(2.0) Recall: a vector in $\mathbb{R}^2$ is given by a triple of numbers

$$\vec{v} = \langle a, b, c \rangle$$

- Magnitude $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$

- Vector addition $\langle a_1, b_1, c_1 \rangle + \langle a_2, b_2, c_2 \rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2 \rangle$

- Scalar multiplication $m \langle a, b, c \rangle = \langle ma, mb, mc \rangle$

Geometrically: $\vec{v} = \overrightarrow{PQ} = (x + a, y + b, z + c)$

$\vec{v} = \langle x, y, z \rangle$

- Vector addition is given by the triangle law

- $m \vec{v}$ has magnitude $|m| \cdot |\vec{v}|$ and direction:
  - $\vec{v}$ if $m > 0$
  - opposite to the direction of $\vec{v}$ if $m < 0$

e.g. $\vec{v}$

$\vec{2v}$

$-\vec{v}$
(2.1) A **unit vector** is a vector of magnitude 1.

- Any non-zero vector is a product of a positive number (magnitude) and a unit vector (direction).

**Proof.** Let \( \mathbf{v} \) be a non-zero vector. \( c = |\mathbf{v}| \) is a positive number

Then \( \mathbf{v} = c \left( \frac{1}{c} \mathbf{v} \right) \)

is a unit vector (in the same direction as that of \( \mathbf{v} \)).

**Standard basis of unit vectors:**

\[
\hat{i} = \langle 1, 0, 0 \rangle \quad \hat{j} = \langle 0, 1, 0 \rangle \quad \hat{k} = \langle 0, 0, 1 \rangle
\]

Thus \( \mathbf{v} = \langle a, b, c \rangle \) can also be written as

\[
\mathbf{v} = a \hat{i} + b \hat{j} + c \hat{k}.
\]

**Example:** Find the sum of the following two vectors in \( \mathbb{R}^2 \)

\[
|\mathbf{v}_1| = 10 \\
|\mathbf{v}_2| = 5
\]

**Sol.** Write \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) in the component form

\[
\mathbf{v}_1 = \langle -10 \cos 60^\circ, 10 \sin 60^\circ \rangle = \langle -5, 5\sqrt{3} \rangle
\]

\[
\mathbf{v}_2 = \langle 5 \cos 30^\circ, 5 \sin 30^\circ \rangle = \langle \frac{5\sqrt{3}}{2}, \frac{5}{2} \rangle
\]

Then \( \mathbf{v}_1 + \mathbf{v}_2 = \langle -5 + \frac{5\sqrt{3}}{2}, 5\sqrt{3} + \frac{5}{2} \rangle \)
(2.2) Dot product: Let \( \mathbf{u} = \langle a_1, b_1, c_1 \rangle \) and \( \mathbf{v} = \langle a_2, b_2, c_2 \rangle \)

Define \( \mathbf{u} \cdot \mathbf{v} = a_1a_2 + b_1b_2 + c_1c_2 \in \mathbb{R} \) (a number)

dot product of \( \mathbf{u} \) and \( \mathbf{v} \).

eg. \( \langle 1, -1, 1 \rangle \cdot \langle 0, 3, 5 \rangle = 1(0) + (-1)(3) + 1(5) = -3 + 5 = 2 \).

Properties of the dot product: for \( \mathbf{u}, \mathbf{v} \) vectors and \( m \in \mathbb{R} \) scalar:

(i) \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)

(ii) \( (m \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (m \mathbf{v}) = m(\mathbf{u} \cdot \mathbf{v}) \)

(iii) \( (\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{v} = \mathbf{u}_1 \cdot \mathbf{v} + \mathbf{u}_2 \cdot \mathbf{v} \)

(iv) \( \mathbf{v} \cdot \mathbf{v} = \| \mathbf{v} \|^2 \)

Geometric form of the dot product:

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors, \( \theta \) be the angle between the two and \( \| \mathbf{u} \| = a, \| \mathbf{v} \| = b \). Let us compute \( \| \mathbf{u} + \mathbf{v} \|^2 \) in two different ways

Algebraically:

\[
\| \mathbf{u} + \mathbf{v} \|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}
\]

\[
= \| \mathbf{u} \|^2 + 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2 = a^2 + b^2 + 2 \mathbf{u} \cdot \mathbf{v} - (1)
\]

Geometrically:

\[
\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{PR} \|^2 = \| \mathbf{PS} \|^2 + \| \mathbf{RS} \|^2
\]

\[
= (a + b \cos \theta)^2 + (b \sin \theta)^2
\]

\[
= a^2 + 2ab \cos \theta + b^2 \cos^2 \theta + b^2 \sin^2 \theta
\]

\[
= a^2 + b^2 + 2ab \cos \theta \quad - (2)
\]
Comparing (1) and (2) we get

\[ \overrightarrow{u} \cdot \overrightarrow{v} = ab \cos \theta \]

(equivently)

Example. If \( \overrightarrow{u} \) and \( \overrightarrow{v} \) are of lengths 3 and 6 respectively and angle between \( \overrightarrow{u} \) and \( \overrightarrow{v} \) is 60°, then

\[ \overrightarrow{u} \cdot \overrightarrow{v} = 3 \times 6 \times \cos 60° \]

\[ = 9 \]

Example. Find the angle between \( \langle 2, 1, 0 \rangle \) and \( \langle -1, 1, 5 \rangle \).

\[ \cos \theta = \frac{\langle 2, 1, 0 \rangle \cdot \langle -1, 1, 5 \rangle}{|\langle 2, 1, 0 \rangle| \cdot |\langle -1, 1, 5 \rangle|} \]

\[ = \frac{-1}{\sqrt{5} \cdot \sqrt{27}} = \frac{-1}{3 \sqrt{15}} \]

Note: Two vectors are perpendicular (or orthogonal), that is, the angle between them is 90°, if and only if their dot product is zero.

\( \overrightarrow{v} \perp \overrightarrow{u} \) \( \iff \) \( \overrightarrow{v} \cdot \overrightarrow{u} = 0 \)

(read \( \overrightarrow{v} \) is perpendicular to \( \overrightarrow{u} \)) (if and only if)
Projection. Let \( \vec{u} \) and \( \vec{v} \) be two non-zero vectors.

\[
\text{proj}_{\vec{v}} \vec{u} = \text{projection of } \vec{u} \text{ onto } \vec{v} \\
\text{or vector projection of } \vec{u} \text{ along } \vec{v}
\]

\[
\text{Comp}_{\vec{v}} \vec{u} = (\text{signed}) \text{ magnitude of } \text{proj}_{\vec{v}} \vec{u}.
\]

\[
\text{proj}_{\vec{v}} \vec{u} = (\text{comp}_{\vec{v}} \vec{u}) \cdot \text{unit vector along } \vec{v}
\]

\[
= (\text{comp}_{\vec{v}} \vec{u}) \cdot \frac{\vec{v}}{|\vec{v}|}
\]

And

\[
\text{comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta
\]

\[
= |\vec{u}| \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}
\]

\[
\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \quad \text{and} \quad \text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v}
\]
Example. Compute projections of $\hat{i}$, $\hat{j}$, $\hat{k}$ along $\langle 1, -1, 2 \rangle$.

\[
\text{proj } \langle 1, -1, 2 \rangle^{\hat{i}} = \frac{\langle 1, 0, 0 \rangle \cdot \langle 1, -1, 2 \rangle}{|\langle 1, -1, 2 \rangle|^2} \langle 1, -1, 2 \rangle = \frac{1}{6} \langle 1, -1, 2 \rangle
\]

\[
\text{proj } \langle 1, -1, 2 \rangle^{\hat{j}} = -\frac{1}{6} \langle 1, -1, 2 \rangle, \quad \text{proj } \langle 1, -1, 2 \rangle^{\hat{k}} = \frac{2}{6} \langle 1, -1, 2 \rangle
\]

\* Matrix of projection.

\[
\text{proj } \langle 1, -1, 2 \rangle = \begin{bmatrix}
\frac{1}{6} & -\frac{1}{6} & \frac{2}{6} \\
-\frac{1}{6} & \frac{1}{6} & -\frac{2}{6} \\
\frac{2}{6} & -\frac{2}{6} & \frac{4}{6}
\end{bmatrix}
\]

\[
\text{comp } \langle 1, -1, 2 \rangle^{\hat{i}} = \frac{1}{6}, \quad \text{comp } \langle 1, -1, 2 \rangle^{\hat{j}} = -\frac{1}{6}, \quad \text{comp } \langle 1, -1, 2 \rangle^{\hat{k}} = \frac{2}{6}
\]
Direction angles

\[ \alpha = \text{angle between } \vec{v} \text{ and } \hat{i} \]
\[ \beta = \text{angle between } \vec{v} \text{ and } \hat{j} \]
\[ \gamma = \text{angle between } \vec{v} \text{ and } \hat{k} \]

\[ \cos \alpha = \frac{\vec{v} \cdot \hat{i}}{|\vec{v}| |\hat{i}|} \quad \cos \beta = \frac{\vec{v} \cdot \hat{j}}{|\vec{v}| |\hat{j}|} \quad \cos \gamma = \frac{\vec{v} \cdot \hat{k}}{|\vec{v}| |\hat{k}|} \]

If \( \vec{v} = \langle a_1, a_2, a_3 \rangle \) then

\[ \cos \alpha = \frac{a_1}{|\vec{v}|} \quad \cos \beta = \frac{a_2}{|\vec{v}|} \quad \cos \gamma = \frac{a_3}{|\vec{v}|} \]

\( \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \) is the unique unit vector in the direction of \( \vec{v} \).

In physics, work done by a force \( \vec{F} \) in moving an object by \( \vec{S} \) is given by

\[ W = \vec{F} \cdot \vec{S} \]

Example.

Find the work done by \( \vec{F} \) in moving an object from A to B

Answer: \[ 5 \times 10 \times \cos 45^\circ = 25 \sqrt{2} \ \text{J} \]

\[ \text{Nm} \]