

MINIMIZERS OF CONVEX FUNCTIONALS ARISING IN RANDOM SURFACES

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ABSTRACT. We investigate C^1 regularity of minimizers to $\int F(\nabla u)dx$ in two dimensions for certain classes of non-smooth convex functionals F . In particular our results apply to the surface tensions that appear in recent works on random surfaces and random tilings of Kenyon, Okounkov and others.

1. INTRODUCTION

The classical problem in the calculus of variations consists in minimizing the functional

$$\min_{\mathcal{A}} \int_{\Omega} F(\nabla u)dx, \quad \Omega \subset \mathbb{R}^n$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a given convex function and \mathcal{A} is a family of admissible functions $u : \Omega \rightarrow \mathbb{R}$ (typically the elements of a functional space satisfying a boundary condition $u = \varphi$ on $\partial\Omega$). In general it is not hard to show the existence and uniqueness for this problem, but the main difficulty arises when trying to prove further smoothness properties of the minimizers. For example, one would like to show that a minimizer has continuous second derivatives. Then it satisfies the Euler-Lagrange equation

$$(1.1) \quad \operatorname{div}(\nabla F(\nabla u)) = F_{ij}(\nabla u)u_{ij} = 0.$$

Hilbert's 19th problem posed at the beginning of the last century refers precisely to this question of regularity: *Are the solutions of regular problems in the calculus of variations always necessarily analytic?*

In two dimensions the problem was solved in 1943 by C.B. Morrey [M] who showed analyticity of solutions by the use of complex analysis and quasiconformal mappings. A partial answer to Hilbert's problem in the general case was given in the 1930s by the use of Schauder's estimates for linear equations [Sc1],[Sc2], which guarantee that minimizers are smooth once they have Hölder continuous derivatives. On the other hand it follows from the comparison principle that minimizers are Lipschitz continuous under quite general conditions on the domain and boundary data. One piece of the puzzle remained to be proved: that Lipschitz minimizers are in fact $C^{1,\alpha}$.

The breakthrough came in the 1950s from the work of E. De Giorgi [D] on the minimal surface equation and independently from J. Nash [N]. Differentiating the equation (1.1) with respect to x_k we see that the derivative u_k satisfies the nonlinear elliptic PDE

$$(1.2) \quad \partial_i(F_{ij}(\nabla u)u_{kj}) = 0.$$

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Ignoring the dependence on ∇u we can write that $v = u_k$ satisfies the linear equation

$$\partial_i(a_{ij}(x)v_j) = 0.$$

De Giorgi observed that the uniform ellipticity condition

$$\lambda I \leq [a_{ij}(x)]_{i,j} \leq \Lambda I$$

on the coefficients is sufficient to obtain continuity for v (in fact Hölder continuity due to scaling). The proof is based on an iteration scheme for various Caccioppoli type inequalities of the form

$$(1.3) \quad \int_{\Omega} |\nabla v|^2 \eta^2 dx \leq C(\Lambda, \lambda) \int_{\Omega} v^2 |\nabla \eta|^2 dx, \quad \forall \eta \in C_0^\infty(\Omega).$$

This result led to extensive study of the theory of linear second order elliptic equations with measurable coefficients (see for example [CFMS], [LSW], [GT]).

With the question of regularity being understood in the case when F is smooth and strictly uniformly convex, it is natural to further investigate what happens when the uniform ellipticity condition on $D^2 F$ fails on a certain set. In the particular case of the p -Laplace equation, i.e $F(\xi) = |\xi|^p$, $1 < p < \infty$, when the degeneracy set consists of only one point (the origin) the regularity of minimizers is well-understood (see for example [E]).

In this paper we direct our attention to the case when the degeneracy set of F can possibly be very large. The first step towards proving regularity of minimizers would be obtaining C^1 continuity. Then the Euler Lagrange equation becomes less ambiguous and the classical theory can be applied near the points whose gradients lie outside the degeneracy set of F .

If F^* denotes the Legendre transform of F then

$$F_{ij}(\nabla F^*) F_{ij}^* = \text{tr}((D^2 F^*)^{-1} D^2 F^*) = n,$$

thus F^* is a particular solution of the nonlinear PDE (1.2). Without further evidence we can ask whether or not minimizers have the same regularity as F^* . In particular: *Is it true that if F is strictly convex then Lipschitz minimizers are of class C^1 ?* This seems to be a difficult question but there is some evidence that suggests that the result might be true at least in two dimensions.

In the first part of the paper we prove *a priori* estimates that answer the question above in two dimensions for two large classes of strictly convex functionals. Our first theorem shows that if \mathbb{R}^2 can be covered by two open sets O_λ, V_Λ ,

$$\begin{aligned} O_\lambda &\subset \{p \in \mathbb{R}^2, D^2 F(p) > \lambda I\}, \\ V_\Lambda &\subset \{p \in \mathbb{R}^2, D^2 F(p) < \Lambda I\}, \end{aligned}$$

then ∇u has a uniform modulus of continuity in the interior of Ω .

Theorem 1.1. *Let u be a minimizer with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that*

$$\overline{B}_M \subset O_\lambda \cup V_\Lambda.$$

Then in $B_{1/2}$, ∇u has a uniform modulus of continuity depending on the modulus of convexity ω_F of F , O_λ, V_Λ, M and $\|\nabla F\|_{L^\infty(B_M)}$.

In the second theorem we show that if $D^2 F > 0$ except at a finite number of points then the same result holds.

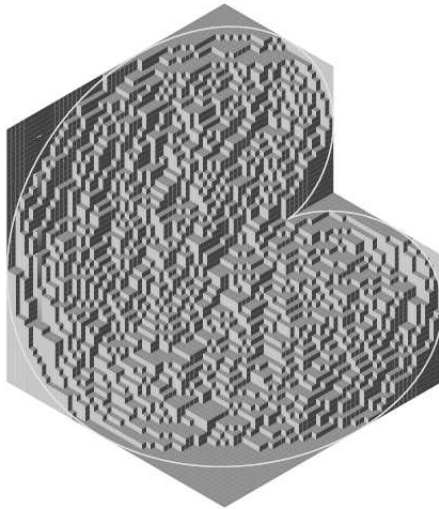


FIGURE 1.

Theorem 1.2. *Let u be a minimizer with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that $\bar{B}_M \setminus \bigcup O_{1/n}$ is a finite set. Then in $B_{1/2}$, ∇u has a uniform modulus of continuity depending on $O_{1/n}$, M and $\|\nabla F\|_{L^\infty(B_M)}$.*

Our interest in this two dimensional regularity problem is motivated by a series of recent papers in combinatorics and statistical mechanics about random tilings and random surfaces by Cohn, Kenyon, Okounkov, Propp, Sheffield and others. Let us briefly explain the connection between our problem and these results.

We present the simplest model of a random surface. For small ε consider the points $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, and call the $(1, 1, 1)$ direction vertical and the plane $P = \{x + y + z = 0\}$ horizontal. An ε -stepped surface is a polygonal surface whose faces are squares in the 2-skeleton of $\varepsilon\mathbb{Z}^3$ and which is a graph in the vertical direction. In other words, the subgraph of an ε -stepped surface is a collection of ε -cubes of $\varepsilon\mathbb{Z}^3$.

Obviously, stepped surfaces can approximate only Lipschitz graphs with gradients lying in the equilateral triangle N with vertices given by the slopes of the three planes $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$.

Consider the “random surfaces” given by all ε -stepped surfaces above a domain $\Omega \subset P$ which stay $O(\varepsilon)$ away from a given function φ defined on $\partial\Omega$. In connection with problems that arise in statistical mechanics, it is interesting to investigate the limiting behavior as $\varepsilon \rightarrow 0$. Cohn, Kenyon and Propp [CKP] proved that as $\varepsilon \rightarrow 0$ the random surfaces almost surely converge uniformly to a nonrandom function u which solves the variational problem

$$(1.4) \quad \min_u \int_{\Omega} F(\nabla u) dx, \quad \nabla u \in \bar{N}, \quad u = \varphi \text{ on } \partial\Omega.$$

The surface tension $F : \bar{N} \rightarrow \mathbb{R}$ can be computed from combinatorial considerations and is strictly convex in the interior of N and constant on ∂N .

The picture of such a random surface is shown in Figure 1 and it appeared on the cover of the Notices of the AMS, March 2005, Volume 52.

Other types of random surfaces can be obtained from random tiling of the plane with various geometric figures. For example there is a one to one correspondence between the tilings with 60° rhombi of an ε - honeycomb lattice and the $\sqrt{3}\varepsilon$ - stepped surfaces: the tilings are the projections of the faces (squares) along the vertical direction onto the plane P .

More generally, one can construct random surfaces corresponding to random perfect matchings (also called *dimer configurations*) on a weighted, bipartite, periodic, planar graph G . Kenyon, Okounkov and Sheffield [KOS] showed that these random surfaces converge *a.s.* to a solution of (1.4) with F and the polygon N (called the Newton polygon) depending on G . The function F is piecewise linear on ∂N , strictly convex in the interior, smooth except at a finite number of points where it can have singularities.

In the second part of this paper we investigate precisely the C^1 regularity of minimizers of (1.4) in which the functional F is, in the interior of a polygon N , strictly convex and smooth except at a finite number of points, but without any further assumptions about the behavior of F on ∂N . In this case the set of degeneracy of D^2F is the union of a finite set with ∂N .

The variational problem (1.4) is equivalent to an obstacle problem. Indeed, let $\underline{\varphi}$ ($\overline{\varphi}$) be the minimum (maximum) of all admissible functions u and define F to be ∞ outside \overline{N} , then problem (1.4) is equivalent to

$$\min_{\underline{\varphi} \leq u \leq \overline{\varphi}} \int_{\Omega} F(\nabla u) dx.$$

The obstacle problem has been extensively studied for the Laplace equation i.e $F(\xi) = |\xi|^2$ (see for example [C]).

The regularity of minimizers in our case is quite delicate. Near the points where the solution separates from the obstacle the equation becomes degenerate. Moreover, even in the set where $\underline{\varphi} < u < \overline{\varphi}$ it cannot be concluded (as in Laplace equation case) that ∇u is in the interior of N . Thus the results from the first part of the paper cannot be directly applied in any reasonable set.

Our main regularity result says that the minimizers are C^1 in Ω except on a number of segments which have an end point on $\partial\Omega$ and have directions perpendicular to the sides of N . On these segments the minimizer coincides with either the lower or the upper obstacle.

Theorem 1.3. *If ∇u is discontinuous at $x_0 \in \Omega$, then there exists a direction ν_i perpendicular to one of the sides $[p_i, p_{i+1}]$ of N such that u is linear on a segment of direction ν_i connecting x_0 and $\partial\Omega$. Precisely, there exists $x_1 = x_0 + t_1\nu_i \in \partial\Omega$ such that*

$$u(x) = u(x_0) + p_i \cdot (x - x_0), \quad \text{for all } x \in [x_0, x_1] \subset \overline{\Omega}.$$

In particular, $u \in C^1$ away from the obstacles.

We also prove a type of continuity result at the points where the minimizer is not differentiable. We show that if a sequence of points converges to a point of non-differentiability then their corresponding gradients approach ∂N (see Theorem 4.1).

As seen in Figure 1, minimizers of (1.4) may have “flat” regions where they are linear and their gradient belongs to the set of vertices of N . We show that these regions must lie between a convex and a concave graph (see Theorem 4.2).

The proof of our results relies on two dimensional techniques. For the results in the first part we prove weak versions of Caccioppoli inequalities in which the right hand side of (1.3) is replaced by a constant. This is done using the precise form of the nonlinearity in equation (1.2). As it turns out, such inequalities are sufficient for proving C^1 continuity. The second part of the paper is more involved. We use the ideas of the first part and an approximation technique together with the fact that any tangent plane splits the graph of a smooth minimizer into at least four connected components. These methods could be exploited further to include other examples of functionals, however most examples would satisfy our assumptions.

As in the work of De Giorgi the key step in proving C^1 continuity consists in obtaining “localization” lemmas (see Lemmas 2.1, 2.2, 3.1, and Theorem 6.1). Roughly speaking, they say that when we restrict to smaller and smaller domains the image of the gradient restricts itself to either one of two known sets (that decompose the plane).

The paper is organized as follows. In Section 2 and Section 3 we prove respectively Theorem 1.1 and Theorem 1.2. In Section 4 we present the degenerate obstacle problem and state the main results. In Section 5 we introduce the approximation problem. In Section 6 we prove Theorem 1.3. In Section 7 we prove Theorem 4.1. Finally in the last section we prove two flatness theorems Theorems 4.3 and 4.4 that are used in the course of the proof of Theorem 1.3.

2. THE PROOF OF THEOREM 1.1.

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth strictly convex function, with modulus of convexity $\omega_F : (0, +\infty) \rightarrow (0, +\infty)$, i.e.

$$F(q) - F(p) - \nabla F(p) \cdot (q - p) \geq \omega_F(|q - p|), \quad q \neq p.$$

Let $0 < \lambda \leq \Lambda$, and denote by O_λ, V_Λ two open sets such that

$$(2.1) \quad O_\lambda \subset \{p \in \mathbb{R}^2, D^2 F(p) > \lambda I\},$$

$$(2.2) \quad V_\Lambda \subset \{p \in \mathbb{R}^2, D^2 F(p) < \Lambda I\}.$$

Throughout this paper balls (of radius r) in the x -variable space are denoted by B_r , while balls in the p -variable space (the gradient space), are denoted by \mathcal{B}_r .

Consider the integral functional

$$I(u) = \int_{B_1} F(\nabla u) dx.$$

We prove the following a priori estimate.

Theorem 1.1. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that*

$$\bar{\mathcal{B}}_M \subset O_\lambda \cup V_\Lambda.$$

Then in $B_{1/2}$, ∇u has a uniform modulus of continuity depending on $\omega_F, O_\lambda, V_\Lambda, M$ and $\|\nabla F\|_{L^\infty(\mathcal{B}_M)}$.

2.1. Statement of the localization Lemmas. The proof of Theorem 1.1 relies on the next two Lemmas. Let \mathbf{e} be an arbitrary unit vector (direction) in \mathbb{R}^2 and denote by

$$\begin{aligned} H_{\mathbf{e}}^+(c) &:= \{p \in \mathbb{R}^2 \mid p \cdot \mathbf{e} > c\}, \\ H_{\mathbf{e}}^-(c) &:= \{p \in \mathbb{R}^2 \mid p \cdot \mathbf{e} < c\}, \\ S_{\mathbf{e}}(c_0, c_1) &:= \{p \in \mathbb{R}^2 \mid c_0 < p \cdot \mathbf{e} < c_1\}, \end{aligned}$$

for some constants $c, c_0, c_1 \in \mathbb{R}$, with $c_0 < c_1$.

Lemma 2.1. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that there exist a direction \mathbf{e} and constants c_0, c_1 such that*

$$(2.3) \quad S_{\mathbf{e}}(c_0, c_1) \cap \nabla u(B_1) \subset O_\lambda.$$

Then, there exists $\delta > 0$ depending on $c_1 - c_0, \lambda, M, \|\nabla F\|_{L^\infty(B_M)}$, such that either

$$(2.4) \quad \nabla u(B_\delta) \subset H_{\mathbf{e}}^+(c_0)$$

or

$$(2.5) \quad \nabla u(B_\delta) \subset H_{\mathbf{e}}^-(c_1).$$

Lemma 2.2. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that there exist a direction \mathbf{e} and constants \tilde{c}, ε such that*

$$(2.6) \quad H_{\mathbf{e}}^+(\tilde{c} - \varepsilon) \cap \nabla u(B_1) \subset V_\Lambda.$$

Then, there exists $\delta > 0$ depending on $\varepsilon, \Lambda, \omega_F, M$, such that either

$$(2.7) \quad \nabla u(B_\delta) \subset H_{\mathbf{e}}^+(\tilde{c} - \varepsilon)$$

or

$$(2.8) \quad \nabla u(B_\delta) \subset H_{\mathbf{e}}^-(\tilde{c} + \varepsilon).$$

As observed in the introduction, these Lemmas say that as we restrict to smaller and smaller domains in the x -space the image of the gradient restricts itself to either one of two half-planes. The Caccioppoli-type inequalities in the next subsection will be the key tool towards the proof of these Lemmas.

2.2. Caccioppoli-type inequalities. Let u be a smooth minimizer to $I(u)$, then u satisfies the Euler-Lagrange equation

$$(2.9) \quad \operatorname{div}(\nabla F(\nabla u)) = 0 \text{ in } B_1.$$

Differentiating the equation in the direction \mathbf{e} we obtain

$$(2.10) \quad \int_{B_1} F_{ij}(\nabla u) u_{ej} \phi_i dx = 0, \quad \forall \phi \in C_0^\infty(B_1).$$

Let $c_0, c_1 \in \mathbb{R}$ with $c_0 < c_1$ and denote by

$$(2.11) \quad G(t) := \begin{cases} 0 & t \leq c_0, \\ 1 & t \geq c_1 \end{cases}$$

a smooth function with bounded slope $|G'(t)| < C$.

Proposition 2.3. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that there exist a direction e and constants c_0, c_1 such that*

$$(2.12) \quad S_e(c_0, c_1) \cap \nabla u(B_1) \subset O_\lambda.$$

Then,

$$(2.13) \quad \int_{B_{1/2}} |\nabla(G(u_e))|^2 dx \leq C$$

for some constant C depending on $c_0, c_1, \lambda, M, \|\nabla F\|_{L^\infty(\mathcal{B}_M)}$.

Proof. In formula (2.10), let us choose as test function $\phi = \xi^2 G(u_e)$, with $\xi \in C_0^\infty(B_1)$, $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $B_{1/2}$. We obtain

$$(2.14) \quad \int_{B_1} F_{ij}(\nabla u) u_{ej} \xi^2 (G(u_e))_i dx = -2 \int_{B_1} F_{ij}(\nabla u) u_{ej} \xi \xi_i G(u_e) dx.$$

Now we analyze the left-hand side of (2.14).

$$(2.15) \quad LHS = \int_{B_1} F_{ij}(\nabla u) u_{ej} u_{ei} \xi^2 G'(u_e) dx$$

$$(2.16) \quad \geq \lambda \int_{B_1} |\nabla u_e|^2 \xi^2 G'(u_e) dx \geq c\lambda \int_{B_1} |\nabla(G(u_e))|^2 \xi^2 dx,$$

where in the first inequality in (2.16) we used the assumption (2.12) and the definition of O_λ , while in the second one we used that G' is bounded. Thus,

$$(2.17) \quad LHS \geq c\lambda \int_{B_1} |\nabla(G(u_e))|^2 \xi^2 dx.$$

On the other hand, using integration by parts we have that the right-hand side of (2.14) is given by

$$\begin{aligned} RHS &= -2 \int_{B_1} \partial_e(F_i(\nabla u)) \xi \xi_i G(u_e) dx \\ &= 2 \int_{B_1} F_i(\nabla u) \partial_e(\xi \xi_i) G(u_e) dx + \int_{B_1} F_i(\nabla u) \xi \xi_i \partial_e(G(u_e)) dx. \end{aligned}$$

Thus, for any $\gamma \leq 1$,

$$(2.18) \quad |RHS| \leq \gamma \int_{B_1} |\nabla(G(u_e))|^2 \xi^2 dx + C/\gamma$$

with C depending on $\|\nabla F\|_{L^\infty(\mathcal{B}_M)}$. Combining (2.17) with (2.18) and choosing γ sufficiently small we obtain the desired claim. \square

Proposition 2.4. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that there exist a direction e and constants \tilde{c}, ε such that*

$$(2.19) \quad H_e^+(\tilde{c} - \varepsilon) \cap \nabla u(B_1) \subset V_\Lambda.$$

Then, set $w = \nabla F(\nabla u)$ and $U = (\nabla u)^{-1}(H_e^+(\tilde{c}))$,

$$(2.20) \quad \int_{B_{1/2} \cap U} |\nabla w|^2 dx \leq C$$

for some constant C depending on $\tilde{c}, \varepsilon, \Lambda, M$.

Proof. In formula (2.10) choose as test function $\phi = \xi^2(u_e - (\tilde{c} - \varepsilon))^+$ with $\xi \in C_0^\infty(B_1)$, $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $B_{1/2}$. Set $U^\varepsilon := (\nabla u)^{-1}(H_e^+(\tilde{c} - \varepsilon)) \supset U$.

Then,

$$(2.21) \quad \int_{U^\varepsilon} F_{ij}(\nabla u) u_{ej} u_{ei} \xi^2 dx = -2 \int_{B_1} F_{ij}(\nabla u) u_{ej} \xi \xi_i (u_e - (\tilde{c} - \varepsilon))^+ dx.$$

According to assumption (2.19) and the definition of V_Λ , on the set U^ε we have that

$$(2.22) \quad F_{ij}(\nabla u) u_{ej} u_{ei} = (\nabla u_e)^T \cdot D^2 F(\nabla u) \cdot \nabla u_e \geq \frac{1}{\Lambda} |D^2 F(\nabla u) \cdot \nabla u_e|^2.$$

Hence, after bounding the right-hand side of (2.21) using Young's inequality we obtain

$$(2.23) \quad \int_{U^\varepsilon} |D^2 F(\nabla u) \cdot \nabla u_e|^2 \xi^2 dx \leq \gamma \int_{U^\varepsilon} |D^2 F(\nabla u) \cdot \nabla u_e|^2 \xi^2 dx + C/\gamma$$

for some constant C depending on $\Lambda, M, \tilde{c}, \varepsilon$ and any $\gamma > 0$. Hence for γ small enough we obtain

$$(2.24) \quad \int_{B_{1/2} \cap U^\varepsilon} |D^2 F(\nabla u) \cdot \nabla u_e|^2 dx \leq C.$$

Now, let \mathbf{f} be a direction close to \mathbf{e} , $\mathbf{f} \neq \mathbf{e}$, such that for some constant $k \in \mathbb{R}$,

$$\{p \cdot \mathbf{f} = k\} \cap \mathcal{B}_M \subset S_e(\tilde{c} - \varepsilon, \tilde{c}).$$

Then,

$$H_{\mathbf{f}}^+(k) \cap \nabla u(B_1) \subset V_\Lambda,$$

and we can repeat the same argument as above with $U^k := (\nabla u)^{-1}(H_{\mathbf{f}}^+(k)) \supset U$ to conclude that

$$(2.25) \quad \int_{B_{1/2} \cap U^k} |D^2 F(\nabla u) \cdot \nabla u_{\mathbf{f}}|^2 dx \leq C.$$

Combining (2.24) with (2.25) we obtain

$$\int_{B_{1/2} \cap U} |D^2 F(\nabla u) D^2 u|^2 dx \leq C,$$

which gives the desired inequality. \square

2.3. The proof of the localization Lemmas. We are now ready to prove the localization Lemmas, using the Caccioppoli-type inequalities above and the fact that our problem is two dimensional.

Proof of Lemma 2.1. Let $\delta > 0$ and assume that there exist $x_0, x_1 \in B_\delta$ such that

$$\nabla u(x_0) \cdot \mathbf{e} \leq c_0, \quad \nabla u(x_1) \cdot \mathbf{e} \geq c_1.$$

Then, by the maximum principle

$$\{\nabla u \cdot \mathbf{e} \leq c_0\} \cap \partial B_r \neq \emptyset,$$

$$\{\nabla u \cdot \mathbf{e} \geq c_1\} \cap \partial B_r \neq \emptyset,$$

for all r , with $\delta \leq r \leq 1/2$. Let

$$x_0^r \in \{\nabla u \cdot \mathbf{e} \leq c_0\} \cap \partial B_r, \quad x_1^r \in \{\nabla u \cdot \mathbf{e} \geq c_1\} \cap \partial B_r$$

for all $\delta \leq r \leq 1/2$. Then, if G is the function defined in (2.11), we have

$$(2.26) \quad 1 = G(u_e(x_1^r)) - G(u_e(x_0^r)) \leq \int_{\partial B_r(x_0^r, x_1^r)} |\nabla(G(u_e))| ds.$$

Applying Cauchy-Schwartz we obtain

$$(2.27) \quad \int_{\partial B_r(x_0^r, x_1^r)} |\nabla(G(u_e))|^2 ds \geq \frac{1}{2\pi r}.$$

Thus,

$$(2.28) \quad \int_{\delta}^{1/2} \int_{\partial B_r(x_0^r, x_1^r)} |\nabla(G(u_e))|^2 ds dr \geq \frac{1}{2\pi} \int_{\delta}^{1/2} \frac{1}{r} dr = \frac{1}{2\pi} \ln \frac{1}{2\delta}.$$

Hence,

$$(2.29) \quad \int_{B_{1/2}} |\nabla(G(u_e))|^2 dx \geq \frac{1}{2\pi} \ln \frac{1}{2\delta},$$

which for δ sufficiently small depending on $c_0, c_1, \lambda, M, \|\nabla F\|_{L^\infty(B_M)}$, contradicts the inequality (2.13). \square

Proof of Lemma 2.2. Let $\delta > 0$ and assume that there exist $x_0, x_1 \in B_\delta$ such that

$$\nabla u(x_0) \cdot e \leq \tilde{c}, \quad \nabla u(x_1) \cdot e \geq \tilde{c} + \varepsilon.$$

Then, as in the previous lemma, for all r with $\delta \leq r \leq 1/2$ there are points $x_0^r, x_1^r \in \partial B_r$ such that

$$x_0^r \in \{\nabla u \cdot e \leq \tilde{c}\}, \quad x_1^r \in \{\nabla u \cdot e \geq \tilde{c} + \varepsilon\}.$$

Denote by

$$\underline{H} := \nabla F(\{p \cdot e \leq \tilde{c}\}) \\ \overline{H} := \nabla F(\{p \cdot e \geq \tilde{c} + \varepsilon\}).$$

Set $w = \nabla F(\nabla u)$, then w maps x_0^r and x_1^r respectively in \underline{H} and \overline{H} .

Now, since F is strictly convex, there exists $\eta = \eta(\varepsilon) > 0$ depending on ω_F such that

$$|p_1 - p_2| \geq \varepsilon \Rightarrow |\nabla F(p_1) - \nabla F(p_2)| \geq 2\eta.$$

Let \mathcal{G} be a smooth function on \mathbb{R}^2 with $|\nabla \mathcal{G}| \leq 1$ such that

$$\mathcal{G}(p) = \begin{cases} 0 & p \in \underline{H}, \\ \eta & p \in \overline{H}. \end{cases}$$

Notice that \mathcal{G} can be obtained by a mollification of $\min\{\eta, (\text{dist}(p, \underline{H}) - \gamma)^+\}$, with γ small.

Then, we have

$$(2.30) \quad \eta = \mathcal{G}(w(x_1^r)) - \mathcal{G}(w(x_0^r)) \leq \int_{\partial B_r(x_0^r, x_1^r)} |\nabla(\mathcal{G}(w))| ds.$$

Arguing as in the proof of the previous Proposition (see computations following (2.26)), we get

$$(2.31) \quad \int_{B_{1/2}} |\nabla(\mathcal{G}(w))|^2 dx \geq \frac{\eta}{2\pi} \ln \frac{1}{2\delta}.$$

However,

$$(2.32) \quad \int_{B_{1/2}} |\nabla(\mathcal{G}(w))|^2 dx \leq \int_{B_{1/2}} |\nabla \mathcal{G}|^2 |\nabla w|^2 dx \leq \int_{B_{1/2} \cap (\nabla u)^{-1}(S_e(\tilde{c}, \tilde{c} + \varepsilon))} |\nabla w|^2 dx.$$

Therefore, combining (2.31) with (2.32), we obtain a contradiction to (2.20) as long as δ is sufficiently small. \square

2.4. The proof of Theorem 1.1. We wish to prove the following claim:

$\forall \varepsilon > 0, \exists \delta > 0, \delta = \delta(\varepsilon, \omega_F, O_\lambda, V_\Lambda, M, \|\nabla F\|_{L^\infty(\mathcal{B}_M)})$ such that $\text{diam}\{\nabla u(B_\delta)\} \leq \varepsilon$ for any minimizer u to $I(u)$.

Since O_λ, V_Λ are open sets covering \mathcal{B}_M , there exists $\varepsilon^* > 0$ such that any ball $\mathcal{B}_{\varepsilon^*}(p) \subset \mathcal{B}_{2M}$ is either contained in O_λ or in V_Λ .

Now, let $\varepsilon < \varepsilon^*$ and let P_N be an N -th regular polygon with sides of length $\varepsilon/2$, such that

$$\mathcal{B}_M \subset P_N \subset \mathcal{B}_{2M}.$$

Thus $N = N(\varepsilon, M)$. Denote by p_i the vertices of P_N , and let T_i be the triangle with vertices p_i, p_{i+1} and $p_{i+2}, i = 1, \dots, N (p_{N+1} = p_1, p_{N+2} = p_2)$. Since the length of the sides of P_N is $\varepsilon/2$, clearly T_i is included in a ball $\mathcal{B}_{\varepsilon^*}(p) \subset \mathcal{B}_{2M}$. Thus,

$$\text{either } T_i \subset O_\lambda \text{ or } T_i \subset V_\Lambda.$$

Without loss of generality we can assume that for some $i = 1, \dots, N$,

$$(2.33) \quad T_i \cap \nabla u(B_1) \neq \emptyset,$$

otherwise we can work with the polygon $P_N \setminus \bigcup_i T_i$. Let $[p, q]$ be the closed segment joining two points p and q . Denote by m_i the middle point of the segments $[p_i, p_{i+1}]$. Also let \mathbf{e}_i be the direction such that

$$[p_i, p_{i+2}] \subset \{p \cdot \mathbf{e}_i = l_i\}, \quad [m_i, m_{i+1}] \subset \{p \cdot \mathbf{e}_i = l_i + \gamma\},$$

for some l_i, γ , with $0 < \gamma < \varepsilon/2$. Then according to either Lemma 2.1 or Lemma 2.2 we have that either

$$(2.34) \quad \nabla u(B_\delta) \subset H^-(\mathbf{e}_i, l_i + \gamma),$$

or

$$(2.35) \quad \nabla u(B_\delta) \subset H^+(\mathbf{e}_i, l_i),$$

for some $\delta = \delta(\varepsilon, \lambda, \Lambda, \omega_F, M, \|\nabla F\|_{L^\infty(\mathcal{B}_M)})$.

If the latter holds then immediately $\text{diam}\{\nabla u(B_\delta)\} \leq \varepsilon$ as desired.

Assume that (2.35) does not hold for any choice of i . Then, according to (2.34) we have that

$$\nabla u(B_{\delta N}) \subset \tilde{P}_N$$

where \tilde{P}_N is the polygon with vertices the middle points $m_i, i = 1, \dots, N$. Now, we repeat this argument with \tilde{P}_N instead of P_N and iterate this a finite number of times till we obtain a polygon of diameter smaller than ε . \square

3. THE PROOF OF THEOREM 1.2

As in the previous Section, let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth strictly convex function, and denote by $O_{1/n}$ an increasing sequence of open sets such that

$$O_{1/n} \subset \{p \in \mathbb{R}^2, D^2F(p) > \frac{1}{n}I\}.$$

Recall that,

$$I(u) = \int_{B_1} F(\nabla u) dx.$$

In this Section we prove the following a priori estimate.

Theorem 1.2. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that $\bar{B}_M \setminus \bigcup O_{1/n}$ is a finite set. Then in $B_{1/2}$, ∇u has a uniform modulus of continuity depending on the sets $O_{1/n}$, M and $\|\nabla F\|_{L^\infty(B_M)}$.*

The proof of Theorem 1.2 follows the same strategy as the proof of Theorem 1.1. Precisely, we determine a localization Lemma which allows us to apply an iteration argument and obtain the desired modulus of continuity.

Here is the statement of the localization Lemma.

Lemma 3.1. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Assume that*

$$(3.1) \quad \nabla u(B_1) \cap \mathcal{B}_\rho(p_0) = \emptyset, \quad \mathcal{B}_{4\rho}(p_0) \cap \bar{B}_M \subset O_{1/n}.$$

Then, there exists $\delta > 0$ depending on $n, M, \rho, \|\nabla F\|_{L^\infty(B_M)}$, such that either

$$(3.2) \quad \nabla u(B_\delta) \subset \mathcal{B}_{4\rho}(p_0)$$

or

$$(3.3) \quad \nabla u(B_\delta) \cap \mathcal{B}_{3\rho}(p_0) = \emptyset.$$

In order to prove Lemma 3.1, we use the following preliminary Lemma.

Lemma 3.2. *Let u be a minimizer to $I(u)$ in B_2 such that*

$$(3.4) \quad \nabla u(B_1) \cap \mathcal{B}_\varepsilon(p_0) = \emptyset, \quad \nabla u(\partial B_1) \cap \mathcal{B}_\delta(p_0) = \emptyset,$$

with $0 < \varepsilon < \delta$. Then,

$$\nabla u(B_1) \cap \mathcal{B}_\delta(p_0) = \emptyset.$$

Proof. For simplicity, we take $p_0 = 0$.

In formula (2.10), let us choose $e = e_k$ and as test function $\phi = \eta_k(\nabla u)$, with η being 0 outside the ball \mathcal{B}_δ . Notice that ϕ is an admissible test function in view of our hypothesis. We obtain,

$$\int_{B_1} F_{ij}(\nabla u) u_{kj} (\eta_k(\nabla u))_i dx = 0.$$

Hence differentiating and then summing over all k 's, we get

$$(3.5) \quad \int_{B_1} \text{Tr}(D^2F D^2u D^2\eta D^2u) dx = 0.$$

To estimate the integrand above, we use the characteristic polynomial equation for a 2×2 matrix,

$$(3.6) \quad A^2 - (\text{Tr}A)A + (\det A)I = 0.$$

In particular,

$$(3.7) \quad \text{Tr}A = 0 \Rightarrow A^2 = -(\det A)I.$$

Thus, since u solves (2.9), we can apply this identity to $A = D^2FD^2u$ to obtain,

$$(3.8) \quad \begin{aligned} \text{Tr}(D^2FD^2uD^2\eta D^2u) &= \text{Tr}((D^2FD^2u)^2 D^2\eta (D^2F)^{-1}) \\ &= \det D^2F |\det D^2u| \text{Tr}((D^2F)^{-1} D^2\eta) \end{aligned}$$

where in the first equality we used that

$$\text{Tr}(AB) = \text{Tr}(BA).$$

We choose η so that

$$\text{Tr}((D^2F)^{-1} D^2\eta) > 0 \quad \text{in } \mathcal{B}_\delta \setminus \overline{\mathcal{B}}_{\varepsilon/2}.$$

For example, for $\delta = 1$

$$\eta(p) = \begin{cases} e^{-k|p|} - ke^{-k|p|} + (k-1)e^{-k} & |p| \leq 1, \\ 0 & |p| > 1, \end{cases}$$

with k a large enough constant depending on the ellipticity constants for D^2F on \mathcal{B}_δ . Combining (3.5) with (3.8), we get that

$$(3.9) \quad \det D^2u = 0, \quad \text{on } (\nabla u)^{-1}(\mathcal{B}_\delta \setminus \overline{\mathcal{B}}_{\varepsilon/2}) \cap B_1.$$

Since u solves the Euler-Lagrange equation we obtain that $D^2u = 0$ and ∇u is constant on the set above. Hence this set is both open and closed in B_1 , therefore it is either empty (and we are done) or it coincides with B_1 . In the latter case, we conclude by continuity that also $\nabla u(\partial B_1) \cap \mathcal{B}_\delta \neq \emptyset$ and we reach a contradiction. \square

Next we obtain a Caccioppoli-type inequality.

Proposition 3.3. *Let u be a minimizer to $I(u)$ with $\|\nabla u\|_{L^\infty(B_1)} \leq M$. Then,*

$$(3.10) \quad \int_{B_{1/2} \cap (\nabla u)^{-1}(O_{1/n})} |D^2u|^2 dx \leq C$$

for some constant C depending on $n, M, \|\nabla F\|_{L^\infty(\mathcal{B}_M)}$.

Proof. In formula (2.10), let us choose $e = e_k$ and as test function $\phi = \xi^2 F_k(\nabla u)$, with $\xi \in C_0^\infty(B_1)$, $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $B_{1/2}$. We obtain, after summing over all k 's,

$$(3.11) \quad \int_{B_1} F_{ij}(\nabla u) u_{kj} \xi^2 (F_k(\nabla u))_i dx = -2 \int_{B_1} F_{ij}(\nabla u) u_{kj} \xi \xi_i F_k(\nabla u) dx.$$

Now we analyze the left-hand side of (3.11). We use again (3.7) for $A = D^2FD^2u$ and we obtain

$$\begin{aligned} LHS &= \int_{B_1} F_{ij}(\nabla u) u_{kj} \xi^2 F_{kl}(\nabla u) u_{li} dx = \int_{B_1} \xi^2 \text{Tr}((D^2FD^2u)^2) dx \\ &= 2 \int_{B_1} \xi^2 (\det D^2F) |\det D^2u| dx. \end{aligned}$$

Moreover, again using (3.7) we have that

$$|D^2u|^2 \leq |(D^2F)^{-1}|^2 |D^2FD^2u|^2 = |(D^2F)^{-1}|^2 |\det(D^2FD^2u)|$$

and hence,

$$\det D^2F |\det D^2u| \geq c |D^2u|^2 \quad \text{on } (\nabla u)^{-1}(O_{1/n}),$$

for some constant c depending on n . Thus,

$$(3.12) \quad LHS \geq c \int_{B_{1/2} \cap (\nabla u)^{-1}(O_{1/n})} |D^2u|^2 dx$$

with c depending on n .

On the other hand, using integration by parts together with the Euler-Lagrange equation $\partial_k(F_k(\nabla u)) = 0$, we have that the right-hand side of (3.11) is given by

$$\begin{aligned} RHS &= -2 \int_{B_1} \partial_k(F_j(\nabla u)) \xi \xi_i F_k(\nabla u) dx \\ &= 2 \int_{B_1} F_j(\nabla u) \partial_k(\xi \xi_i) F_k(\nabla u) dx, \end{aligned}$$

and hence

$$(3.13) \quad |RHS| \leq C$$

with C depending on $\|\nabla F\|_{L^\infty(\mathcal{B}_M)}$. Combining (3.11), (3.12) and (3.13) we obtain the desired claim. \square

Finally, we are ready to exhibit the proof of the localization Lemma 3.1.

Proof of Lemma 3.1. Let $\delta > 0$ and assume that there exist $x_0, x_1 \in B_\delta$ such that

$$|\nabla u(x_0) - p_0| \geq 4\rho, \quad |\nabla u(x_1) - p_0| < 3\rho.$$

Then, by the maximum principle for each $\delta \leq r \leq 1/2$ there exists a point $x_0^r \in \partial B_r$ such that

$$|\nabla u(x_0^r) - p_0| \geq 4\rho.$$

Also, from Lemma 3.2, for each $\delta \leq r \leq 1/2$ there exists a point $x_1^r \in \partial B_r$ such that

$$|\nabla u(x_1^r) - p_0| < 3\rho.$$

Let \mathcal{G} be a smooth function on \mathbb{R}^2 with $|\nabla \mathcal{G}| \leq 1/\rho$ such that

$$\mathcal{G}(p) = \begin{cases} 0 & |p - p_0| \geq 4\rho, \\ 1 & |p - p_0| \leq 3\rho. \end{cases}$$

We proceed as in Lemma 2.2 to obtain,

$$(3.14) \quad \int_{B_{1/2}} |\nabla(\mathcal{G}(\nabla u))|^2 dx \geq \frac{1}{2\pi} \ln \frac{1}{2\delta}.$$

However, since $\mathcal{B}_{4\rho}(p_0) \cap \overline{\mathcal{B}}_M \subset O_{1/n}$, by the definition of \mathcal{G} we get

$$(3.15) \quad \int_{B_{1/2}} |\nabla(\mathcal{G}(\nabla u))|^2 dx \leq \int_{B_{1/2}} |\nabla \mathcal{G}|^2 |D^2u|^2 dx \leq \frac{1}{\rho^2} \int_{B_{1/2} \cap (\nabla u)^{-1}(O_{1/n})} |D^2u|^2 dx.$$

Therefore, combining (3.14) with (3.15), we obtain a contradiction to the Caccioppoli inequality (3.10) as long as δ is sufficiently small. \square

We now combine all the ingredients above and provide the proof of Theorem 1.2.

Proof of Theorem 1.2. We wish to prove the following claim:

$\forall \varepsilon > 0, \exists \delta > 0, \delta = \delta(\varepsilon, O_{1/n}, M, \|\nabla F\|_{L^\infty(\mathcal{B}_M)})$ such that $\text{diam}\{\nabla u(B_\delta)\} \leq \varepsilon$ for any minimizer u to $I(u)$.

Fix $\varepsilon > 0$. Let $\overline{\mathcal{B}}_M \setminus \bigcup O_{1/n} = \{q_1, \dots, q_m\}$. We cover the set $\mathcal{B}_{2M} \setminus \bigcup_{i=1}^m \mathcal{B}_{5\rho}(q_i)$ with a finite number of balls \mathcal{B}_ρ^k of radius ρ , with ρ small, say $\rho = \varepsilon/5$. Notice that $\mathcal{B}_{4\rho}^k \cap \overline{\mathcal{B}}_M \subset O_{1/n}$, for a large n .

Clearly, since $\nabla u(B_1) \subset \overline{\mathcal{B}}_M$, then $\nabla u(B_1) \cap \mathcal{B}_\rho^k = \emptyset$, for some $\mathcal{B}_\rho^k \subset \mathcal{B}_{2M} \setminus \overline{\mathcal{B}}_M$. Then, according to Lemma 3.1 we have that either

$$(3.16) \quad \nabla u(B_\delta) \subset \mathcal{B}_{4\rho}^k$$

or

$$(3.17) \quad \nabla u(B_\delta) \cap \mathcal{B}_{3\rho}^k = \emptyset,$$

for some $\delta > 0$. If (3.16) occurs, then we reached our conclusion. Otherwise, if (3.17) occurs, we conclude that $\nabla u(B_\delta) \cap \mathcal{B}_\rho^j = \emptyset$, for all \mathcal{B}_ρ^j such that $\mathcal{B}_\rho^j \cap \mathcal{B}_\rho^k \neq \emptyset$. Hence we can apply again Lemma 3.1 to the balls \mathcal{B}_ρ^j . We iterate this argument. If at some step we reach the conclusion (3.16), then we are done. If at each step we reach the conclusion (3.17), then after a finite number of steps (because our balls cover a connected domain) we obtain that for some small $\tilde{\delta}$, $\nabla u(B_{\tilde{\delta}}) \cap \mathcal{B}_\rho^k = \emptyset$ for all k . Since $\nabla u(B_{\tilde{\delta}})$ is connected, we conclude that $\nabla u(B_{\tilde{\delta}}) \subset \mathcal{B}_{5\rho}(q_i)$ for some i . Again we reach the desired conclusion. \square

4. A DEGENERATE OBSTACLE PROBLEM

4.1. The statement of the problem and preliminaries. Let N be a convex (open) polygon in \mathbb{R}^2 with n vertices $\{p_1, \dots, p_n\} = \mathcal{P}$ (also, set $p_{n+1} = p_1$). Let $[p, q]$ be the closed segment joining two points p and q and let (p, q) be the open segment joining them. Denote by ν_i the direction perpendicular to the side $[p_i, p_{i+1}]$, $i = 1, \dots, n$. Finally, let $\mathcal{Q} = \{q_1, \dots, q_m\}$ be a finite subset of N .

Let $F : \overline{N} \rightarrow \mathbb{R}$ be a convex function such that

- (i) $F \in C^2(N \setminus \mathcal{Q})$, $D^2F > 0$ on $N \setminus \mathcal{Q}$;
- (ii) F is bounded.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $\varphi : \partial\Omega \rightarrow \mathbb{R}$ be a function that admits an extension $\tilde{\varphi}$ with $\nabla \tilde{\varphi} \in \overline{N}$.

We consider the following problem (P): minimize the functional

$$I(u) = \int_{\Omega} F(\nabla u) dx,$$

among all Lipschitz competitors u , with $u = \varphi$ on $\partial\Omega$, $\nabla u \in \overline{N}$ (we think of F to be equal to $+\infty$ outside \overline{N} .)

Denote by,

$$\underline{\varphi} = \inf\{v : v = \varphi \text{ on } \partial\Omega, \nabla v \in \overline{N}\};$$

$$\overline{\varphi} = \sup\{v : v = \varphi \text{ on } \partial\Omega, \nabla v \in \overline{N}\}.$$

We refer to $\underline{\varphi}, \overline{\varphi}$ as to respectively, the lower obstacle and upper obstacle.

Also, in what follows we want to distinguish when gradients are close or not to ∂N . One way of doing this is to consider the compactification of N with one point. We adopt a slightly different approach by introducing the following function

$$(4.1) \quad H : \overline{N} \rightarrow S^2$$

such that, H is continuous, ∂N is mapped to a point, and H is a homeomorphism between N and $H(N)$. DEFINE H OUTSIDE N.

Remark. Notice that our minimization problem is equivalent to the following obstacle problem: minimizing

$$\tilde{I}(u) = \int_{\Omega} \tilde{F}(\nabla u) dx,$$

among all competitors v such that $\underline{\varphi} \leq v \leq \overline{\varphi}$, with \tilde{F} a convex extension of F outside \overline{N} . □

4.2. The main results. We state now our main results. Let u be the unique minimizer to the problem (P) .

Our first main result says that the minimizers are C^1 in Ω except on a number of segments which have an end point on $\partial\Omega$ and have directions perpendicular to the sides of N . On these segments the minimizer coincides with one of the lower or upper obstacle.

Theorem 1.3. *If ∇u is discontinuous at $x_0 \in \Omega$, then there exists a direction ν_i perpendicular to one of the sides $[p_i, p_{i+1}]$ such that u is linear on a segment of direction ν_i connecting x_0 and $\partial\Omega$. Precisely, there exists $x_1 = x_0 + t_1\nu_i \in \partial\Omega$ such that*

$$u(x) = u(x_0) + p_i \cdot (x - x_0), \quad \text{for all } x \in [x_0, x_1] \subset \overline{\Omega}.$$

In particular, $u \in C^1$ away from the obstacles.

Our second main result says that if a sequence of points converges to a point of non-differentiability of u then their corresponding gradients approach ∂N . Let H be as in (4.1); the precise statement of our result reads as follows.

Theorem 4.1. *$H(\nabla u)$ is continuous in Ω .*

Minimizers of (P) may develop “flat” regions where they are linear and their gradient belongs to the set of vertices of N . The next theorem describes the shape of these regions.

Theorem 4.2. *Let p_i be a vertex of N and let ω_i be a direction so that*

$$\omega_i \cdot (p - p_i) > 0 \quad \forall p \in \overline{N} \setminus \{p_i\}.$$

Let S be the interior of the set $\{u(x) = c + p_i \cdot x\}$ and assume that $S \neq \emptyset$. Then, $\partial S \cap \Omega$ consists of a convex graph (by above) and a concave graph (by below) in the ω_i direction.

Finally, the next two results deal with the case when u is a perturbation of a linear function. They will be used in the proof of our main Theorem 1.3.

Theorem 4.3. *Assume $B_1 \subset \Omega$ and*

$$\mathcal{B}_\delta(p_0) \subset N.$$

Then, there exists ε depending on δ, p_0, F , such that if

$$|u - p_0 \cdot x| \leq \varepsilon, \quad x \in B_1$$

then $u \in C^1(B_{1/2})$ and

$$\nabla u(B_{1/2}) \subset \mathcal{B}_\delta(p_0).$$

Theorem 4.4. *Assume $B_1 \subset \Omega$ and let $p_0 \in \partial N$. Then, there exists ε depending on δ, F , such that if*

$$|u - p_0 \cdot x| \leq \varepsilon, \quad x \in B_1$$

then

$$[\nabla u(x), p_0] \text{ is in a } \delta\text{-neighborhood of } \partial N, \text{ for a.e. } x \in B_{1/2}.$$

We finish this Section by showing that minimizers to the problem (P) are unique. We use the abbreviation L.P. for Lebesgue point.

Proposition 4.5. *The minimization problem (P) admits a unique solution.*

Proof. Let u_1, u_2 be two distinct solutions. By the convexity of F , we have that $(u_1 + u_2)/2$ is also a solution and

$$\frac{F(\nabla u_1) + F(\nabla u_2)}{2} = F\left(\frac{\nabla u_1 + \nabla u_2}{2}\right), \quad \text{a.e. in } \Omega.$$

Hence, since F is strictly convex in N , we have that if x is a L.P. for ∇u_1 and ∇u_2 then either

$$(4.2) \quad \nabla u_1(x) = \nabla u_2(x)$$

or

$$(4.3) \quad \nabla u_1(x), \nabla u_2(x) \in [p_i, p_{i+1}], \quad \text{for some } i = 1, \dots, n.$$

Now, let us assume by contradiction that $u_1(0) > u_2(0)$, $0 \in \Omega$, say $u_1(0) - u_2(0) = \varepsilon$. Then, since u_1, u_2 are Lipschitz and they coincide on the boundary, there exists $\rho = c\varepsilon$ such that if $|\tau| < \rho$, then

$$B_\rho \subset \{u_2(x + \tau) + \varepsilon/2 < u_1\} \subset \subset \Omega.$$

The minimality of $u_2(x + \tau)$ together with the inclusion above imply that

$$v_\tau(x) = \min\{u_1(x), u_2(x + \tau) + \varepsilon/2\}$$

is also a minimizer of the problem (P).

Thus, if $x_1 \in B_{\rho/2}$ is a L.P. for ∇u_1 and $x_2 \in B_{\rho/2}$ is a L.P. for ∇u_2 , we can translate x_2 by τ to coincide with x_1 , so that x_1 is a L.P. for v_τ and according to (4.2)-(4.3) we have either

$$(4.4) \quad \nabla u_1(x_1) = \nabla u_2(x_2)$$

or

$$(4.5) \quad \nabla u_1(x_1), \nabla u_2(x_2) \in [p_i, p_{i+1}], \quad \text{for some } i = 1, \dots, n.$$

Similarly, by appropriate translations we obtain that if $x_1 \in B_{\rho/2}$ is a L.P. for ∇u_1 and $x_2, x_3, x_4 \in B_{\rho/2}$ are L.P. for ∇u_2 then

$$\begin{aligned} & F\left(\frac{1}{4}(\nabla u_1(x_1) + \nabla u_2(x_2) + \nabla u_2(x_3) + \nabla u_2(x_4))\right) \\ &= \frac{1}{4}(F(\nabla u_1(x_1)) + F(\nabla u_2(x_2)) + F(\nabla u_2(x_3)) + F(\nabla u_2(x_4))). \end{aligned}$$

Hence

$$(4.6) \quad \nabla u_1(x_1), \nabla u_2(x_2), \nabla u_2(x_3), \nabla u_2(x_4) \text{ are collinear.}$$

We distinguish two cases.

Case 1. There exists at least one L.P. $\bar{x} \in B_{\rho/2}$ for either ∇u_1 or ∇u_2 such that $\bar{p} = \nabla u_i(\bar{x}) \in N$. Then, we conclude that $\nabla u_1 = \nabla u_2 = \bar{p}$ in $B_{\rho/2}$. Hence $u_1 - u_2 = \varepsilon$ on $B_{\rho/2}$. Now we can proceed as above with 0 replaced by any point in $B_{\rho/2}$. By iterating this argument a finite number of times we conclude that $\{u_1 - u_2 = \varepsilon\}$ must coincide with Ω . This contradicts that $u_1 = u_2$ on $\partial\Omega$.

Case 2. All Lebesgue points $x \in B_{\rho/2}$ for $\nabla u_i, i = 1, 2$ are mapped on ∂N .

First we claim that all Lebesgue points $x \in B_{\rho/2}$ for $\nabla u_i, i = 1, 2$ are mapped on the same side. Indeed, according to (4.6) any three such points for ∇u_2 are collinear. The claim follows by interchanging u_2 and u_1 .

Thus all Lebesgue points of $\nabla u_1, \nabla u_2$ in $B_{\rho/2}$ must lie on the same side, say $[p_1, p_2]$. Therefore $u_1 - u_2$ is constant ε in $B_{\rho/2}$ on the segment with middle point at zero, in the direction perpendicular to $[p_1, p_2]$. This implies that the function $\nu \cdot x$ cannot achieve a maximum (minimum) on the set $\{u_1 - u_2 = \varepsilon\}$, where ν is a direction which differs from any of the perpendicular directions to the sides $[p_i, p_{i+1}]$. This contradicts that Ω is bounded. □

5. THE APPROXIMATION.

Let \bar{F} be a convex function on \mathbb{R}^2 , $\bar{F} = 0$ on N , $\bar{F}(p) = |p|^3$ for $|p|$ large, $\bar{F} \in C^\infty(\mathbb{R}^2 \setminus \bar{N})$ with $D^2\bar{F} > 0$ on $\mathbb{R}^2 \setminus \bar{N}$.

Let $F_m \in C^\infty(\mathbb{R}^2)$ with $D^2F_m > 0$, be such that

- (i) $F_m \rightarrow F$ uniformly on \bar{N} ;
- (ii) $D^2F_m \rightarrow D^2F$ uniformly on compacts on $N \setminus \mathcal{Q}$;
- (iii) $F_m(p) = C_m\bar{F}$ in $D_m := \{p \in \mathbb{R}^2 \mid \bar{F}(p) > 1/m\}$, with $C_m \rightarrow +\infty$.

Let u_m be the minimizer to

$$\int_{\Omega} F_m(\nabla v) dx$$

among all competitors $v = \varphi$ on $\partial\Omega$. Here φ is the boundary data for the minimizer u of the problem (P).

We will show that the u_m 's are a smooth approximation u . Using the lower obstacle $\underline{\varphi}$ as competitor, we see that $\int_{\Omega} F_m(\nabla u_m) dx$ is bounded above by a fixed constant. Hence, since $C_m > 1$ we conclude

$$\int_{\Omega} |\nabla u_m|^3 dx \leq C.$$

By Sobolev embedding theorem we have that u_m is uniformly Holder continuous in $\bar{\Omega}$. Thus, by Ascoli-Arzelà (a subsequence of) u_m converges uniformly on $\bar{\Omega}$ to a function u . First we show the following Proposition (recall the definition (4.1) of H from the previous Section).

Proposition 5.1. *u is the minimizer to the problem (P). Also*

$$\nabla u_m \rightarrow \nabla u \text{ in measure on } A := \{x|x \text{ is a L.P. for } \nabla u \text{ and } \nabla u(x) \in N\},$$

and

$$H(\nabla u_m) \rightarrow H(\nabla u) \text{ in measure on } \Omega.$$

Proof. Let v be a function on $\bar{\Omega}$, $v = \varphi$ on $\partial\Omega$ and $\nabla v \in \bar{N}$. Let \tilde{F} be a convex function on \mathbb{R}^2 with bounded gradient, which approximates F in N . Then

$$\begin{aligned} \int_{\Omega} F(\nabla v) dx &\geq \int_{\Omega} F_m(\nabla v) dx - \varepsilon \geq \int_{\Omega} F_m(\nabla u_m) dx - \varepsilon \geq \int_{\Omega} \tilde{F}(\nabla u_m) dx - 2\varepsilon \\ &\geq \int_{\Omega} (\tilde{F}(\nabla u) + \nabla \tilde{F}(\nabla u)(\nabla u_m - \nabla u) + \omega_{\tilde{F}}(\nabla u, \nabla u_m - \nabla u)) dx - 2\varepsilon, \end{aligned}$$

with $\omega_{\tilde{F}} \geq 0$ the modulus of convexity of \tilde{F} . Using that $\nabla u_m \rightarrow \nabla u$ weakly in L^3 , we conclude that

$$\int_{\Omega} F(\nabla v) dx \geq \int_{\Omega} \tilde{F}(\nabla u) dx - 2\varepsilon.$$

Since \tilde{F} is arbitrary outside \bar{N} , we deduce that $\nabla u \in \bar{N}$ and u is the minimizer of the problem (P).

Since F is strictly convex in N and

$$\int_{\Omega} \omega_{\tilde{F}}(\nabla u, \nabla u_m - \nabla u) dx \rightarrow 0, \quad m \rightarrow \infty$$

we also obtain that ∇u_m converges in measure to ∇u in A , and $H(\nabla u_m)$ converges in measure to $H(\nabla u)$. □

We continue with the following Proposition.

Proposition 5.2. *$u_m \in C^\infty(\Omega)$. Also, for any compact $K \subset \Omega$ and $\delta > 0$, $\nabla u_m(K)$ is in a δ -neighborhood of \bar{N} for m large enough.*

Proof. Using the lower (or upper) obstacle $\underline{\varphi}$ as competitor, we see that $\int_{\Omega} F_m(\nabla u_m) dx$ is bounded above by a fixed constant. Hence,

$$(5.1) \quad \int_{\Omega} \bar{F}(\nabla u_m) dx \rightarrow 0, \quad \text{as } m \rightarrow +\infty.$$

For notational simplicity we denote $F_m = \tilde{F}$ and $u_m = \tilde{u}$. We have

$$\int_{\Omega} \tilde{F}_{ij}(\nabla \tilde{u}) \tilde{u}_{kj} \phi_i dx = 0, \quad \forall \phi \in C_0^\infty(\Omega).$$

In this formula, let us choose $\phi = \xi^2 \eta(\nabla \tilde{u}) \eta_k(\nabla \tilde{u})$, with η a convex function, $\eta = 0$ outside D_{m_0} for some fixed m_0 , $\eta(p) = |p|$ for large $|p|$, and $\xi \in C_0^\infty(\Omega)$.

If $m > m_0$ we have, (for notational simplicity we drop the dependence on $\nabla \tilde{u}$)

$$(5.2) \quad \int_{\Omega} \tilde{F}_{ij} \tilde{u}_{kj} \xi^2 (\eta \eta_k)_i dx = -2 \int_{\Omega} \tilde{F}_{ij} \tilde{u}_{kj} \xi \xi_i \eta \eta_k dx.$$

Since η is convex,

$$\tilde{F}_{ij} \tilde{u}_{kj} \eta_{kl} \tilde{u}_{li} \geq 0.$$

Also, in D_{m_0} we have

$$\lambda_{m_0}(1 + \eta)I \leq D^2 \tilde{F} \leq \Lambda_{m_0}(1 + \eta)I.$$

Combining these facts with (5.2) we deduce that the function $\psi = \eta^{3/2} + \eta$ satisfies the following Caccioppoli-type inequality,

$$(5.3) \quad \int_{\Omega} |\nabla \psi|^2 \xi^2 \leq C(m_0) \int_{\Omega} \psi^2 |\nabla \xi|^2.$$

By the classical theory we get that for any compact $K \subset \Omega$,

$$\|\psi(\nabla u_m)\|_{L^\infty(K)} \leq C(m_0, K) \|\psi(\nabla u_m)\|_{L^2(\Omega)} \leq C(m_0, K) \left(\int_{\Omega} \bar{F}(\nabla u_m) dx \right)^{1/2}.$$

In the set where ∇u_m is bounded above we obtain that $u_m \in C^\infty$ by the standard theory. The second part follows from (5.1) (recall that $\psi \equiv 0$ on D_{m_0}). \square

Remark. The smoothness of u_m follows also from the fact that u_m solves a uniformly elliptic equation in 2D, therefore u_m is $C^{1,\alpha}$ in the interior and hence it is C^∞ . \square

Our analysis will rely on the following classical theorem that was also used in other two dimensional results (see for example [GT], [H], [S]).

Theorem 5.3. *Let v be a solution to*

$$a_{ij}(x)v_{ij}(x) = 0 \quad \text{in } D \subset \mathbb{R}^2,$$

with $A(x) = a_{ij}(x) > 0$, $A(x) \in C^\infty$, and D simply connected. Assume v is not linear. Then in each neighborhood U there exists a point x_U such that each set

$$\{v > l_U\}, \quad \{v < l_U\}$$

with

$$l_U(x) := v(x_U) + \nabla v(x_U) \cdot (x - x_U)$$

has at least two connected components in D that intersect U . Moreover these components are not compactly supported in D .

Proof. Let $x_U \in U$ be such that $\det D^2 v(x_U) < 0$. Such a point exists otherwise if $\det D^2 v = 0$ in U , then $D^2 v = 0$ in U and by unique continuation v is linear in D .

Clearly, the sets $\{v > l_U\}, \{v < l_U\}$ intersect a small ball around x_U precisely in four disjoint connected components.

On the other hand, it follows from the maximum principle that any connected component of the sets $\{v > l_U\}, \{v < l_U\}$ cannot be compactly supported in D .

Hence, since $D \subset \mathbb{R}^2$ and D is simply connected, the components in the small ball belong to four disjoint connected components in D . \square

6. THE PROOF OF THEOREM 1.3.

In this Section we exhibit the proof of Theorem 1.3. We start by obtaining a result to which we refer as to the localization Theorem. From now on we tacitly assume that our statements hold for all m sufficiently large.

Theorem 6.1. *Assume $B_1 \subset \Omega$ and*

$$(6.1) \quad \nabla u_m(B_1) \cap \mathcal{B}_\rho(p_0) = \emptyset, \quad p_0 \in \overline{N}.$$

Then for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, F, \mathcal{B}_\rho(p_0))$ such that either

$$\nabla u_m(B_\delta) \subset \mathcal{B}_\varepsilon(p), \text{ for some } p \in N,$$

or

$$\nabla u_m(B_\delta) \subset \mathcal{N}_\varepsilon$$

with \mathcal{N}_ε the ε neighborhood of ∂N .

Remark. Another way of stating the conclusion of this theorem is to say that $H(\nabla u_m)$ is continuous with a uniform modulus of continuity.

Also recall that $\nabla u_m \rightarrow \nabla u$ in measure, thus if the hypothesis of Theorem 6.1 holds for all large m 's, then u satisfies the same conclusion of the theorem. \square

Proof. Assume

$$\nabla u_m(B_1) \cap \mathcal{B}_r(p) = \emptyset, \quad \overline{\mathcal{B}_{4r}(p)} \subset N \setminus \mathcal{Q}.$$

We wish to prove that there exists $\delta > 0$ depending on $F, \mathcal{B}_r(p)$, such that either

$$(6.2) \quad \nabla u_m(B_\delta) \subset \mathcal{B}_{4r}(p),$$

or

$$(6.3) \quad \nabla u_m(B_\delta) \cap \mathcal{B}_{3r}(p) = \emptyset.$$

We argue similarly as for Lemma 3.1. It suffices to prove the following Caccioppoli-type inequality,

$$(6.4) \quad \int_{B_{1/2} \cap (\nabla u_m)^{-1}(\mathcal{B}_{4r}(p))} |D^2 u_m|^2 dx \leq C,$$

for some constant C depending on $F, \mathcal{B}_r(p)$.

For notational simplicity, let $F_m = \tilde{F}$ and $u_m = \tilde{u}$. We have

$$(6.5) \quad \int_{B_1} \tilde{F}_{ij}(\nabla \tilde{u}) \tilde{u}_{kj} \phi_i dx = 0, \quad \forall \phi \in C_0^\infty(B_1).$$

Let us choose $\phi = \xi^2 \eta_k(\nabla \tilde{u})$, with η compactly supported on $\overline{\mathcal{B}_{4r}(p)}$ and $\xi \in C_0^\infty(B_1), 0 \leq \xi \leq 1, \xi \equiv 1$ on $B_{1/2}$. We obtain, after summing over all k 's (when clear, we drop the dependence on $\nabla \tilde{u}$),

$$(6.6) \quad \int_{B_1} \tilde{F}_{ij} \tilde{u}_{kj} \xi^2 (\eta_k(\nabla \tilde{u}))_i dx = -2 \int_{B_1} \tilde{F}_{ij} u_{kj} \xi \xi_i \eta_k dx.$$

Now we analyze the left-hand side of (6.6). We proceed as in Lemma 3.2 and we obtain

$$\begin{aligned} LHS &= \int_{B_1} \tilde{F}_{ij} \tilde{u}_{kj} \xi^2 \eta_{kl} \tilde{u}_l dx \\ &= \int_{B_1} \xi^2 (\det D^2 \tilde{F}) |\det D^2 \tilde{u}| \text{Tr}((D^2 \tilde{F})^{-1} D^2 \eta) dx. \end{aligned}$$

Choose η such that

$$\text{Tr}((D^2 \tilde{F})^{-1} D^2 \eta) \geq c, \quad \text{on } \mathcal{B}_{4r}(p) \setminus \mathcal{B}_r(p),$$

for a small constant c depending on the ellipticity constants of $D^2 F$ on $\mathcal{B}_{4r}(p)$, say λ, Λ . We conclude

$$(6.7) \quad LHS \geq c \int_{B_1 \cap (\nabla \tilde{u})^{-1}(\mathcal{B}_{4r}(p))} |D^2 \tilde{u}|^2 \xi^2 dx$$

with c depending on λ, Λ .

On the other hand, the right-hand side of (6.6) is bounded by

$$(6.8) \quad |RHS| \leq \gamma C \int_{B_1 \cap (\nabla \tilde{u})^{-1}(\mathcal{B}_{4r}(p))} |D^2 \tilde{u}|^2 \xi^2 dx + \tilde{C}$$

with C depending on Λ . Combining (6.6), (6.7) and (6.8) we obtain the desired inequality.

We now proceed similarly as in the proof of Theorem 1.2. We cover the set $N \setminus (\mathcal{N}_\varepsilon \cup \mathcal{B}_\varepsilon(q_i))$ as a finite union of balls \mathcal{B}_r^k of radius $r = \varepsilon/5$ (with centers in the set). From our assumption, there exists k such that

$$\nabla u_m(B_1) \cap \mathcal{B}_r^k = \emptyset,$$

provided that ε is small enough depending on ρ, F . Now the conclusion follows from the same iteration argument as in Theorem 1.2 (notice that by Proposition 5.2, $\nabla u_m(B_\delta)$ is in an ε neighborhood of \bar{N} for large m). \square

6.1. The proof of Theorem 1.3. We now present a series of Propositions which will all be combined towards the proof of Theorem 1.3. We start by stating two Propositions for the approximation u_m which correspond to the flatness Theorems 4.3-4.4 for the minimizer u . We present their proofs in the last Section.

Proposition 6.2. *Assume $B_1 \subset \Omega$ and*

$$\mathcal{B}_\delta(p_0) \subset N.$$

Then, there exists ε depending on δ, p_0, F , such that if

$$|u_m - p_0 \cdot x| \leq \varepsilon, \quad x \in B_1$$

then

$$\nabla u_m(B_{1/2}) \subset \mathcal{B}_\delta(p_0).$$

Proposition 6.3. *Let $p_0 \in \partial N$. Then, there exists ε depending on δ, F , such that if*

$$|u_m - p_0 \cdot x| \leq \varepsilon, \quad x \in B_1$$

then

$$[\nabla u_m(x), p_0] \text{ is in a } \delta\text{-neighborhood of } \partial N, \text{ for all } x \in B_{1/2}.$$

Next, we present another Proposition which will be used to prove Theorem 1.3, and relies on the previous flatness results.

Proposition 6.4. *Assume $0 \in (p_1, p_2)$ and e_2 is normal to (p_1, p_2) and points inside N . Let $0 \in [a_1, a_2] \subset (p_1, p_2)$ and assume that for each $r > 0$ and $p \in [a_1, a_2]$, there exists a sequence of $m \rightarrow \infty$ such that*

$$\nabla u_m(B_r) \cap \mathcal{B}_r(p) \neq \emptyset.$$

Then u is constant on a segment of direction e_2 connecting 0 and $\partial\Omega$. Precisely, there exists $\tilde{x} = se_2 \in \partial\Omega$ such that

$$u(x) = u(0), \quad \text{for all } x \in [0, \tilde{x}] \subset \bar{\Omega}.$$

In order to prove Proposition 6.4 we will need the following Lemma.

Lemma 6.5. *Assume $0 \in \partial N$, e_2 points inside N ,*

$$R := \{|x_1| < \delta, |x_2| < 1\} \Subset \Omega,$$

and

$$u(x) > 0, \quad x \in \bar{R} \cap \{x_2 = 1\}, \quad u(x) < 0, \quad x \in \bar{R} \cap \{x_2 = -1\}, \quad u(0) = 0.$$

Assume that for each $r > 0$, there exists a sequence of $m \rightarrow \infty$ such that

$$\nabla u_m(B_r) \cap \mathcal{B}_r(0) \neq \emptyset.$$

Then either the set

$$\{u = 0\} \cap \{x_1 > 0\}$$

or the set

$$\{u = 0\} \cap \{x_1 < 0\}$$

is given by the region between a convex graph (from above) and a concave graph (from below) in the e_2 direction.

Proof. The lemma holds trivially if u is linear. Assume u is not linear. Then, for m large enough, also u_m is not linear. Then, by the assumptions together with Theorem 5.3, we have that there exists $x_m \rightarrow 0$ with $\nabla u_m(x_m) \rightarrow 0$ such that the set

$$\{u_m < l_m := u_m(x_m) + \nabla u_m(x_m) \cdot (x - x_m)\}$$

has at least two distinct components in \bar{R} . One of the components that does not contain the segment $\bar{R} \cap \{x_2 = -1\}$ must intersect one of the lateral sides, say $x_1 = \delta$ (for infinitely many m 's.) Then we can find a polygonal line connecting any neighborhood of x_m with $x_1 = \delta$ that is included in this component.

Now, for each $t \in (0, \delta]$ we define $\underline{h}(t), \bar{h}(t)$ to be the points in \mathbb{R}^2 on the line $x_1 = t$ such that

$$[\underline{h}(t), \bar{h}(t)] = \{u = 0\} \cap \{x_1 = t\},$$

(notice that u is increasing in the e_2 direction.) Let $c \in (a, b) \subset (0, \delta]$.

Claim. The segment $[\bar{h}(a), \bar{h}(b)]$ is above $\underline{h}(c)$ in the e_2 direction.

Indeed, assume by contradiction that our claim does not hold. Then, there exists a linear function l increasing in the e_2 direction, such that

$$[\bar{h}(a), \bar{h}(b)] \subset \{l < -1\}, \quad \underline{h}(c) \in \{l > 1\}.$$

Let $R_{ab} := \{a < t < b, |x_2| < 1\}$. Denote by U_m the connected component of $\{u_m < l_m\}$ in $\overline{R_{ab}}$ that contains the segment $\overline{R_{ab}} \cap \{x_2 = -1\}$. We compare u_m and $l_m + \varepsilon l$ in the set $\overline{R_{ab}} \setminus U_m$, with ε small enough depending on u and l . We have

$$u - \varepsilon l > 0 \text{ on } \overline{R_{ab}} \cap \{x_2 = 1\}$$

and

$$u - \varepsilon l > 0 \text{ on } \{u \geq -\varepsilon/2\} \cap (\{x_1 = a\} \cup \{x_1 = b\}).$$

Notice that, since $u_m \rightarrow u, l_m \rightarrow 0$ uniformly, we have that

$$\{u < -\varepsilon/2\} \cap \overline{R_{ab}} \subset U_m$$

for all m large enough, thus

$$u_m - l_m - \varepsilon l > 0 \text{ on } \partial R_{ab} \setminus U_m.$$

On the other hand, u is strictly negative on the segment $\{x_1 = c\} \cap \{l < 0\} \cap \overline{R_{ab}}$. Thus this segment is included in U_m , which implies that

$$u_m - l_m - \varepsilon l < 0 \text{ on } \partial U \cap \{x_1 = c\} \cap R_{ab}.$$

Hence the minimum of $u_m - l_m - \varepsilon l$ in $\overline{R_{ab}} \setminus U_m$ is negative and by the maximum principle it occurs at some point $x_0 \in \partial U_m \cap R_{ab}$. Thus,

$$(6.9) \quad u_m - l_m \geq \varepsilon(l - l(x_0)) \text{ in } \overline{R_{ab}} \setminus U_m.$$

Recall that there exists a polygonal line included in $(\overline{R_{ab}} \setminus U_m) \cap \{u_m < l_m\}$ that connects the lines $x_1 = a$ and $x_2 = b$. Now the right-hand side of (6.9) is increasing in the e_2 direction and we obtain a contradiction at a point where $x_0 + se_2, s \geq 0$ intersects this polygonal line. Thus the claim is proved.

Next we prove that $\overline{h}(t)$ is a convex curve and $\underline{h}(t)$ is a concave curve, if $t > 0$. Indeed, let $0 < t_1 < t_2 \leq \delta$, and let Q be the convex set generated by $\overline{h}(t_i), \underline{h}(t_i), i = 1, 2$. From the claim above we see that

$$Q \cap [\underline{h}(t), \overline{h}(t)] \neq \emptyset, \quad t_1 \leq t \leq t_2.$$

Thus,

$$u \leq 0 \text{ on } [\underline{h}(t_1), \underline{h}(t_2)].$$

Let l' be the linear function that is 0 on the line passing through $\underline{h}(t_i) - \gamma e_2, i = 1, 2$ and has slope ε in the e_2 direction. Clearly, if ε is small depending on u and γ , then $u \leq l'$ on the boundary of $\{t_1 < x_1 < t_2, x_2 \geq -1\} \cap \{l' \leq 0\}$. Hence the same inequality is true in the interior. Thus

$$u < 0, \quad \text{below } [\underline{h}(t_1), \underline{h}(t_2)] - \gamma e_2.$$

By letting γ tend to 0 and repeating the same argument from above, we find that $[\underline{h}(t), \overline{h}(t)] \subset Q$. □

Let $[a_1, a_2] \subset (p_1, p_2)$ with e_2 normal to (p_1, p_2) and pointing inside N . For all $p \in [a_1, a_2]$ with the property that for each $r > 0$ there exists a sequence of $m \rightarrow \infty$ such that

$$\nabla u_m(B_r) \cap \mathcal{B}_r(p) \neq \emptyset,$$

we define

$$(6.10) \quad C_p := \{u = p \cdot x\}.$$

Assume that $R := \{|x_1| < \delta, |x_2| < 1\} \subset \Omega$ with

$$u > 0 \text{ on } \overline{R} \cap \{x_2 = 1\}, \quad u < 0, \text{ on } \overline{R} \cap \{x_2 = -1\}, \quad u(0) = 0.$$

Then for each $p \in [a_1, a_2]$, possibly by taking δ smaller, we also have

$$u(x) > p \cdot x, \text{ on } \overline{R} \cap \{x_2 = 1\}, \quad u(x) < p \cdot x, \text{ on } \overline{R} \cap \{x_2 = -1\}.$$

Thus applying Lemma 6.5 (with the origin replaced by p) for each slope p as above, either the set

$$C_p^+ := C_p \cap \{0 < x_1 < \delta\}$$

or the set

$$C_p^- := C_p \cap \{0 < x_1 < \delta\}$$

is given by the region between a convex function and a concave function. Let A^+ (resp. A^-) be the set of $p \in [a_1, a_2]$ such C_p^+ (resp. C_p^-) is given by such region.

With this notation, we state and prove the following Lemma which will be used for the proof of Proposition 6.4.

Lemma 6.6. *Let $p \in A^+$ and $\alpha \in (0, \delta)$. Then, for any neighborhoods V of $C_p^+ \cap \{x_1 = \alpha\}$ and W of p , there exist a point $\tilde{x} \in V$ and m large such that $\nabla u_m(\tilde{x}) \in W$.*

Proof. Without loss of generality we assume $p = 0 \in A^+$. We refer to the proof of the previous lemma. We can assume that u_m is not linear, otherwise the statement is trivial. Let $\alpha \in (a, b) \subset (0, \delta)$, and let us focus on the connected component \tilde{U}_m of $\{u_m < l_m\}$ which contains the polygonal line connecting $x_1 = a$ with $x_1 = b$. Since $u_m \rightarrow u, l_m \rightarrow 0$ uniformly, we obtain that \tilde{U}_m is in any neighborhood of C_p , if m is large enough. Now, consider the function $u_m - l_m + \frac{1}{2}(x_1 - \alpha)^2$. Then the minimum of this function in \tilde{U}_m is negative and is achieved at an interior point \tilde{x} . The desired conclusion follows by taking the interval (a, b) sufficiently small. \square

We are now ready to exhibit the proof of Proposition 6.4.

Proof of Proposition 6.4. We take $u(0) = 0$ and use the notation of the previous Lemmas. Assume by contradiction that $\{u = 0\}$ does not contain either of the segments in the direction e_2 connecting 0 with $\partial\Omega$. Then there exists δ small such that $R \Subset \Omega$, $u > 0$ on $\overline{R} \cap \{x_2 = 1\}$ and $u < 0$ on $\overline{R} \cap \{x_2 = -1\}$. Therefore, given $[a_1, a_2] \subset (p_1, p_2)$, we can define A^+ and A^- as above (see discussion before Lemma 6.6.)

Since $\overline{A^+} \cup \overline{A^-} = [a_1, a_2]$, then there exists an open interval contained in either $\overline{A^+}$ or $\overline{A^-}$, say $(-\rho e_1, \rho e_1) \subset \overline{A^+}$.

Now set

$$D_t := \{u = t e_1 \cdot x\} \cap \{0 < x_1 < \delta\}, \quad -\rho < t < \rho.$$

Notice that if $t e_1 \in A^+$ then D_t is exactly $C_{t e_1}^+$.

We claim that D_t is precisely the region between two segments with one endpoint on $\{x_1 = 0\}$ and the other on $\{x_1 = \delta\}$.

Indeed, let $t_n e_1 \in A^+, t_n \uparrow t \in (-\rho, \rho)$, t_n strictly increasing. Notice that, if $x_1 > 0$

$$(6.11) \quad \{u < t x_1\} = \bigcup_n \{u < t_n x_1\} = \bigcup_n \{u \leq t_n x_1\}.$$

Using that $C_{t_n e_1}^+$ is the region between a convex and a concave graph, we obtain from (6.11) that $\partial\{u < tx_1\} \cap \{0 < x_1 < \delta\}$ is both a concave and a convex graph. This implies the claim.

Next, choose a point of differentiability x^* for u in the open set

$$\{-\rho e_1 \cdot x < u < \rho e_1 \cdot x\} \cap \{0 < x_1 < \delta\},$$

such that

$$(6.12) \quad \nabla u(x^*) \cdot e_2 > \gamma > 0.$$

Set $p^* = \nabla u(x^*)$. Without loss of generality we can assume $u(x^*) = 0$ and thus the set D_0 consists of just one segment. Therefore, if $t_n \uparrow 0$, t_n increasing, $t_n e_1 \in A^+$, then

$$(6.13) \quad C_{t_n e_1}^+ \cap \{e_1 \cdot (x - x^*) = 0\} \rightarrow x^*.$$

Since x^* is a point of differentiability of u and $u_m \rightarrow u$ uniformly, we obtain that for any $\varepsilon > 0$ we can find ρ, m such that

$$|u_m - p^* \cdot (x - x^*)| \leq \varepsilon \rho \quad \text{in } B_\rho(x^*).$$

On the other hand from (6.13) together with Lemma 6.6, we can find a point $x_m \in B_{\rho/2}$ such that

$$|\nabla u_m(x_m)| \leq \gamma/2.$$

Then, using (6.12) we see that the middle point of $[\nabla u_m(x_m), p^*]$ is outside a $\gamma/4$ neighborhood of $\partial N \cup \{p^*\}$ (provided that γ is chosen small enough depending on N). This contradicts either Proposition 6.2 (if $p^* \in N$) or Proposition 6.3 (if $p^* \in \partial N$), by choosing $\varepsilon = \varepsilon(\gamma, p^*, N)$ sufficiently small. \square

Now, we are finally ready to present the proof of Theorem 1.3.

Proof of Theorem 1.3. For simplicity take $x_0 = 0$. Assume that there are no segments connecting 0 with $\partial\Omega$ as in the statement of the Theorem. We wish to prove that given $\rho > 0$ there exists δ (depending also on u) small such that $\text{diam}(\nabla u(B_\delta)) \leq \rho$.

By Proposition 6.4 for each side $[p_i, p_{i+1}]$ and each subinterval $(a_i, a_{i+1}) \subset [p_i, p_{i+1}]$ there exist a point $a \in (a_i, a_{i+1})$ and $r > 0$ such that

$$\nabla u_m(B_r) \cap \mathcal{B}_r(a) = \emptyset.$$

Clearly we can find $\varepsilon > 0$ and points $a_k \in \partial N$ such that any ball $\mathcal{B}_{\rho/2}(p)$ centered at $p \in \partial N$ contains one of the balls $\mathcal{B}_\varepsilon(a_k)$ and

$$\nabla u_m(B_\varepsilon) \cap \mathcal{B}_\varepsilon(a_k) = \emptyset.$$

We cover the open set

$$O = \{x \in N \mid \text{dist}(x, \partial N) > \varepsilon/2\}$$

with a finite number of balls of radius $\varepsilon/10$ centered in the set. According to the localization Theorem 6.1, if δ is small enough depending only on F, ε and say $\mathcal{B}_\varepsilon(a_1)$ we find that either

$$(6.14) \quad \text{diam}(\nabla u_m(B_\delta)) \leq \varepsilon/5$$

or

$$(6.15) \quad \nabla u_m(B_\delta) \cap O = \emptyset.$$

In the latter case we know from Proposition 5.2 that for m large enough

$$(6.16) \quad \nabla u_m(B_\delta) \text{ is included in a } \varepsilon/2\text{-neighborhood of } N.$$

Since $\nabla u_m(B_\delta)$ is connected, we conclude from the choice of ε , (6.15) and (6.16) that

$$\text{diam}(\nabla u_m(B_\delta)) \leq \rho.$$

The conclusion follows by letting $m \rightarrow \infty$. □

We conclude this Section with the proof of Theorem 4.2.

Proof of Theorem 4.2. For simplicity assume $p_i = 0$, $c = 0$ and $\omega = e_2$. Since u is increasing in a cone of directions around e_2 we conclude that S is between two Lipschitz graphs in the e_2 direction. Assume by contradiction that the lower graph is not concave. Then we can find two points z_1, z_2 such that $u(z_i) < 0$ and the segment $[z_1, z_2]$ is tangent from below to S . Let l be the linear function which vanishes at the points $z_i + \varepsilon e_2$ with $\nabla l \cdot e_2 = \varepsilon$. We compare u and l in the set $\{x \cdot z_1 < x_1 < x \cdot z_2\} \cap \{l < 0\} \cap \Omega$. If ε is small enough then $u \leq l$ on the boundary, hence $u \leq l$ in this set. This implies that S is above the segment $[z_1, z_2] + \varepsilon e_2$ and we contradict the fact that $[z_1, z_2]$ was tangent to S . □

7. THE PROOF OF THEOREM 4.1

We distinguish two cases, when the polygon N has $n \geq 4$ vertices and when it has only $n = 3$ vertices. The latter is more involved and we only present a sketch of the proof.

Lemma 7.1. *Assume N has more than 3 vertices. Then $H(\nabla u)$ is continuous.*

Proof. If $H(\nabla u)$ is not continuous at the origin, then for each side $[p_i, p_{i+1}]$ there exists a segment of direction ν_i perpendicular to that side, starting from the origin and ending at $x_i \in \partial\Omega$ such that $u(x) = u(0) + p_i \cdot x, x \in [0, x_i]$. This follows from the localization Theorem and from Proposition 6.4.

For simplicity of exposition, assume that

$$p_1 = |p_1|(-\cos \theta, \sin \theta), \quad p_2 = 0, \quad p_3 = |p_3|(\cos \theta, \sin \theta), \quad \theta \in (0, \pi/2).$$

Recall that in our notation ν_i is either of the two directions perpendicular to $[p_i, p_{i+1}]$. Let us choose ν_1, ν_2 pointing inside N , that is

$$\nu_1 = (\sin \theta, \cos \theta), \quad \nu_2 = (-\sin \theta, \cos \theta).$$

Assume $u(0) = 0$, then $u = 0$ on two segments starting at 0 of directions $\pm\nu_1$ and $\pm\nu_2$ respectively. Denote by ϑ_2 the closed angle (smaller than π) generated by these two directions. We wish to prove that

$$(7.1) \quad \lim_{x \rightarrow 0, x \in \vartheta_2} H(\nabla u) = H(0),$$

and moreover that no segment of direction $\nu_j, j \neq 1, 2$ along which $u = p_j \cdot x$ can intersect the angle ϑ_2 .

We distinguish two cases.

Case 1. $u = 0$ along segments of directions ν_i , $i = 1, 2$ (or analogously $-\nu_i$, $i = 1, 2$.)

Since $\nabla u \in \overline{N}$, then u is increasing in the cone of directions in ϑ_2 , thus $u \geq 0$ in ϑ_2 . We also have that $u - p_1 \cdot x$ is increasing in the direction $(\sin(\theta + \delta), \cos(\theta + \delta))$ with $\delta > 0$ small depending on N . This implies that near the origin in ϑ_2 we have that $u \leq p_1 \cdot x$, and analogously $u \leq p_3 \cdot x$. Thus, in $B_\rho \cap \vartheta_2$ we have

$$0 \leq u \leq \min\{p_1 \cdot x, p_3 \cdot x\}.$$

We use the hypothesis $n \geq 4$ to prove that no segment of direction ν_j , $j \neq 1, 2$ in which $u = p_j \cdot x$ can intersect the angle ϑ_2 . Indeed, in this angle near the origin we have $u \leq p \cdot x$ for all $p \in [p_1, p_3]$ and also $p \cdot x > 0$ when $p \in (p_1, p_3)$. If $u = p_j \cdot x$ on a segment in the interior of the angle ϑ_2 then we reach a contradiction since there exists $0 < \lambda < 1$ such that $\lambda p_j \in (p_1, p_3)$ and on this segment $u > \lambda p_j \cdot x \geq u$.

In B_ρ we consider the function \tilde{u} such that $\tilde{u} = u$ in the angle ϑ_2 and $\tilde{u} = 0$ outside this angle. Then, \tilde{u} is a minimizer for I in B_ρ . This follows from the fact that the gradient of the minimizer with boundary data \tilde{u} on ∂B_ρ belongs to \overline{N} and therefore the minimizer must be bounded by 0 from below and by $\max\{p_i \cdot x, 0\}$, $i = 1, 3$ from above.

Clearly if $j \geq 3$, then $p_j \cdot \nu_j \neq 0$ while $\tilde{u} = 0$ outside the angle ϑ_2 . This implies that $\tilde{u} \neq p_j \cdot x$, on a segment of direction ν_j . Thus we conclude from the localization Theorem and from Proposition 6.4 that $H(\nabla \tilde{u})$ is continuous and

$$\lim_{x \rightarrow 0, x \in \vartheta_2} H(\nabla u) = \lim_{x \rightarrow 0} H(\nabla \tilde{u}) = H(0).$$

Case 2. $u = 0$ along segments of directions ν_1 , and $-\nu_2$ (or analogously $-\nu_1, \nu_2$.)

Then $u = 0$ in $B_\rho \cap \vartheta_2$ and (7.1) clearly holds. Also, as in the conclusion of the previous case no segment of direction ν_j , $j \geq 3$ in which $u = p_j \cdot x$ can intersect ϑ_2 .

Since the angles generated by consecutive directions cannot overlap, they must cover a neighborhood of the origin and the lemma is proved. \square

Lemma 7.2. *If N has three sides, $H(\nabla u)$ is continuous.*

Sketch of the proof. We use the same notation as in the previous proof. The only case that does not follow from a similar analysis as before, is when u is linear on segments of directions ν_1, ν_2 and $-\nu_3$ (ν_i points inside N). More precisely, $u = \min\{p_1 \cdot x, p_3 \cdot x\}$ in $B_1 \cap \vartheta_2$, and $u \leq 0$ outside ϑ_2 . We can also assume that u is not linear in the direction of ν_3 . We can assume further that u is not linear on the segments of direction $-\nu_1$ (or $-\nu_2$). Otherwise, as in the proof of the previous Lemma, we can construct another minimizer $\tilde{u} = \min\{u, p_1 \cdot x\}$ which coincides with u on $\{p_1 \cdot x \leq 0\}$. The continuity of $H(\nabla \tilde{u})$ (and hence of $H(\nabla u)$) follows as in the case $n \geq 4$.

In the remaining case we wish to prove that the assumption (6.1) of the localization Theorem holds, hence the Lemma follows.

First, suppose that there exists a direction $p \in (p_1, p_3)$ and a rectangle

$$(7.2) \quad R_{ab} = \{a < x_1 < b, |x_2| < 1/2\}, \quad 0 \in (a, b) \subset [-1/2, 1/2]$$

such that

$$(7.3) \quad u < 0 \text{ on } \partial R_{ab} \cap \{p \cdot x \leq 0\}.$$

Then, for small ε (depending on u) any linear function l with

$$\nabla l \in \mathcal{B}_{\varepsilon^2}(\varepsilon p), \quad |l(0)| \leq \varepsilon^2$$

has the property that $\{u < l\} \cap \partial R_{ab}$ consists of one connected component. This implies that if we approximate u in R_{ab} by functions u_m ($u_m = u$ on ∂R_{ab}), then

$$\nabla u_m(B_{\varepsilon^3}) \cap \mathcal{B}_{\varepsilon^2}(\varepsilon p) = \emptyset,$$

otherwise we contradict Theorem Hence the Lemma follows from the localization Theorem.

The only case when we cannot find one pair $p, (a, b)$ satisfying (7.2)-(7.3), is when $u = 0$ in B_ρ (ρ small depending on u) above a line passing through the origin and below the angle ϑ_2 (recall that u is increasing in the e_2 direction). After a dilation, assume $\rho = 1$. Therefore, we can assume that

$$0 \geq u \geq \tilde{p} \cdot x, \quad \text{in } B^- := B_1 \cap \{\tilde{p} \cdot x \leq 0\},$$

for some $\tilde{p} \in (p_1, p_3)$.

Now one can obtain a localization Theorem in B^- around the origin, even though 0 is a boundary point. Indeed, let u_m be the approximation for u in B^- ($u_m = u$ on ∂B^-). Then $\nabla u_m(x) \in [0, \tilde{p}]$ for all $x \in \partial B^- \cap \{\tilde{p} \cdot x = 0\}$. Thus, we can obtain a Caccioppoli-type inequality at the boundary as in the proof of the localization Theorem, as long as the function η in formula (6.6) (the domain of integration is now B^-) is 0 on the segment $[0, \tilde{p}]$ and $\xi \in C_0^\infty(B_1)$. This implies that the iteration argument in the proof of the localization Theorem is valid provided that $\mathcal{B}_{4r}^k \cap [0, \tilde{p}] = \emptyset$.

Also, a boundary version of Proposition 6.4 holds because the ‘‘thin’’ connected components of $\{u_m < l_m\}$ cannot intersect $\tilde{p} \cdot x = 0$. More precisely, given $a_i \in \partial N \setminus \{\tilde{p}\}$, one can show that there exists r depending on u and a_i such that

$$\nabla u_m(B_r \cap B^-) \cap \mathcal{B}_r(a_i) = \emptyset.$$

Thus for any δ , there exists ε depending on δ and u such that $\nabla u_m(B_\varepsilon \cap B^-)$ is in a δ neighborhood of the segment $[0, \tilde{p}]$. Letting $m \rightarrow \infty$ we obtain the same result for u .

Consider now a sequence of blow-up minimizers

$$\frac{1}{r_k} u(r_k x), \quad x \in B_1,$$

that converges to \bar{u} . Clearly, \bar{u} is still a minimizer for I and by the conclusion above $\nabla \bar{u}(B^-) \subset [0, \tilde{p}]$. This implies that $\bar{u}(x) = p \cdot x$ in B^- with $p \in [0, \tilde{p}]$. If $p \neq 0$, then we reach a contradiction since \bar{u} is not a minimizer in a neighborhood of points x such that $\tilde{p} \cdot x = 0, x \neq 0$.

In conclusion for any $\varepsilon > 0$ there exists r_ε depending on u and ε such that

$$\left| \frac{1}{r} u(rx) \right| \leq \varepsilon, \quad x \in B^-$$

for all $r \leq r_\varepsilon$. Now the result follows applying the flatness Theorem 4.4 in balls $B_{c|x|}(x)$, for c small enough and $x \in B^-$.

□

8. THE PROOF OF THE FLATNESS THEOREMS

We finally present the proofs of Theorem 4.3 and Theorem 4.4. Since $u_m \rightarrow u$ uniformly, it suffices to prove Proposition 6.2 and Proposition 6.3.

Proposition 6.2. *Assume $B_1 \subset \Omega$ and*

$$\mathcal{B}_\delta(p_0) \subset N.$$

Then, there exists ε depending on δ, p_0, F , such that if

$$|u_m - p_0 \cdot x| \leq \varepsilon, \quad x \in B_1$$

then

$$\nabla u_m(B_{1/2}) \subset \mathcal{B}_\delta(p_0).$$

Proof. We can assume that u_m is not linear, otherwise the result is obvious. Also, it suffices to show that the conclusion of our statement holds at 0, i.e.

$$\nabla u_m(0) \in \mathcal{B}_\delta(p_0), \quad \text{for large } m.$$

First notice that

$$(8.1) \quad \nabla u_m(B_{2\sqrt{\varepsilon}}(0)) \cap \mathcal{B}_{2\sqrt{\varepsilon}}(p_0) \neq \emptyset$$

by considering $\min\{\frac{1}{2}|x|^2 + u_m - p_0 \cdot x\}$.

Let $\mathcal{B}_{2\rho}(p_1) \subset N$ such that $F \in C^2(\overline{\mathcal{B}_{2\rho}(p_1)})$, p_0 not in $\mathcal{B}_{2\rho}(p_1)$. We claim that

$$(8.2) \quad \nabla u_m(B_{1/2}) \cap \mathcal{B}_\rho(p_1) = \emptyset, \quad \text{for small } \varepsilon.$$

Clearly, the proposition follows from the claim, together with (8.1) and the localization Theorem. We are left with the proof of the claim.

Assume by contradiction that there exists $x_0 \in B_{1/2}$ such that

$$p_2 := \nabla u_m(x_0) \in \mathcal{B}_\rho(p_1).$$

We know that the set

$$\{u_m(x) < l_m := u_m(x_0) + p_2 \cdot (x - x_0)\}$$

has in B_1 at least two distinct connected components that intersect any neighborhood of x_0 .

Notice that by the flatness assumption

$$\{(x - x_0) \cdot (p_0 - p_2) < -2\varepsilon\} \subset \{u_m < l_m\}$$

and

$$\{(x - x_0) \cdot (p_0 - p_2) > 2\varepsilon\} \subset \{u_m > l_m\}.$$

This implies that one of the connected components of $\{u_m < l_m\}$ is included in the strip $\{|(x - x_0) \cdot (p_0 - p_2)| \leq 2\varepsilon\}$.

By changing the system of coordinates in the x and p spaces, we can assume that we have the following situation:

$$p_2 = 0, \quad p_0 = \alpha e_2, \quad x_0 = \pm e_1/4, \quad u_m(x_0) = 0$$

$$|u_m - \alpha e_2 \cdot x| \leq 2\varepsilon \quad \text{in } \overline{R} := \{|x_1| \leq 1/8, |x_2| \leq 1/8\}, \quad \alpha > \rho.$$

Moreover, the set $\{u_m < 0\}$ has one connected component in \overline{R} included in the strip $\{|x \cdot \alpha e_2| \leq 2\varepsilon\}$, that intersects both $\{x_1 = \pm 1/8\}$.

Let U be the connected component that contains $\bar{R} \cap \{x_2 = -1/8\}$. In $\bar{R} \setminus U$ we compare u_m with the function w defined by

$$\begin{aligned} w &= \delta' g(v), \quad v(x) = x_2 - 20x_1^2 \\ g(v) &= e^{kv} - \text{constant}, \quad g(-1/8) = 0, \end{aligned}$$

for some δ', k to be chosen later.

Now notice that $D^2 F_m$ is uniformly elliptic in \mathcal{B}_ρ with ellipticity constants λ, Λ depending only on F . Since

$$D^2 w = \delta' e^{kv} (D^2 v + k \nabla v \otimes \nabla v)$$

we see that if the constant δ' is chosen small enough so that $\nabla w \in \mathcal{B}_\rho$, and k is sufficiently large depending only on λ, Λ we have that

$$\text{Tr}(D^2 F_m D^2 w) \geq \lambda |(D^2 w)^+| - \Lambda |(D^2 w)^-| > 0.$$

Therefore, w is a subsolution and the minimum of $u_m - w$ must occur on $\partial(\bar{R} \setminus U)$.

Notice that by choosing δ' possibly smaller depending on ρ and k , we get that

$$\alpha e_2 \cdot x > w \quad \text{on } \{x_2 = 1/8, |x_1| \leq 1/8\} \cup \{x_1 = \pm 1/8, x_2 \geq -\delta''\},$$

and therefore

$$u_m - w > 0 \quad \text{on } \{x_2 = 1/8, |x_1| \leq 1/8\} \cup \{x_1 = \pm 1/8, x_2 \geq -2\varepsilon/\rho\},$$

for small ε . However, $u_m - w < 0$ on $\partial U \cap \{x_1 = 0\}$. Hence the minimum must occur at some point $z_0 \in \partial U \cap R$. Then,

$$u_m(x) \geq w(x) - w(z_0), \quad x \in \bar{R} \setminus U.$$

As in Proposition 6.4, this is a contradiction since the line $z_0 + te_2, t \geq 0$ intersects the other connected component of $\{u_m < 0\}$, while on this line the function w is increasing. □

Proposition 6.3. *Let $p_0 \in \partial N$. Then, there exists ε depending on δ, F , such that if*

$$|u_m - p_0 \cdot x| \leq \varepsilon, \quad x \in B_1$$

then

$$[\nabla u_m(x), p_0] \quad \text{is in a } \delta\text{-neighborhood of } \partial N, \text{ for all } x \in B_{1/2}.$$

Proof. We can argue as in the previous proof. Notice that in the proof of (8.1) and (8.2), we do not use that $p_0 \in N$, thus they hold also for $p_0 \in \partial N$. Then, when applying the localization Theorem, using also Proposition 5.2, we conclude that $\nabla u_m(B_{3/4})$ is included in a δ -neighborhood \mathcal{N}_δ of ∂N if ε is small depending only on δ, F .

Again, it suffices to prove our conclusion at 0, that is $[\nabla u_m(0), p_0] \subset \mathcal{N}_{4\delta}$. By changing the system of coordinates in the x and p spaces, we can assume that we have the following situation:

$$\nabla u_m(x_0) = 0, \quad u_m(x_0) = 0, \quad x_0 = \pm e_1/4, \quad p_0 = \alpha e_2$$

$$|u_m - \alpha e_2 \cdot x| \leq 2\varepsilon \quad \text{in } \bar{R} := \{|x_1| \leq 1/8, |x_2| \leq 1/8\},$$

and $\nabla u_m(\bar{R}) \subset \mathcal{N}_\delta$. We need to show that $[0, p_0] \subset \mathcal{N}_{4\delta}$. We can assume that $\alpha > \delta$, otherwise the conclusion clearly holds since $p_0 \in \partial N$. Moreover, the set

$\{u_m < 0\}$ has one connected component in \bar{R} included in the strip $\{|x \cdot \alpha e_2| \leq 2\varepsilon\}$ that intersects both sides $\{x_1 = \pm 1/8\}$.

In the set $\bar{R} \setminus U$ (U as in the previous proof) we compare u_m with the function w given by

$$w(x) = \frac{\alpha}{2}(x_2 - 20x_1^2) + 2\varepsilon.$$

For small ε we get

$$u_m - w > 0 \quad \text{on } \{x_2 = 1/8, |x_1| \leq 1/8\} \cup (\{x_1 = \pm 1/8\} \cap (\bar{R} \setminus U)).$$

However, $u_m - w < 0$ on $\partial U \cap \{x_1 = 0\}$. Hence the minimum of $u_m - w$ is negative and must occur at some point $z_0 \in R \setminus U$. On the other hand, since w is increasing in the x_2 direction, we obtain as in the previous proof that the minimum cannot occur on $R \cap \partial U$. Thus z_0 is an interior point and

$$\nabla u_m(z_0) = \nabla w(z_0).$$

Moreover $|z_0| \leq \sqrt{\varepsilon/\alpha}$. Indeed, it is straightforward to check that if

$$x \in \{\alpha x_2 \geq -2\varepsilon, |x| \geq \sqrt{\varepsilon/\alpha}\} \supset (\bar{R} \setminus U) \setminus B_{\sqrt{\varepsilon/\alpha}}(0),$$

then

$$u_m - w \geq \alpha x_2 - w - 2\varepsilon \geq \varepsilon,$$

and the claim follows. Therefore, as $\varepsilon \rightarrow 0$, $\nabla u_m(z_0) \rightarrow \alpha e_2/2$. If ε is small enough (depending on δ and F), $\nabla u_m(z_0) \in \mathcal{B}_\delta(p_0/2)$ and $\nabla u_m(z_0) \in \mathcal{N}_\delta$ since $z_0 \in \bar{R}$. Hence $p_0/2 \in \mathcal{N}_{2\delta}$ which implies $[0, p_0] \subset \mathcal{N}_{4\delta}$, and the proposition is proved. \square

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