Problem 1 (15 points) Determine whether the following lines are parallel, intersecting or skew.

\[ L_1 : x = 3 - t \quad y = 2 + t \quad z = 4t \]
\[ L_2 : \frac{x+1}{2} = \frac{y-6}{2} = z - 10 \]

- Direction vector of \( L_1 = (-1, 1, 4) \)  Direction vector of \( L_2 = (2, 2, 1) \)  
  These vectors are not parallel.
- \( L_1 \) and \( L_2 \) intersect if there is \( t \) such that
  \[
  \frac{3-t+1}{2} = \frac{2+t-6}{2} = 4t-10
  \]
  \[
  \Rightarrow \quad 4-t = t-4 \quad \text{and} \quad \frac{t-4}{2} = 4t-10 \Rightarrow t=4 \quad \text{and} \quad t-4 = 8t-20
  \]
  There is no such \( t \). Hence \( L_1 \) and \( L_2 \) do not intersect \( t = \frac{16}{7} \)

\[ L_1 \text{ and } L_2 \text{ are skew} \]

Problem 2 (10 points) Find the equation of the plane consisting of all the points which are equidistant from \((1, 0, -2)\) and \((3, 4, 0)\).

Distance of \((x, y, z)\) from \((1, 0, -2)\) = \(\sqrt{(x-1)^2 + y^2 + (z+2)^2}\)
Distance of \((x, y, z)\) from \((3, 4, 0)\) = \(\sqrt{(x-3)^2 + (y-4)^2 + z^2}\)

These distances are equal if
\[
(x-1)^2 + y^2 + (z+2)^2 = (x-3)^2 + (y-4)^2 + z^2
\]
\[
\Rightarrow \quad x^2 - 2x + 1 + y^2 + z^2 + 4 + 4 = x^2 - 6x + 9 + y^2 - 8y + 16 + z^2
\]
\[
\Rightarrow \quad 4x + 8y + 4z = 20
\]

or
\[
x + 2y + z = 5
\]
Problem 3 (20 points) True/False. Justify your answer.

(a) If \( \bar{u} \cdot \bar{v} = 0 \) and \( \bar{u} \times \bar{v} = \bar{0} \), then either \( \bar{u} = \bar{0} \) or \( \bar{v} = \bar{0} \).

Since \( |\bar{u}|^2 |\bar{v}|^2 = (\bar{u} \cdot \bar{v})^2 + |\bar{u} \times \bar{v}|^2 = 0 \),
either \( |\bar{u}| = 0 \) or \( |\bar{v}| = 0 \)

(b) There exists a vector \( \bar{v} \) such that \( \langle 1, 2, 1 \rangle \times \bar{v} = \langle 3, 1, 5 \rangle \) \[ \text{FALSE.} \]
If there were such a \( \bar{v} \) then \( \langle 3, 1, 5 \rangle \) would be \( \perp \) to \( \langle 1, 2, 1 \rangle \).
But \( \langle 3, 1, 5 \rangle \cdot \langle 1, 2, 1 \rangle = 10 \neq 0 \)

(c) An object starting from \( (1, 1, 1) \) and moving along \( (-3, 0, 2) \) never passes through \( (-5, 1, 4) \). \[ \text{TRUE.} \]
The object moves along the line \( x = 1-3t, \ y = 1+0t, \ z = 1+2t \)
This line will pass through \( (-5, 1, 4) \) if there is \( t \) such that \( 1-3t = -5 \Rightarrow t = 2 \)
\( 1 = 1 \)
\( 1+2t = 4 \Rightarrow t = \frac{3}{2} \) \( \Rightarrow \) there is no such \( t \). So the line never passes through \( (-5, 1, 4) \)

(d) The following three points lie on the same line
\( P = (1, 2, -1) \quad Q = (4, 0, 2) \quad R = (10, -4, 8) \) \[ \text{TRUE.} \]
\( \overrightarrow{PQ} = \langle 3, -2, 3 \rangle \quad \overrightarrow{PR} = \langle 9, -6, 9 \rangle \)
\( = 3 \overrightarrow{PQ} \)
\( \Rightarrow \) \( P, Q, R \) are on the same line
Problem 4 (15 points) Find symmetric equations describing the line $L$ which passes through $(0, 1, 2)$, is parallel to the plane $x + y + z = 2$ and is perpendicular to the line $x = 1 + t, y = 1 - t, z = 2t$.

$L$ is parallel to $x + y + z = 2 \implies$ Direction vector of $L$ is $L \parallel \langle 1, 1, 1 \rangle$

Direction vector of $L$ is also $L \parallel \langle 1, -1, 2 \rangle$

$\implies L$ is parallel to the cross product $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle$

$$\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 2\hat{k}$$

Equation (symmetric) of the line $L$:

$$\frac{x - 0}{3} = \frac{y - 1}{-1} = \frac{z - 2}{-2}$$
Problem 5 (20 points)

(a) Prove that $\vec{AB} \times \vec{AC} = \vec{AB} \times \vec{AD}$ where $A, B, C, D$ are vertices of the following parallelogram.

$$
\vec{AC} = \vec{AB} + \vec{BC} = \vec{AB} + \vec{AD}
$$

$$
\Rightarrow \vec{AB} \times \vec{AC} = \vec{AB} \times \vec{AB} + \vec{AB} \times \vec{AD}
$$

$$
= \vec{0} + \vec{AB} \times \vec{AD} \quad (\text{because } \vec{u} \times \vec{u} = \vec{0})
$$

$$
= \vec{AB} \times \vec{AD}
$$

(b) For a non–zero vector $\vec{v}$ and a vector $\vec{u}$ prove that

$$
\text{Proj}_v(\vec{u}) \cdot \vec{v} = \vec{u} \cdot \vec{v}
$$

$$
\text{Proj}_v(\vec{u}) = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}
$$

$$
\text{Proj}_v(\vec{u}) \cdot \vec{v} = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) (\vec{v} \cdot \vec{v}) = \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) |\vec{v}|^2
$$

$$
= \vec{u} \cdot \vec{v}
$$
(c) Prove that the following three vectors are coplanar:

\[ \hat{i} + 5\hat{j} - 2\hat{k}, \quad 3\hat{i} - \hat{j}, \quad 5\hat{i} + 9\hat{j} - 4\hat{k} \]

\[
\left( 3\hat{i} - \hat{j} \right) \times \left( 5\hat{i} + 9\hat{j} - 4\hat{k} \right) = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
3 & -1 & 0 \\
5 & 9 & -4
\end{vmatrix}
\]

\[
(\hat{i} + 5\hat{j} - 2\hat{k}) \cdot \left( (3\hat{i} - \hat{j}) \times (5\hat{i} + 9\hat{j} - 4\hat{k}) \right) = 4\hat{i} + 12\hat{j} + 32\hat{k}
\]

\[
= 4(1) + 12(5) + 32(-2)
\]

\[
= 4 + 60 - 64 = 0
\]

Hence the three vectors are coplanar.

(d) If \( L_1 \) and \( L_2 \) are two parallel lines with the direction vector \( \vec{v} \), passing through points \( P \) and \( Q \) respectively, prove that the distance between \( L_1 \) and \( L_2 \) is given by

\[
\frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|}
\]

Distance = \( |\overrightarrow{PQ}| \sin \Theta \)

\[
\frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|} = \frac{|\overrightarrow{PQ}| |\vec{v}| \sin \Theta}{|\vec{v}|}
\]

\[
= |\overrightarrow{PQ}| \sin \Theta = \text{Distance}
\]
Problem 6 (20 points) Consider the surface given by the following equation

\[ x^2 + 4y^2 = 1 + z^2 \]

(a) Sketch \( x \) and \( z \)-traces of this surface.

\[ \begin{aligned}
\text{\( x \)-trace:} & \quad x = 0 : 4y^2 - z^2 = 1 \\
& \quad x = \pm 2 : 4y^2 - z^2 = 0 \\
& \quad |x| > 1 : z^2 - 4y^2 = x^2 - 1 \\
\text{\( z \)-trace:} & \quad z = 0 : x^2 + 4y^2 = 1 \\
& \quad z = \pm 1 : x^2 + 4y^2 = 2 \\
\end{aligned} \]

(b) Use the traces from the previous part to sketch the surface.