Problem 1. Given two real numbers $x$ and $y$, we set

$$\min(x,y) := \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y \leq x \end{cases}, \quad \max(x,y) := \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y \leq x \end{cases}.$$ 

Prove that

$$\min(x,y) = \frac{x + y - |x - y|}{2}, \quad \max(x,y) = \frac{x + y + |x - y|}{2}.$$ 

Problem 2. We consider the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$. We want to prove that the limit of $f$ at $+\infty$ does not exist.

1. Prove, using a theorem from your notes, that if $f$ has a limit $\ell$ at $+\infty$ then it satisfies $-1 \leq \ell \leq 1$.

2. In this question we prove that if $f$ has a limit $\ell$ at $+\infty$, then $\ell = 0$.
   (a) What is the value of $f$ at a number of the form $n\pi$ for $n \in \mathbb{N}$?
   (b) Let $\ell \in [-1,1] - \{0\}$. Prove that $\ell$ cannot be the limit of $f$. (Use $\epsilon := |\ell|$).

3. In this question we prove that 0 cannot be the limit of $f$ at $+\infty$.
   (a) What is the value of $f$ at a number of the form $\pi/2 + 2n\pi$ for $n \in \mathbb{N}$?
   (b) Prove that 0 cannot be the limit of $f$. (Use $\epsilon := 1/2$ for example).

Problem 3. Let $a \in \mathbb{R}$ a real number such that

for all $\epsilon > 0$, we have $|a| < \epsilon$.

Prove that $a = 0$.

Problem 4. Let $a$ and $b$ be two real numbers such that

for all $x \in \mathbb{R}$, $b < x \Rightarrow a < x$.

Prove that $a \leq b$. 