Problem 1 (15 points) Consider the following function

\[ f(x, y, z) = x \sqrt{y^2 + z^2} \]

(a) Compute the differential of \( f \).

\[
df = \left| \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy + \frac{\partial f}{\partial z} \, dz \right|
\]

\[
= \sqrt{y^2 + z^2} \, dx + \frac{xy}{\sqrt{y^2 + z^2}} \, dy + \frac{xz}{\sqrt{y^2 + z^2}} \, dz
\]

(b) Find the linear approximation of \( f \) at \((2, 3, 4)\).

\[
f_x(2, 3, 4) = 5 \quad f_y(2, 3, 4) = \frac{6}{5} \quad f_z(2, 3, 4) = \frac{8}{5}
\]

\[
L(x, y, z) = 10 + 5(x-2) + 1.2(y-3) + 1.6(z-4)
\]

(c) Use the linear approximation to find an approximate value of \( f(1.98, 3.01, 3.99) \).

\[
f(1.98, 3.01, 3.99) \approx L(1.98, 3.01, 3.99)
\]

\[
= 10 + 5(-0.02) + 1.2(0.01) + 1.6(-0.01)
\]

\[
= 10 - 0.1 - 0.004 \quad = 9.896
\]
Problem 2 (20 points) Find the equation of the tangent plane to a surface $S$ which contains the following two parametric curves, at the point of intersection of these curves.

\[ \mathbf{r}_1(t) = \langle t - 2, (t - 1)^2, 5 \rangle \]
\[ \mathbf{r}_2(s) = \left\langle \cos \left( \frac{\pi}{4} s \right), \sin \left( \frac{\pi}{4} s \right), s^2 + 1 \right\rangle \]

**Point of intersection:**

\[ t - 2 = \cos \left( \frac{\pi}{4} s \right) \quad \Rightarrow \quad t - 2 = 0 \quad \Rightarrow \quad t = 2 \]
\[ (t - 1)^2 = \sin \left( \frac{\pi}{4} s \right) \]
\[ s^2 + 1 = s \quad \Rightarrow \quad s = \pm 2 \quad (s = -2 \text{ is not possible}) \]

\[ \sin \left( \frac{\pi}{4} s \right) = \sin \left( -\frac{\pi}{2} \right) = -1 \]

\[ t = s = 2 \quad \text{gives} \quad (0, 1, 5) \]

\[ \mathbf{r}'_1(t) = \langle 1, 2(t-1), 0 \rangle \quad \Rightarrow \quad \mathbf{r}'_1(2) = \langle 1, 2, 0 \rangle \]

\[ \mathbf{r}'_2(s) = \left\langle -\sin \left( \frac{\pi}{4} s \right) \frac{\pi}{4}, \cos \left( \frac{\pi}{4} s \right) \frac{\pi}{4}, 2s \right\rangle \quad \Rightarrow \quad \mathbf{r}'_2(2) = \left\langle -\frac{\pi}{4}, 0, 4 \right\rangle \]

**Normal:**

\[ \mathbf{n} = \mathbf{r}'_1(2) \times \mathbf{r}'_2(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -\frac{\pi}{4} & 0 & 4 \end{vmatrix} = 8\mathbf{i} - 4\mathbf{j} + \frac{\pi}{2}\mathbf{k} \]

**Equation of the tangent plane:**

\[ 8(x - 0) - 4(y - 1) + \frac{\pi}{2}(2 - 5) = 0 \]
\[ 8x - 4y + \frac{\pi}{2}z + 4 - \frac{5\pi}{2} = 0 \]
Problem 3 (15 points) Prove that the following two surfaces have the same tangent planes at \((1, -1, 0)\)

\[
x^2z + yz + 2xy + 2 = 0
\]
\[
x^2 + y^2 + z^2 - 2 = 0
\]

\[
F(x, y, z) = x^2z + yz + 2xy + 2
\]
\[
G(x, y, z) = x^2 + y^2 + z^2 - 2
\]

\[
\begin{align*}
F_x &= 2xz + 2y \\
F_y &= 2z + 2x \\
F_z &= x^2 + y
\end{align*}
\]

\[
\begin{align*}
G_x &= 2x \\
G_y &= 2y \\
G_z &= 2z
\end{align*}
\]

Normal vector to \(F = 0\) at \((1, -1, 0)\) = \([F_x(1, -1, 0), F_y(1, -1, 0), F_z(1, -1, 0)]\)

= \(\langle 2, 2, 0 \rangle\)

Normal vector to \(G = 0\) at \((1, -1, 0)\) = \([G_x(1, -1, 0), G_y(1, -1, 0), G_z(1, -1, 0)]\)

= \(\langle 2, -2, 0 \rangle\)

Since the two normal vectors are parallel, \(F = 0\) and \(G = 0\) have the same tangent plane at \((1, -1, 0)\)

\[
\begin{align*}
2(x-1) - 2(y+1) &= 0 \\
2x - 2y &= 4 \\
x - y &= 2
\end{align*}
\]

\[\]
Problem 4 (20 points) True/False. Justify your answers with a proof or a counterexample.

(a) For two parametric curves \( \vec{r}_1(t) \) and \( \vec{r}_2(t) \) we have
\[
\frac{d}{dt}(\vec{r}_1(t) \times \vec{r}_2(t)) = \vec{r}_1'(t) \times \vec{r}_2'(t)
\]
E.g., \( \vec{r}_1(t) = \langle t, t, 1 \rangle \) \( \vec{r}_1'(t) = \langle 1, 1, 0 \rangle \)
\( \vec{r}_2(t) = \langle t, t, 2 \rangle \) \( \vec{r}_2'(t) = \langle 1, 1, 0 \rangle \)
\( \vec{r}_1 \times \vec{r}_2 = t\hat{i} - t\hat{j} \Rightarrow \frac{d}{dt}(\vec{r}_1(t) \times \vec{r}_2(t)) = \hat{k} - \hat{j} \)
\( \Rightarrow \vec{r}_1'(t) \times \vec{r}_2'(t) = 0 \)

(b) \( \lim_{(x,y) \to (0,0)} (x^2 + y^2) \ln(\sqrt{x^2 + y^2}) = 0. \)
\[
\text{L.H.S.} = \lim_{r \to 0} r^2 \ln(r) = 0
\]

(c) The line joining the center of a sphere and a point \( P \) on the sphere is always perpendicular to the tangent plane to the sphere at \( P \).
\[
\text{Equation of a sphere} \quad (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 0
\]
\( \text{Normal vector} = \langle 2(x-x_0), 2(y-y_0), 2(z-z_0) \rangle \)
\[= \text{direction vector of the line joining } (x_0, y_0, z_0) \text{ and } (x, y, z) \]

(d) Assume that \( f(x,y) \) is a symmetric function of two variables, that is, \( f(x,y) = f(y,x) \). Then \( f_x(x,y) \) is also symmetric.
\( e.g. \quad f(x,y) = xy \quad \text{is symmetric but} \)
\( f_x = y \quad \text{is not.} \)
Problem 5 (15 points) One side of a triangle is increasing at a rate of 3 cm/sec and a second side is decreasing at a rate of 2 cm/sec. Assume that the area of the triangle remains unchanged, at what rate is the angle between the two sides changing, when the first side is 20 cm, the second side is 30 cm and the angle is $\pi/6$.

$$\frac{dx}{dt} = 3 \quad \frac{dy}{dt} = -2$$

$$\frac{d\theta}{dt} = ?$$

$$A = \frac{1}{2} xy \sin(\theta) \quad \frac{dA}{dt} = 0$$

$$\Rightarrow \quad y \sin(\theta) \frac{dx}{dt} + x \sin \theta \frac{dy}{dt} + xy \cos \theta \frac{d\theta}{dt} = 0$$

At $x = 20 \quad y = 30 \quad \theta = \frac{\pi}{6}$

$$30 \left( \frac{1}{2} \right) (3) + 20 \left( \frac{1}{2} \right) (-2) + (20)(30) \frac{\sqrt{3}}{2} \frac{d\theta}{dt} = 0$$

$$45 - 20 + 300 \sqrt{3} \frac{d\theta}{dt} = 0$$

$$45 - 20 + 300 \sqrt{3} \frac{d\theta}{dt} = 0$$

$$\frac{d\theta}{dt} = \frac{-25}{300 \sqrt{3}} \text{ radians/sec}$$

$$= \frac{-1}{12 \sqrt{3}} \text{ rad/sec}$$
Problem 6 (15 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$yz + x\ln(y) - z^2 = 0$$

Take $\frac{\partial}{\partial x}$:

$$y \frac{\partial z}{\partial x} + \ln(y) - 2z \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{\ln(y)}{y - 2z}$$

Take $\frac{\partial}{\partial y}$:

$$z + y \frac{\partial z}{\partial y} + \frac{x}{y} - 2z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} (y - 2z) = -z - \frac{x}{y} = \frac{-yz - x}{y}$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = -\frac{yz - x}{y(y - 2z)}$$