Diverse questions

(1) Draw an injective map from the set \{A, B, C\} to the set \{1, 2\}.

(2) Draw a surjective map from the set \{A, B, C\} to the set \{1, 2\}.

(3) Draw a surjective map from the set \{A, B, C\} to the set \{1, 2, 3, 4\}.

(4) What is the definition of an equivalence relation?

(5) Define the equivalence relation mod 5 on \mathbb{Z}.

(6) Define the greatest common divisors of two nonzero integers \(a\) and \(b\).

(7) Give a statement that is equivalent to “the \(gcd\) of \(a\) and \(b\) is 1”.

(8) Describe (with quantifiers) the set \(S\) of all numbers in \(\mathbb{Z}\) that are not multiples of 5.

(9) Prove that all elements in \(S\) are invertible mod 5 that is to say: for every \(s \in S\) there is \(k \in \mathbb{Z}\) such that \(sk \equiv 1 \mod 5\).

(10) Are the elements in \(S\) invertible mod 5?

(11) What is the inverse of 3 mod 5? (that is: for which \(k \in \{0, 1, 2, 3, 4\}\) do we have \(3k \equiv 1 \mod 5\)?) Same question for 1 mod 5, 2 mod 5, 3 mod 5, 4 mod 5.

(12) Compute \(4! \mod 5\).

(13) Let \(a, b, c \in \mathbb{Z} - \{0\}\). Under which hypothesis can we claim that
\[ a \text{ divides } bc \Rightarrow a \text{ divides } c? \]

(14) Can you prove the previous statement?

(15) We define the lowest common divisor of two nonzero integers \(a\) and \(b\) to be the smallest positive integer \(m\) such that \(m\) is a multiple of both \(a\) and \(b\). What is the \(lcm\) of 5 and 7? Of 21 and 39? What do you remark about \(21 \times 39, gcd(21, 39), lcm(21, 39)\)?

(16) Let \(a\) and \(b\) be two nonzero integers. Prove that any divisor of \(a\) and \(b\) divides \(gcd(a, b)\).

(17) Does the equation \(57u + 28v = 13\) have a solution \((u, v) \in \mathbb{Z}^2\)? Find one if yes.

(18) Does the equation \(21u + 28v = 1\) have a solution \((u, v) \in \mathbb{Z}^2\)? Find one if yes.

(19) Give a polynomial with degree 2 such that \(P(0) = 0, P(1) = 1, P(2) = 0\).

(20) Is there a polynomial with degree 3 such that \(P(0) = 0, P(1) = 1, P(2) = 0\)?

(21) Is there a polynomial with degree 1 such that \(P(0) = 0, P(1) = 1, P(2) = 0\)?

(22) Prove that \(X - n\) divides \(X^3 - (4 + n)X^2 + (4n + 3)X - 3n\).
(23) Consider the sequence defined by
\[
\begin{cases}
    u_0 = 0 \text{ and } u_1 = 1 \\
    u_{n+1} = 7u_n + 8u_{n-1}
\end{cases}
\]
Compute \( u_{n+1} - u_{n-1} \). Define \( v_n = u_{n+1} + u_n \). Compare \( v_{n+1} \) and \( v_n \). Give and prove the formula for \( v_n \).

(24) Prove that \( 2^{3n} - 1 \) is divisible by 7 for all \( n \in \mathbb{N} \).