Questions to review for the final exam

This is a non-exhaustive list of questions you might want to think of to review the material we covered this term.

1) Prove the following: let \( f, g : \mathbb{R} \to \mathbb{R} \) and suppose that \( f(x) \leq g(x) \) for all \( x \geq 10 \). If \( \lim_{x \to +\infty} f(x) = +\infty \), then \( \lim_{x \to +\infty} g(x) = +\infty \).

2) Give a precise statement of the squeeze theorem for sequences.

3) Prove the squeeze theorem for sequences.

4) Show that a sequence \((u_n)_{n \in \mathbb{N}}\) having a limit \( \ell \in \mathbb{R} \) is bounded (that is, there are \( m, M \in \mathbb{R} \) such that \( m \leq u_n \leq M \) for all \( n \in \mathbb{N} \)).

5) Does any bounded sequence have a limit?

6) Prove that an integer \( n \in \mathbb{N}, n \geq 1 \) is either prime or has a nontrivial divisor which is \( \leq \sqrt{n} \).

7) Let \( p \) be a prime number. Prove that \( \sqrt{p} \) is not a rational number.

8) Is the polynomial \( X^3 + X^2 + 7X + 9 \) irreducible in \( \mathbb{R}[X] \)?

9) Is the polynomial \( 3X^2 + 7X - 9 \) irreducible in \( \mathbb{R}[X] \)? in \( \mathbb{Q}[X] \)?

10) Find all the solutions \((x, y) \in \mathbb{Z}^2\) to the equation \( 45x + 27y = 7 \).

11) Find all the solutions \((x, y) \in \mathbb{Z}^2\) to the equation \( 45x + 27y = 3 \).

12) Prove carefully that for \( a, b, c, d \in \mathbb{Z} \) and \( n \in \mathbb{N}, n \geq 1 \) we have \( a \equiv b \mod n \) and \( c \equiv d \mod n \) implies \( ac \equiv bd \mod n \).

13) Define the relation \( \mathcal{R} \) on \( \mathbb{Z} \) by: \( aRb \) is \( a + b \equiv 0 \mod 3 \). Is it an equivalence relation? If yes, give the set of equivalence classes.

14) Define the relation \( \mathcal{R} \) on \( \mathbb{Z} \) by: \( aRb \) is \( a^2 \equiv b^2 \mod 4 \). Is it an equivalence relation? If yes, give the set of equivalence classes.

15) Is the function \( \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \) well defined?

16) Is the function \( \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \) well defined?
(17) Is the function

\[
\begin{align*}
\mathbb{Z}/4\mathbb{Z} & \to \mathbb{Z}/8\mathbb{Z} \\
[x]_8 & \mapsto [x^2]_4
\end{align*}
\]

well defined?

(18) Prove by induction that if \(X\) is a set with cardinality \(n\) then \(\mathcal{P}(X)\) has cardinality \(2^n\).