PROBLEMS FOR HOMEWORK 11

(1) Find all complex numbers $r$ such that the function $x(t) = e^{rt}$ satisfies the following

$$x''(t) - 6x'(t) + 13x(t) = 0$$

Use this to find a function $x(t)$ satisfying the equation above, such that $x'(0) = 4$ and $x(0) = 0$.

(2) Consider the following subset of the complex plane:

$$\Pi := \{ z \in \mathbb{C} \text{ such that } 0 \leq \text{Im}(z) < 2\pi \}$$

(a) Sketch the subset $\Pi$ on the complex plane.
(b) Prove that for every non-zero complex number $w$, there is one and only one $z \in \Pi$ such that $e^z = w$.
(c) Sketch the image of a horizontal line in $\Pi$ under the exponential map. A horizontal line consists of all complex numbers with fixed imaginary part.
(d) Sketch the image of a vertical line in $\Pi$ under the exponential map. A vertical line consists of complex numbers with fixed real part.

(3) Let Log($z$) be defined by fixing its imaginary part to lie between 0 and $2\pi$: $0 \leq \text{Im}(\text{Log}) < 2\pi$.

(a) Compute

$$\text{Log}(-i) \quad \text{Log}(-1) \quad \text{Log}(-1 - \sqrt{3}i)$$

(b) Consider the following parametric curve in the complex plane

$$z(t) = \cos(t) + i \sin(t)$$

Sketch the graph of the function $f(t) = \text{imaginary part of Log}(z(t))$. Is this function continuous?

(4) Find all complex numbers $z \in \mathbb{C}$ such that $z^5 = 16\sqrt{2} + 16\sqrt{2}i$.

(5) A person wants to go from point $A$ to point $B$, while there is a 10$m$ wide river between the two points. The distance from point(s) $A$ (and $B$) to the (respective) bank(s) of the river is 10$m$ and the horizontal distance from point $A$ to $B$ is again
10m (see the figure below). If the person travels at the speed of 2m/s on ground and 1m/s in water, prove that his/her travel time is minimized when

$$\frac{\sin(a)}{\sin(b)} = 2$$

where a and b are angles as shown in the picture below.
*Hint: this is a problem of Lagrange multipliers. Identify the objective function to be minimized, and the constraint. Phrase the problem in terms of Lagrange multipliers.*

(6) Use Lagrange multipliers to find the max/min values of the determinant of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if \(a^2 + b^2 = c^2 + d^2 = m\) (a constant positive real number).