(1) Give the definition of the continuity of a function $f : \mathbb{R} \to \mathbb{R}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x \to +\infty} f(x) = \ell$ and $(a_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \to +\infty} a_n = +\infty$. Prove that $\lim_{n \to +\infty} f(a_n) = \ell$.

(3) For $n \in \mathbb{N}$, $n \geq 2$, we consider the number

$$u_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$ 

Prove by strong induction that for any $n \geq 2$ there is an odd number $o_n$ and an even number $e_n \neq 0$ such that $u_n = \frac{o_n}{e_n}$.

(4) Consider a sequence $(A_n)_{n \in \mathbb{N}}$ satisfying

$$A_{n+2} = -2A_{n+1} + A_n.$$ 

(a) Prove that if $A_0 = 0$ and $A_1 = 0$ then $A_n = 0$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence given by $a_n = (-1 - \sqrt{2})^n$ satisfies the condition $(\ast)$.

(c) Prove that the sequence given by $b_n = (-1 + \sqrt{2})^n$ satisfies the condition $(\ast)$.

(d) Let $x, y \in \mathbb{R}$. Prove that any sequence of the form $(xa_n + yb_n)_{n \in \mathbb{N}}$ satisfies $(\ast)$.

(e) Compute $a_0, b_0, a_1, b_1$.

(f) Prove that the system

$$\begin{cases} 35 &= a_0x + b_0y \\ -47 &= a_1x + b_1y \end{cases}$$

has a unique solution $(x, y) \in \mathbb{R}^2$.

(g) Give an explicit example of sequence $A_n$ satisfying $(\ast)$ and such that $A_0 = 35$ and $A_1 = -47$.

(h) Prove, using (a) that the example you gave in (g) is the only example of sequence $A_n$ satisfying $(\ast)$ and such that $A_0 = 35$ and $A_1 = -47$. 