h = h \quad \text{on } \partial \tilde{S}_h \setminus \tilde{G}_h;$$

$$\mu |y|^2 \le \tilde{\varphi}_h \le \min\left\{h, \mu^{-1}|y|^2\right\} \qquad \text{on } \tilde{G}_h.$$

Moreover, using that

$$|S_h| \sim h^{n/2}$$

in (5.19), (5.20) and that $0 \in \partial S_h$, we obtain

$$\tilde{x}_h^* + ch^{1/2}AB_1 \subset \tilde{S}_h \subset Ch^{1/2}AB_1, \quad \det A = 1, \tag{5.21}$$

for the matrix A as above and with $\tilde{x}_h^*$ the center of mass of $\tilde{S}_h$.

Next we use the induction hypothesis and show that $\tilde{S}_h$ is equivalent to a ball.

Lemma 5.6. *There exists C_0 such that*

$$T_h S_h = \tilde{S}_h \subset C_0 h^{n/2} B_1^+.$$

Proof. We need to show that

$$|A| \le C.$$

Since $\tilde{S}_h$ satisfies (5.18) we see that

$$\tilde{S}_h \subset \{|(z, x_n)| \le Ch^{1/2}\},$$

which together with the inclusion (5.21) gives $|A_2| \le C$ hence

$$\gamma_j, \theta_n \le C, \quad |\theta_j| \le C.$$

Also, since

$$\tilde{G}_h \subset \tilde{S}_h,$$

we find from (5.21)

$$\beta_i \ge c > 0, \quad i = 1, \cdots, k.$$

We define the rescaling

$$w(x) = \frac{1}{h}\tilde{u}(h^{1/2}Ax)$$

defined in a domain $\Omega_w = S_1(w)$. Then (5.21) gives

$$B_c(x_0) \subset \Omega_w \subset B_C^+,$$

and $w = \varphi_w$ on $\partial\Omega_w$ with

$$\varphi_w = 1 \quad \text{on } \partial\Omega_w \setminus G_w,$$

$$\tilde{\mu} \sum_1^k \beta_i^2 x_i^2 \le \varphi_w \le \min\{1, \tilde{\mu}^{-1} \sum_1^k \beta_i^2 x_i^2\} \quad \text{on } G_w := h^{-1/2} A^{-1} \tilde{G}_h.$$

This implies that

$$w \in \mathcal{D}_0^{\bar{\mu}}(\beta_1, \beta_2, \ldots, \beta_k, \infty, \ldots, \infty)$$

for some small $\bar{\mu}$ depending on $\mu, M, \lambda, \Lambda, n, k$.

We claim that

$$b_u(h) \ge c_\star.$$

First we notice that

$$b_u(h) = b_{\tilde{u}}(h) \sim \theta_n.$$

Since

$$\theta_n \prod \beta_i \prod \gamma_j = \det A = 1$$

and

$$\gamma_j \le C,$$

we see that if $b_u(h)$ (and therefore θ_n) becomes smaller than a critical value c_* then

$$\beta_k \ge C_k(\bar{\mu}, \bar{M}, \lambda, \Lambda, n),$$

with $\bar{M} := 2\bar{\mu}^{-1}$, and by the induction hypothesis

$$b_w(\tilde{h}) \ge \bar{M} \ge 2 b_w(1)$$

for some $\tilde{h} > C_k^{-1}$. This gives

$$\frac{b_u(h\tilde{h})}{b_u(h)} = \frac{b_w(\tilde{h})}{b_w(1)} \ge 2,$$

which implies $b_u(h\tilde{h}) \ge 2 b_u(h)$ and our claim follows.

Next we claim that γ_j are bounded below by the same argument. Indeed, from the claim above θ_n is bounded below and if some γ_j is smaller than a small value $\tilde{c}_*$ then

$$\beta_k \ge C_k(\bar{\mu}, \bar{M}_1, \lambda, \Lambda, n)$$

with

$$\bar{M}_1 := \frac{2M}{\bar{\mu} c_\star}.$$

By the induction hypothesis

$$b_w(\tilde{h}) \ge \bar{M}_1 \ge \frac{2M}{c_\star} b_w(1),$$

hence

$$\frac{b_u(h\tilde{h})}{b_u(h)} \ge \frac{2M}{c_\star}$$

which gives $b_u(h\tilde{h}) \ge 2M$, contradiction. In conclusion θ_n, γ_j are bounded below which implies that β_i are bounded above. This shows that $|A|$ is bounded and the lemma is proved.

$\square$

End of the proof of Proposition 5.2.

The proof is finished since Lemma 5.6, (5.15), (5.17) contradict Lemma 5.4.

□

6. Pogorelov estimate in half-domain

In this section we obtain a version of Pogorelov estimate at the boundary (Theorem 6.4 below). A similar estimate was proved also in [TW]. We start with the following a priori estimate.

Proposition 6.1. *Let $u : \bar{\Omega} \to \mathbb{R}$, $u \in C^4(\bar{\Omega})$ satisfy the Monge-Ampere equation*
$$\det D^2 u = 1 \quad in\ \Omega.$$
Assume that for some constant $k > 0$,
$$B_k^+ \subset \Omega \subset B_{k^{-1}}^+,$$
and
$$\begin{cases} u = \frac{1}{2}|x'|^2 & on \quad \partial\Omega \cap \{x_n = 0\} \\ u = 1 & on \quad \partial\Omega \cap \{x_n > 0\}. \end{cases}$$
Then
$$\|u\|_{C^{3,1}(\{u < \frac{1}{16}k^2\})} \le C(k,n).$$

Proof. We divide the proof into four steps.

Step 1: We show that
$$|\nabla u| \le C(k,n) \quad \text{in the set} \quad D := \{u < k^2/2\}.$$

For each
$$x_0 \in \{|x'| \le k, \quad x_n = 0\},$$
we consider the barrier
$$w_{x_0}(x) := \frac{1}{2}|x_0|^2 + x_0 \cdot (x - x_0) + \delta|x' - x_0|^2 + \delta^{1-n}(x_n^2 - k^{-1}x_n),$$
where δ is small so that
$$w_{x_0} \le 1 \quad \text{in} \quad B_{k^{-1}}^+.$$
Then
$$w_{x_0}(x_0) = u(x_0), \quad w_{x_0} \le u \quad \text{on} \quad \partial\Omega \cap \{x_n = 0\},$$
$$w_{x_0} \le 1 = u \quad \text{on} \quad \partial\Omega \cap \{x_n > 0\},$$
and
$$\det D^2 w_{x_0} > 1,$$
thus in Ω
$$u \ge w_{x_0} \ge u(x_0) + x_0 \cdot (x - x_0) - \delta^{1-n}k^{-1}x_n.$$

This gives a lower bound for $u_n(x_0)$. Moreover, writing the inequality for all x_0 with $|x_0| = k$ we obtain
$$D \subset \{x_n \ge c(|x'| - k)\}.$$
From the values of u on $\{x_n = 0\}$ and the inclusion above we obtain a lower bound on u_n on ∂D in a neighborhood of $\{x_n = 0\}$. Since Ω contains the cone generated by ke_n and $\{|x'| \le 1, x_n = 0\}$ and $u \le 1$ in Ω, we can use the convexity of u and obtain also an upper bound for u_n and all $|u_i|$, $1 \le i \le n-1$, on ∂D in a neighborhood of $\{x_n = 0\}$. We find
$$|\nabla u| \le C \quad \text{on} \quad \partial D \cap \{x_n \le c_0\},$$

where $c_0 > 0$ is a small constant depending on k and n. We obtain a similar bound on $\partial D \cap \{x_n \geq c_0\}$ by bounding below

$$dist(\partial D \cap \{x_n \geq c_0\}, \partial\Omega)$$

by a small positive constant. Indeed, if

$$y \in \partial\Omega \cap \{x_n \geq c_0/2\},$$

then there exists a linear function l_y with bounded gradient so that

$$u(y) = l_y(y), \quad u \geq l_y \quad \text{on} \quad \partial\Omega.$$

Then, using Alexandrov estimate for $(u - l_y)^-$ we obtain

$$u(x) \geq l_y(x) - C d(x)^{1/n}, \qquad d(x) := dist(x, \partial\Omega)$$

hence D stays outside a fixed neighborhood of y.

Step 2: We show that

$$\|D^2 u\| \leq C(k,n) \quad \text{on} \quad E := \{x_n = 0\} \cap \{|x'| \leq k/2\}.$$

It suffices to prove that $|u_{in}|$ are bounded in E with $i = 1, .., n-1$. Let

$$L\,\varphi := u^{ij}\varphi_{ij}$$

denote the linearized Monge-Ampere operator for u. Then

$$\begin{aligned} L\,u_i &= 0, \qquad u_i = x_i \quad \text{on} \quad \{x_n = 0\}, \\ L\,u &= n, \end{aligned}$$

and if we define $P(x) = \delta|x'|^2 + \delta^{1-n}x_n^2$ then

$$\begin{aligned} L\,P &= Tr\left((D^2u)^{-1}D^2P\right) \\ &\geq n\left(\det(D^2u)^{-1}\det D^2P\right)^{\frac{1}{n}} \\ &\geq n. \end{aligned}$$

Fix $x_0 \in E$. We compare u_i and

$$v_{x_0}(x) := x_i + \gamma_1\left[\delta|x' - x_0|^2 + \delta^{1-n}(x_n^2 - \gamma_2 x_n) - (u - l_{x_0})\right],$$

where l_{x_0} denotes the supporting linear function for u at x_0, $\delta = 1/4$, and γ_1, $\gamma_2 \geq 0$. Clearly,

$$L\,v_{x_0} \geq 0,$$

and, since u is Lipschitz in D we can choose γ_1, γ_2 large, depending only on k and n such that

$$v_{x_0} \leq u_i \quad \text{on} \quad \partial D.$$

This shows that the inequality above holds also in D and we obtain a lower bound on $u_{in}(x_0)$. Similarly we obtain an upper bound.

Step 3: We show that

$$\|D^2u\| \leq C \quad \text{on} \quad \{u < k^2/8\}.$$

We apply the classical Pogorelov estimate in the set

$$F := \{u < k^2/4\}.$$

Precisely if the maximal value of

$$\log\left(\frac{1}{4}k^2 - u\right) + \log u_{ii} + \frac{1}{2}u_i^2$$

occurs in the interior of F then this value is bounded by a constant depending only on n and $\max_F |\nabla u|$ (see [C2]). From step 2, the expression is bounded above on ∂F and the estimate follows.

Step 4: The Monge-Ampere equation is uniformly elliptic in $\{u < k^2/8\}$ and by Evans-Krylov theorem and Schauder estimates we obtain the desired $C^{3,1}$ bound.

□

Remark 6.2. Assume the boundary values of u are given by

$$\begin{cases} u = p(x') & \text{on} \quad \partial\Omega \cap \{x_n = 0\} \\ u = 1 & \text{on} \partial\Omega \cap \{x_n > 0\}, \end{cases}$$

with $p(x')$ a quadratic polynomial that satisfies

$$\rho|x'|^2 \le p(x') \le \rho^{-1}|x'|^2,$$

for some $\rho > 0$. Then

$$\|u\|_{C^{3,1}(\{u<\frac{1}{16}k^2\})} \le C(\rho, k, n).$$

Indeed, after an affine transformation we can reduce the problem to the case $p(x') = |x'|^2/2$.

Remark 6.3. Proposition 6.1 holds as well if we replace the half-space $\{x_n \ge 0\}$ with a large ball of radius ε^{-1}

$$\mathcal{B}_\varepsilon := \{\, |x - \varepsilon^{-1}e_n| \le \varepsilon^{-1}\}.$$

Precisely, if

$$B_k \cap \mathcal{B}_\varepsilon \subset \Omega \subset B_{k^{-1}} \cap \mathcal{B}_\varepsilon,$$

and the boundary values of u satisfy

$$\begin{cases} u = \frac{1}{2}|x'|^2 & \text{on} \quad B_1 \cap \partial\mathcal{B}_\varepsilon \subset \partial\Omega \\ u \in [1, 2] & \text{on} \quad \partial\Omega \setminus (B_1 \cap \partial\mathcal{B}_\varepsilon), \end{cases}$$

then for all small ε,

$$\|u\|_{C^{3,1}(\{u<k^2/16\})} \le C,$$

with C depending only on k and n.

The proof is essentially the same except that in the barrier functions w_{x_0}, v_{x_0} we need to replace x_n by $(x - x_0) \cdot \nu_{x_0}$ where ν_{x_0} denotes the inner normal to $\partial\Omega$ at x_0, and in step 2 we work (as in [CNS]) with the tangential derivative

$$T_i := (1 - \varepsilon x_n)\partial_{x_i} + \varepsilon x_i \partial_{x_n},$$

instead of ∂_{x_i}.

As a consequence of the Proposition 6.1 and the remarks above we obtain

Theorem 6.4. *Let $u : \Omega \to \mathbb{R}$ satisfy the Monge-Ampere equation*

$$\det D^2 u = 1 \quad \text{in } \Omega.$$

Assume that for some constants $\rho, k > 0$,

$$B_k^+ \subset \Omega \subset B_{k^{-1}}^+,$$

and (see Definition 2.5) the boundary values of u are given by

$$\begin{cases} u = p(x') & \text{on} \quad \{p(x') \le 1\} \cap \{x_n = 0\} \subset \partial\Omega \\ u = 1 & \text{on the rest of } \partial\Omega, \end{cases}$$

where p is a quadratic polynomial that satisfies

$$\rho|x'|^2 \le p(x') \le \rho^{-1}|x'|^2.$$

Then

$$\|u\|_{C^{3,1}(B^+_{c_0})} \le c_0^{-1}, \tag{6.1}$$

with $c_0 > 0$ small, depending only on k, ρ and n.

Proof. We approximate u on $\partial\Omega$ by a sequence of smooth functions u_m on $\partial\Omega_m$, with Ω_m smooth, uniformly convex, so that u_m, Ω_m satisfy the conditions of Remark 6.3 above. Notice that u_m is smooth up to the boundary by the results in [CNS], thus we can use Proposition 6.1 for u_m. We let $m \to \infty$ and obtain (6.1) since

$$B^+_{c_0} \subset \{u < k^2/16\},$$

by convexity.

□

7. Pointwise $C^{2,\alpha}$ estimates at the boundary

Let Ω be a bounded convex set with

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \ge 0\} \cap B_{\frac{1}{\rho}}, \tag{7.1}$$

for some small $\rho > 0$, that is $\Omega \subset (\mathbb{R}^n)^+$ and Ω contains an interior ball tangent to $\partial\Omega$ at 0.

Let $u : \overline{\Omega} \to \mathbb{R}$ be convex, continuous, satisfying

$$\det D^2 u = f, \qquad 0 < \lambda \le f \le \Lambda \quad \text{in } \Omega, \tag{7.2}$$

and

$$x_{n+1} = 0 \text{ is a tangent plane to } u \text{ at } 0, \tag{7.3}$$

in the following sense:

$$u \ge 0, \quad u(0) = 0,$$

and any hyperplane $x_{n+1} = \varepsilon x_n$, $\varepsilon > 0$ is not a supporting plane for u.

We also assume that on $\partial\Omega$, in a neighborhood of $\{x_n = 0\}$, u separates quadratically from the tangent plane $\{x_{n+1} = 0\}$,

$$\rho|x|^2 \le u(x) \le \rho^{-1}|x|^2 \quad \text{on } \partial\Omega \cap \{x_n \le \rho\}. \tag{7.4}$$

Our main theorem is the following.

Theorem 7.1. *Let Ω, u satisfy* (7.1)-(7.4) *above with $f \in C^\alpha$ at the origin, i.e*

$$|f(x) - f(0)| \le M|x|^\alpha \quad \text{in} \quad \Omega \cap B_\rho,$$

for some $M > 0$, and $\alpha \in (0,1)$. Suppose that $\partial\Omega$ and $u|_{\partial\Omega}$ are $C^{2,\alpha}$ at the origin, i.e we assume that on $\partial\Omega \cap B_\rho$ we satisfy

$$|x_n - q(x')| \le M|x'|^{2+\alpha},$$

$$|u - p(x')| \le M|x'|^{2+\alpha},$$

where $p(x')$, $q(x')$ are quadratic polynomials.

Then $u \in C^{2,\alpha}$ *at the origin, that is there exists a quadratic polynomial* $\mathcal{P}_0$ *with*

$$\det D^2\mathcal{P}_0 = f(0), \quad \|D^2\mathcal{P}_0\| \le C(M),$$

such that

$$|u - \mathcal{P}_0| \le C(M)|x|^{2+\alpha} \quad in \quad \Omega \cap B_\rho,$$

where $C(M)$ *depends on* M, ρ, λ, Λ, n, α.

From (7.1) and (7.4) we see that p, q are homogenous of degree 2 and

$$\|D^2 p\|, \|D^2 q\| \le \rho^{-1}.$$

A consequence of the proof of Theorem 7.1 is that if $f \in C^\alpha$ near the origin, then $u \in C^{2,\alpha}$ in any cone $\mathcal{C}_\theta$ of opening $\theta < \pi/2$ around the x_n-axis i.e

$$\mathcal{C}_\theta := \{x \in (\mathbb{R}^n)^+ | \quad |x'| \le x_n \tan\theta\}.$$

Corollary 7.2. *Assume* u *satisfies the hypotheses of Theorem 7.1 and*

$$\|f\|_{C^\alpha(\bar\Omega)} \le M.$$

Given any $\theta < \pi/2$ *there exists* $\delta(M,\theta)$ *small, such that*

$$\|u\|_{C^{2,\alpha}(\mathcal{C}_\theta \cap B_\delta)} \le C(M,\theta).$$

We also mention the global version of Theorem 7.1.

Theorem 7.3. *Let* Ω *be a bounded, convex domain and let* $u : \overline{\Omega} \to \mathbb{R}$ *be convex, Lipschitz continuous, satisfying*

$$\det D^2 u = f, \qquad 0 < \lambda \le f \le \Lambda \quad in\ \Omega.$$

Assume that

$$\partial\Omega, \quad u|_{\partial\Omega} \in C^{2,\alpha}, \qquad f \in C^\alpha(\bar\Omega),$$

for some $\alpha \in (0,1)$ *and there exists a constant* $\rho > 0$ *such that*

$$u(y) - u(x) - \nabla u(x) \cdot (y-x) \ge \rho |y-x|^2 \qquad \forall x, y \in \partial\Omega,$$

where $\nabla u(x)$ *is understood in the sense of* (7.3). *Then* $u \in C^{2,\alpha}(\bar\Omega)$ *and*

$$\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C,$$

with C *depending on* $\|\partial\Omega\|_{C^{2,\alpha}}$, $\|u|_{\partial\Omega}\|_{C^{2,\alpha}}$, $\|u\|_{C^{0,1}(\bar\Omega)}$, $\|f\|_{C^\alpha(\bar\Omega)}$, ρ, λ, Λ, n, α.

In general, the Lipschitz bound is easily obtained from the boundary data $u|_{\partial\Omega}$. We can always do this if for example Ω is uniformly convex.

The proof of Theorem 7.1 is similar to the proof of the interior $C^{2,\alpha}$ estimate from [C2], and it has three steps. First we use the localization theorem to show that after a rescaling it suffices to prove the theorem only in the case when M is arbitrarily small (see Lemma 7.4). Then we use Pogorelov estimate in half-domain (Theorem 6.4) and reduce further the problem to the case when u is arbitrarily close to a quadratic polynomial (see Lemma 7.5). In the last step we use a standard iteration argument to show that u is well-approximated by quadratic polynomials at all scales.

We assume for simplicity that

$$f(0) = 1,$$

otherwise we divide u by $f(0)$.

Constants depending on ρ, λ, Λ, n and α are called universal. We denote them by C, c and they may change from line to line whenever there is no possibility of confusion. Constants depending on universal constants and other parameters i.e M, σ, δ, etc. are denoted as $C(M, \sigma, \delta)$.

We denote linear functions by $l(x)$ and quadratic polynomials which are homogenous and convex we denote by $p(x')$, $q(x')$, $P(x)$.

The localization theorem says that the section S_h is comparable to an ellipsoid E_h which is obtained from $B_{h^{1/2}}$ by a sliding along $\{x_n = 0\}$. Using an affine transformation we can normalize S_h so that it is comparable to B_1. In the next lemma we show that, if h is sufficiently small, the corresponding rescaling u_h satisfies the hypotheses of u in which the constant M is replaced by an arbitrary small constant σ.

Lemma 7.4. *Given any $\sigma > 0$, there exist small constants $h = h_0(M, \sigma)$, $k > 0$ depending only on ρ, λ, Λ, n, and a rescaling of u*

$$u_h(x) := \frac{u(h^{1/2} A_h^{-1} x)}{h}$$

where A_h is a linear transformation with

$$\det A_h = 1, \quad \|A_h^{-1}\|, \, \|A_h\| \le k^{-1} |\log h|,$$

so that

a)

$$B_k \cap \bar{\Omega}_h \subset S_1(u_h) \subset B^+_{k^{-1}}, \qquad S_1(u_h) := \{u_h < 1\},$$

b)

$$\det D^2 u_h = f_h, \qquad |f_h(x) - 1| \le \sigma |x|^\alpha \quad \text{in} \quad \Omega_h \cap B_{k^{-1}},$$

c) On $\partial\Omega_h \cap B_{k^{-1}}$ we have

$$|x_n - q_h(x')| \le \sigma |x'|^{2+\alpha}, \qquad |q_h(x')| \le \sigma,$$

$$|u_h - p(x')| \le \sigma |x'|^{2+\alpha},$$

where q_h is a quadratic polynomial.

Proof. By the localization theorem Theorem 3.1, for all $h \le c$,

$$S_h := \{u < h\} \cap \bar{\Omega},$$

satisfies

$$k E_h \cap \bar{\Omega} \subset S_h \subset k^{-1} E_h,$$

with

$$E_h = A_h^{-1} B_{h^{1/2}}, \qquad A_h x = x - \nu_h x_n$$

$$\nu_h \cdot e_n = 0, \quad \|A_h^{-1}\|, \, \|A_h\| \le k^{-1} |\log h|.$$

Then we define u_h as above and obtain

$$S_1(u_h) = h^{-1/2} A_h S_h,$$

hence

$$B_k \cap \bar{\Omega}_h \subset S_1(u_h) \subset B^+_{k^{-1}},$$

where

$$\Omega_h := h^{-1/2} A_h \Omega.$$

Then

$$\det D^2 u_h = f_h(x) = f(h^{1/2} A_h^{-1} x),$$

and

$$\begin{aligned}|f_h(x)-1| &\le M|h^{1/2}A_h^{-1}x|^\alpha \\ &\le M(h^{1/2}k^{-1}|\log h|)^\alpha |x|^\alpha \\ &\le \sigma |x|^\alpha\end{aligned}$$

if $h_0(M,\sigma)$ is sufficiently small.

Next we estimate $|x_n - h^{1/2}q(x')|$ and $|u_h - p(x')|$ on $\partial\Omega_h \cap B_{k^{-1}}$. We have

$$x \in \partial\Omega_h \quad \Leftrightarrow \quad y := h^{1/2}A_h^{-1}x \in \Omega,$$

or

$$h^{1/2}x_n = y_n, \qquad h^{1/2}x' = y' - \nu_h y_n.$$

If $|x| \le k^{-1}$ then

$$|y| \le k^{-1}h^{1/2}|\log h||x| \le h^{1/4},$$

if h_0 is small hence, since Ω has an interior tangent ball of radius ρ, we have

$$|y_n| \le \rho^{-1}|y'|^2.$$

Then

$$|\nu_h y_n| \le k^{-1}|\log h||y'|^2 \le |y'|/2,$$

thus

$$\frac{1}{2}|y'| \le |h^{1/2}x'| \le \frac{3}{2}|y'|.$$

We obtain

$$\begin{aligned}|x_n - h^{1/2}q(x')| &\le h^{-1/2}|y_n - q(y')| + h^{1/2}|q(h^{-1/2}y') - q(x')| \\ &\le Mh^{-1/2}|y'|^{2+\alpha} + Ch^{1/2}\left(|x'||\nu_h x_n| + |\nu_h x_n|^2\right) \\ &\le 2Mh^{(\alpha+1)/2}|x'|^{2+\alpha} + Ch^{1/2}\left(h^{1/2}|\log h||x'|^3 + h|\log h|^2|x'|^4\right) \\ &\le \sigma|x'|^{2+\alpha},\end{aligned}$$

if h_0 is chosen small. Hence on $\partial\Omega_h \cap B_{k^{-1}}$,

$$|x_n - q_h(x')| \le \sigma|x'|^{2+\alpha}, \qquad q_h := h^{1/2}q(x'),$$

$$|q_h| \le \sigma,$$

and also

$$\begin{aligned}|u_h - p(x')| &\le h^{-1}|u(y) - p(y')| + |p(h^{-1/2}y') - p(x')| \\ &\le Mh^{-1}|y'|^{2+\alpha} + C\left(|x'||\nu_h x_n| + |\nu_h x_n|^2\right) \\ &\le 2Mh^{\alpha/2}|x'|^{2+\alpha} + C\left(h^{1/2}|\log h||x'|^3 + h|\log h|^2|x'|^4\right) \\ &\le \sigma|x'|^{2+\alpha}.\end{aligned}$$

∎

In the next lemma we show that if σ is sufficiently small, then u_h can be well-approximated by a quadratic polynomial near the origin.

Lemma 7.5. *For any δ_0, ε_0 there exist $\sigma_0(\delta_0,\varepsilon_0)$, $\mu_0(\varepsilon_0)$ such that for any function u_h satisfying properties a), b), c) of Lemma 7.4 with $\sigma \le \sigma_0$ we can find a rescaling*

$$\tilde u(x) := \frac{(u_h - l_h)(\mu_0 x)}{\mu_0^2},$$

with

$$l_h(x) = \gamma_h x_n, \quad |\gamma_h| \le C_0, \qquad C_0 \text{ universal},$$

that satisfies

a) in $\tilde\Omega \cap B_1$,

$$\det D^2 \tilde u = \tilde f, \qquad |\tilde f(x) - 1| \le \delta_0 \varepsilon_0 |x|^\alpha \quad in \quad \tilde\Omega \cap B_1,$$

and

$$|\tilde u - P_0| \le \varepsilon_0 \quad in \quad \tilde\Omega \cap B_1,$$

for some P_0, quadratic polynomial,

$$\det D^2 P_0 = 1, \quad \|D^2 P_0\| \le C_0;$$

b) On $\partial\tilde\Omega \cap B_1$ there exist $\tilde p_0$, $\tilde q_0$ such that

$$|x_n - \tilde q_0(x')| \le \delta_0 \varepsilon_0 |x'|^{2+\alpha}, \qquad |\tilde q_0(x')| \le \delta_0 \varepsilon_0,$$

and

$$|\tilde u - \tilde p_0(x')| \le \delta_0 \varepsilon_0 |x'|^{2+\alpha},$$
$$\tilde p_0(x') = P_0(x'), \qquad \frac{\rho}{2}|x'|^2 \le \tilde p_0(x') \le 2\rho |x'|^2.$$

Proof. We prove the lemma by compactness. Assume by contradiction that the statement is false for a sequence u_m satisfying a), b), c) of Lemma 7.4 with $\sigma_m \to 0$. Then, we may assume after passing to a subsequence if necessary that

$$p_m \to p_\infty, \quad q_m \to 0 \quad \text{uniformly on} \quad B_{k^{-1}},$$

and

$$u_m : S_1(u_m) \to \mathbb{R}$$

converges to (see Definition 2.3)

$$u_\infty : \Omega_\infty \to \mathbb{R}.$$

Then, by Theorem 2.6, u_∞ satisfies

$$B_k^+ \subset \Omega_\infty \subset B_{k^{-1}}^+, \qquad \det D^2 u_\infty = 1,$$

$$\begin{cases} u_\infty = p_\infty(x') & \text{on} \quad \{p_\infty(x') < 1\} \cap \{x_n = 0\} \subset \partial\Omega_\infty \\ u_\infty = 1 & \text{on the rest of} \quad \partial\Omega_\infty. \end{cases}$$

From Pogorelov estimate in half-domain (Theorem 6.4) there exists c_0 universal such that

$$|u_\infty - l_\infty - P_\infty| \le c_0^{-1}|x|^3 \quad \text{in} \quad \mathrm{B}_{c_0}^+,$$

where

$$l_\infty := \gamma_\infty x_n, \quad |\gamma_\infty| \le c_0^{-1},$$

and P_∞ is a quadratic polynomial such that

$$P_\infty(x') = p_\infty(x'), \quad \det D^2 P_\infty = 1, \quad \|D^2 P_\infty\| \le c_0^{-1}.$$

Choose μ_0 small such that

$$c_0^{-1}\mu_0 = \varepsilon_0/32,$$

hence

$$|u_\infty - l_\infty - P_\infty| \le \frac{1}{4}\varepsilon_0\mu_0^2 \quad \text{in} \quad B^+_{2\mu_0},$$

which together with $p_m \to p_\infty$ implies that for all large m

$$|u_m - l_\infty - P_\infty| \le \frac{1}{2}\varepsilon_0\mu_0^2 \quad \text{in} \quad S_1(u_m) \cap B^+_{\mu_0}.$$

Then, for all large m,

$$\tilde{u}_m := \frac{(u_m - l_\infty)(\mu_0 x)}{\mu_0^2}$$

satisfies in $\tilde{\Omega}_m \cap B_1$

$$|\tilde{u}_m - P_\infty| \le \varepsilon_0/2,$$

and

$$\det D^2\tilde{u}_m = \tilde{f}_m(x) = f_m(\mu_0 x),$$
$$|\tilde{f}_m(x) - 1| \le \sigma_m(\mu_0|x|)^\alpha \le \delta_0\varepsilon_0|x|^\alpha.$$

We define

$$\tilde{q}_m := \mu_0 q_m, \qquad \tilde{p}_m := p_m - \gamma_\infty q_m,$$

and clearly

$$\tilde{p}_m \to p_\infty, \quad \tilde{q}_m \to 0 \quad \text{uniformly in} \quad B_1.$$

On $\partial\tilde{\Omega}_m \cap B_1$ we have

$$\begin{aligned} |x_n - \tilde{q}_m(x')| &= \mu_0^{-1}|\mu_0 x_n - q_m(\mu_0 x')| \\ &\le \mu_0^{-1}\sigma_m|\mu_0 x'|^{2+\alpha} \\ &\le \delta_0\varepsilon_0|x'|^{2+\alpha}, \end{aligned}$$

and

$$\begin{aligned} |\tilde{u}_m - \tilde{p}_m(x')| &= \mu_0^{-2}|(u_m - l_\infty)(\mu_0 x) - p_m(\mu_0 x') + \gamma_\infty q_m(\mu_0 x')| \\ &\le \mu_0^{-2}(|(u_m - p_m)(\mu_0 x')| + |\gamma_\infty||\mu_0 x_n - q_m(\mu_0 x')|) \\ &\le \sigma_m\mu_0^\alpha(1 + |\gamma_\infty|)|x'|^{2+\alpha} \\ &\le \delta_0\varepsilon_0|x'|^{2+\alpha}. \end{aligned}$$

Finally, we let P_m be a perturbation of P_∞ such that

$$P_m(x') = \tilde{p}_m(x'), \quad \det D^2 P_m = 1, \quad P_m \to P_\infty \quad \text{uniformly in } B_1.$$

Then $\tilde{u}_m$, P_m, $\tilde{p}_m$, $\tilde{q}_m$ satisfy the conclusion of the lemma for all large m, and we reached a contradiction. ☐

From Lemma 7.4 and Lemma 7.5 we see that given any δ_0, ε_0 there exist a linear transformation

$$T := \mu_0 h_0^{1/2} A_{h_0}^{-1}$$

and a linear function

$$l(x) := \gamma x_n$$

with

$$|\gamma|, \|T^{-1}\|, \|T\| \le C(M, \delta_0, \varepsilon_0),$$

such that the rescaling

$$\tilde{u}(x) := \frac{(u-l)(Tx)}{(\det T)^{2/n}},$$

defined in $\tilde{\Omega} \subset \mathbb{R}^n$ satisfies

1) in $\tilde{\Omega} \cap B_1$

$$\det D^2 \tilde{u} = \tilde{f}, \quad |\tilde{f} - 1| \le \delta_0 \varepsilon_0 |x|^\alpha,$$

and

$$|\tilde{u} - P_0| \le \varepsilon_0,$$

for some P_0 with

$$\det D^2 P_0 = 1, \quad \|D^2 P_0\| \le C_0;$$

2) on $\partial\tilde{\Omega} \cap B_1$ we have $\tilde{p}$, $\tilde{q}$ so that

$$|x_n - \tilde{q}(x')| \le \delta_0 \varepsilon_0 |x'|^{2+\alpha}, \qquad |\tilde{q}(x')| \le \delta_0 \varepsilon_0,$$

$$|\tilde{u} - \tilde{p}(x')| \le \delta_0 \varepsilon_0 |x'|^{2+\alpha}, \qquad \frac{\rho}{2}|x'|^2 \le \tilde{p}(x') = P_0(x') \le 2\rho |x'|^2.$$

By choosing δ_0, ε_0 appropriately small, universal, we show in Lemma 7.6 that there exist $\tilde{l}$, $\tilde{P}$ such that

$$|\tilde{u} - \tilde{l} - \tilde{P}| \le C|x|^{2+\alpha} \quad \text{in} \quad \tilde{\Omega} \cap B_1, \qquad \text{and} \quad |\nabla \tilde{l}|, \, \|D^2 \tilde{P}\| \le C,$$

with C a universal constant. Rescaling back, we obtain that u is well approximated by a quadratic polynomial at the origin i.e

$$|u - l - P| \le C(M)|x|^{2+\alpha} \quad \text{in} \quad \Omega \cap B_\rho, \qquad \text{and} \quad |\nabla l|, \, \|D^2 P\| \le C(M)$$

which, by (7.3), proves Theorem 7.1.

Since $\alpha \in (0,1)$, in order to prove that $\tilde{u} \in C^{2,\alpha}(0)$ it suffices to show that $\tilde{u}$ is approximated of order $2+\alpha$ by quadratic polynomials $l_m + P_m$ in each ball of radius r_0^m for some small $r_0 > 0$, and then $\tilde{l} + \tilde{P}$ is obtained in the limit as $m \to \infty$ (see [C2], [CC]). Thus Theorem 7.1 follows from the next lemma.

Lemma 7.6. *Assume $\tilde{u}$ satisfies the properties 1), 2) above. There exist ε_0, δ_0, r_0 small, universal, such that for all $m \ge 0$ we can find l_m, P_m so that*

$$|\tilde{u} - l_m - P_m| \le \varepsilon_0 r^{2+\alpha} \quad \textit{in} \quad \tilde{\Omega} \cap B_r, \qquad \textit{with} \quad r = r_0^m.$$

Proof. We prove by induction on m that the inequality above is satisfied with

$$l_m = \gamma_m x_n, \quad |\gamma_m| \le 1,$$

$$P_m(x') = \tilde{p}(x') - \gamma_m \tilde{q}(x'), \quad \det D^2 P_m = 1, \qquad \|D^2 P_m\| \le 2C_0.$$

From properties 1),2) above we see that this holds for $m = 0$ with $\gamma_0 = 0$.

Assume the conclusion holds for m and we prove it for $m+1$. Let

$$v(x) := \frac{(\tilde{u} - l_m)(rx)}{r^2}, \quad \text{with} \quad r := r_0^m,$$

and define

$$\varepsilon := \varepsilon_0 r^\alpha.$$

Then

$$|v - P_m| \le \varepsilon \quad \text{in} \quad \Omega_v \cap B_1, \quad \Omega_v := r^{-1}\tilde{\Omega}, \tag{7.5}$$

$$|\det D^2 v - 1| = |\tilde{f}(rx) - 1| \le \delta_0 \varepsilon.$$

On $\partial\Omega_v \cap B_1$ we have

$$\begin{aligned}\left|\frac{x_n}{r} - \tilde{q}(x')\right| &= r^{-2}|rx_n - \tilde{q}(rx')| \\ &\le \delta_0\varepsilon|x'|^{2+\alpha}, \\ &\le \delta_0\varepsilon,\end{aligned}$$

which also gives

$$|x_n| \le 2\delta_0\varepsilon \quad \text{on} \quad \partial\Omega_v \cap B_1. \tag{7.6}$$

From the definition of v and the properties of P_m we see that in B_1

$$|v - P_m| \le r^{-2}|(\tilde{u} - \tilde{p})(rx)| + |\gamma_m||x_n/r - \tilde{q}| + 2nC_0|x_n|,$$

and the inequalities above and property 2) imply

$$|v - P_m| \le C_1\delta_0\varepsilon \quad \text{in} \quad \partial\Omega_v \cap B_1, \tag{7.7}$$

with C_1 universal constant (depending only on n and C_0).

We want to compare v with the solution

$$w : B^+_{1/8} \to \mathbb{R}, \qquad \det D^2 w = 1,$$

which has the boundary conditions

$$\begin{cases} w = v \quad \text{on} \quad \partial B^+_{1/8} \cap \Omega_v \\ w = P_m \qquad \text{on} \quad \partial B^+_{1/8} \setminus \Omega_v. \end{cases}$$

In order to estimate $|u - w|$ we introduce a barrier ϕ defined as

$$\phi : \bar{B}_{1/2} \setminus B_{1/4} \to \mathbb{R}, \qquad \phi(x) := c(\beta)\left(4^\beta - |x|^{-\beta}\right),$$

where $c(\beta)$ is chosen such that $\phi = 1$ on $\partial B_{1/2}$ and $\phi = 0$ on $\partial B_{1/4}$.

We choose the exponent $\beta > 0$ depending only on C_0 and n such that for any symmetric matrix A with

$$(2C_0)^{1-n} I \le A \le (2C_0)^{n-1} I,$$

we have

$$Tr\, A(D^2\phi) \le -\eta_0 < 0,$$

for some η_0 small, depending only on C_0 and n.

For each y with $y_n = -1/4$, $|y'| \le 1/8$ the function

$$\phi_y(x) := P_m + \varepsilon(C_1\delta + \phi(x - y))$$

satisfies

$$\det D^2\phi_y \le 1 - \frac{\eta_0}{2}\varepsilon \qquad \text{in} \quad B_{1/2}(y) \setminus B_{1/4}(y),$$

if $\varepsilon \le \varepsilon_0$ is sufficiently small. From (7.5), (7.7) we see that

$$v \le \phi_y \quad \text{on} \quad \partial(\Omega_v \cap B_{1/2}(y)),$$

and if $\delta_0 \le \eta_0/2$,

$$\det D^2 v \ge \det D^2\phi_y.$$

This gives

$$v \le \phi_y \quad \text{in} \quad \Omega_v \cap B_{1/2}(y),$$

and using the definition of w we obtain

$$w \le \phi_y \quad \text{on} \quad \partial B^+_{1/8}.$$

The maximum principle yields

$$w \leq \phi_y \quad \text{in} \quad B^+_{1/8},$$

and by varying y we obtain

$$w(x) \leq P_m + \varepsilon(C_1\delta_0 + Cx_n) \quad \text{in} \quad B^+_{1/8}.$$

Recalling (7.6), this implies

$$w - P_m \leq 2C_1\delta_0\varepsilon \quad \text{on} \quad B^+_{1/8} \setminus \Omega_v.$$

The opposite inequality holds similarly, hence

$$|w - P_m| \leq 2C_1\delta_0\varepsilon \quad \text{on} \quad B^+_{1/8} \setminus \Omega_v. \tag{7.8}$$

From the definition of w and (7.7) we also obtain

$$|v - w| \leq 3C_1\delta_0\varepsilon \quad \text{on} \quad \partial(\Omega_v \cap B^+_{1/8}). \tag{7.9}$$

Now we claim that

$$|v - w| \leq C_2\delta_0\varepsilon \quad \text{in} \quad \Omega_v \cap B^+_{1/8}, \quad C_2 \text{ universal.} \tag{7.10}$$

For this, we use the following inequality. If $A \geq 0$ is a symmetric matrix with

$$1/2 \leq \det A \leq 2,$$

and $a \geq 0$, then

$$\begin{aligned} \det(A + aI) &= \det A \det(I + aA^{-1}) \\ &\geq \det A(1 + Tr(aA^{-1})) \\ &\geq \det A(1 + a/2) \\ &\geq \det A + a/4. \end{aligned}$$

This and (7.9) give that in $\Omega_v \cap B^+_{1/8}$

$$\begin{aligned} w + 2\delta_0\varepsilon(|x|^2 - 2C_1) &\leq v, \\ v + 2\delta_0\varepsilon(|x|^2 - 2C_1) &\leq w, \end{aligned}$$

and the claim (7.10) is proved.

Next we approximate w by a quadratic polynomial near 0. From (7.5),(7.8), (7.10) we can conclude that

$$|w - P_m| \leq 2\varepsilon \quad \text{in} \quad B^+_{1/8},$$

if δ_0 is sufficiently small. Since $w = P_m$ on $\{x_n = 0\}$, and

$$\frac{\rho}{4}|x'|^2 \leq P_m(x') \leq 4\rho|x'|^2, \quad \det D^2P_m = 1, \quad \|D^2P_m\| \leq 2C_0,$$

we conclude from Pogorelov estimate (Theorem 6.4) that

$$\|D^2w\|_{C^{1,1}(B^+_{c_0})} \leq c_0^{-1},$$

for some small universal constant c_0. Thus in $B^+_{c_0}$, $w - P_m$ solves a uniformly elliptic equation

$$Tr\, A(x)D^2(w - P_m) = 0,$$

with the $C^{1,1}$ norm of the coefficients $A(x)$ bounded by a universal constant. Since

$$w - P_m = 0 \quad \text{on} \quad \{x_n = 0\},$$

we obtain

$$\|w - P_m\|_{C^{2,1}(B^+_{c_0/2})} \le C_3 \|w - P_m\|_{L^\infty(B^+_{c_0})} \le 2C_3\varepsilon,$$

with C_3 a universal constant. Then

$$|w - P_m - \tilde{l}_m - \tilde{P}_m| \le 2C_3\varepsilon|x|^3 \quad \text{if } |x| \le c_0/2, \tag{7.11}$$

with

$$\tilde{P}_m(x') = 0, \quad \tilde{l}_m = \tilde{\gamma}_m x_n, \qquad |\tilde{\gamma}_m|, \, \|D^2\tilde{P}_m\| \le 2C_3\varepsilon.$$

Since $\tilde{l}_m + P_m + \tilde{P}_m$ is the quadratic expansion for w at 0 we also have

$$\det D^2(P_m + \tilde{P}_m) = 1.$$

We define

$$P_{m+1}(x) := P_m(x) + \tilde{P}_m(x) - r\tilde{\gamma}_m\tilde{q}(x') + \sigma_m x_n^2$$

with σ_m so that

$$\det D^2 P_{m+1} = 1,$$

and let

$$l_{m+1}(x) := \gamma_{m+1}x_n, \qquad \gamma_{m+1} = \gamma_m + r\tilde{\gamma}_m.$$

Notice that

$$|\gamma_{m+1} - \gamma_m|, \; \|D^2P_{m+1} - D^2P_m\| \le C_4\varepsilon = C_4\varepsilon_0 r_0^{m\alpha}, \tag{7.12}$$

and

$$\|D^2P_{m+1} - D^2(P_m + \tilde{P}_m)\| \le C_4\delta_0\varepsilon,$$

for some C_4 universal. From the last inequality and (7.10), (7.11) we find

$$|v - \tilde{l}_m - P_{m+1}| \le (2C_3r_0^3 + C_2\delta_0 + C_4\delta_0)\varepsilon \quad \text{in} \quad \Omega_v \cap B^+_{r_0}.$$

This gives

$$|v - \tilde{l}_m - P_{m+1}| \le \varepsilon r_0^{2+\alpha} \quad \text{in} \quad \Omega_v \cap B^+_{r_0},$$

if we first choose r_0 small (depending on C_3) and then δ_0 depending on r_0, C_2, C_4, hence

$$|\tilde{u} - l_{m+1} - P_{m+1}| \le \varepsilon r^2 r_0^{2+\alpha} = \varepsilon_0(rr_0)^\alpha \quad \text{in} \quad \Omega \cap B^+_{rr_0}.$$

Finally we choose ε_0 small such that (7.12) and

$$\gamma_0 = 0, \quad \|D^2P_0\| \le C_0,$$

guarantee that

$$|\gamma_m| \le 1, \quad \|D^2P_m\| \le 2C_0$$

for all m. This shows that the induction hypotheses hold for $m+1$ and the lemma is proved.

□

Remark 7.7. The proof of Lemma 7.6 applies also at interior points. More precisely, if $\tilde{u}$ satisfies the hypotheses in $B_1(x_0) \subset \tilde{\Omega}$ instead of $B_1 \cap \tilde{\Omega}$ then the conclusion holds in $B_1(x_0)$. The proof is in fact simpler since, in this case we take w so that

$$w = v \quad \text{on} \quad \partial B_1(x_0),$$

and then (7.9) is automatically satisfied, so there is no need for the barrier ϕ. Also, at the end we apply the classical interior estimate of Pogorelov instead of the estimate in half-domain.

Now we can sketch a proof of Corollary 7.2 and Theorem 7.3.

If u satisfies the conclusion of Theorem 7.1 then, after an appropriate dilation, any point in $\mathcal{C}_\theta \cap B_\delta$ becomes an interior point x_0 as in Remark 7.7 above for the rescaled function $\tilde{u}$. Moreover, the hypotheses of Lemma 7.6 hold in $B_1(x_0)$ for some appropriate $\varepsilon \le \varepsilon_0$. Then Corollary 7.2 follows easily from Remark 7.7.

If u satisfies the hypotheses of Theorem 7.3 then we obtain as above that

$$\|u\|_{C^{2,\alpha}(D_\delta)} \le C, \qquad D_\delta := \{x \in \Omega| \quad dist(x, \partial\Omega) \le \delta\},$$

for some δ and C depending on the data. We combine this with the interior $C^{2,\alpha}$ estimate of Caffarelli in [C2] and obtain the desired bound.

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Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: savin@math.columbia.edu