From the book: 4.18. You can use Problem 4 (and its solution) below as a guideline.

**Problem 1.**  (1) Find all polynomials $P \in \mathbb{R}[X]$ such that $P(x) = P(x - 3)$ for any $x \in \mathbb{R}$.

(2) We consider the map

$$
G : \mathbb{R}[X] \rightarrow \mathbb{R}[X]
$$

$$
P \rightarrow P(X) - P(X - 3).
$$

(a) If $P$ has degree $n \in \mathbb{N}$, what is the degree of $G(P)$ ?

(b) Is $G$ injective ?

**Problem 2.** We recall that for any $n \in \mathbb{N}, n \geq 1$ we have $X^n - 1 = (X - 1) \times \sum_{k=0}^{n-1} X^k$.

Let $a,b \in \mathbb{R}$ and $n \in \mathbb{N}, n \geq 1$. Let $P = aX^{n+1} + bX^n + 1$.

(1) Find a *necessary and sufficient condition* on $a$ and $b$ for $P$ to have 1 as a root.

(2) Under this condition, give explicitly the polynomial $Q$ (that will depend on $n$ and $a$) such that

$$
P = (X - 1)Q.
$$

(3) Under which condition does $Q$ have 1 as a root ?

**Problem 3.**  (1) Find all polynomials with degree 3 such that

$$
P(1) = P(2) = P(3) = 0.
$$

First discuss how many of them there can be (up to multiplication by a constant). Then find one.... and conclude.

(2) Find all polynomials $P \in \mathbb{R}[X]$ such that for all $n \in \mathbb{Z}$ we have

$$
P(n) = \cos(2n\pi/3).
$$

(3) Find all polynomials $P \in \mathbb{R}[X]$ such that for all $n \in \mathbb{Z}$ we have

$$
P(n) = \sin(n\pi) + \sqrt{5}.
$$
**Problem 4.** Prove that there is a unique polynomial $P$ with degree 3 such that for any $N \in \mathbb{N}$ we have

$$P(N) = \sum_{n=0}^{N} n^2.$$

**Solution.** First we are going to ”guess” what this polynomial can be. Let $P$ be a polynomial with degree 3. It can be written

$$P = a_3 X^3 + a_2 X^2 + a_1 X + a_0$$

with $a_0, a_1, a_2, a_3 \in \mathbb{R}$. If $P$ satisfies the requirement, we have in particular

$$\begin{cases}
0 & = a_0 \\
0 + 1 & = a_0 + a_1 + a_2 + a_3 \\
0 + 1 + 2^2 & = a_0 + 2a_1 + 4a_2 + 8a_3 \\
0 + 1 + 2^2 + 3^2 & = a_0 + 3a_1 + 9a_2 + 27a_3
\end{cases}$$

We solve this system (you should be able to do it properly – otherwise try this during recitation) and find

$$a_0 = 0, \ a_1 = 1/6, \ a_2 = 1/2, \ a_3 = 1/3.$$ 

Now we have found the only possible candidate for the solution of our problem: it is

$$P = \frac{1}{3} X^3 + \frac{1}{2} X^2 + \frac{1}{6} X = \frac{X(2X + 1)(X + 1)}{6}.$$ 

We still have to prove that this candidate actually works, that it to say that it satisfies Equation (1) for all $N \in \mathbb{N}$.

So let us prove by induction on $N \in \mathbb{N}$ that – with the $P$ we found – we have $P(N) = \sum_{n=0}^{N} n^2$ for all $N \in \mathbb{N}$.

- Base step: for $N = 0$ it is true.
- Suppose that for some fixed $N_0 \in \mathbb{N}$ we know that $P(N_0) = \sum_{n=0}^{N_0} n^2$. Then, by induction hypothesis, we have

$$\sum_{n=0}^{N_0+1} n^2 = P(N_0) + (N_0 + 1)^2.$$
and we notice that
\[
P(N_0) + (N_0 + 1)^2 = \frac{N_0(2N_0 + 1)(N_0 + 1)}{6} + (N_0 + 1)^2
\]
\[
= \frac{N_0(2N_0 + 1)(N_0 + 1) + 6(N_0 + 1)^2}{6}
\]
\[
= \frac{(N_0 + 1)[(2N_0 + 1)N_0 + 6(N_0 + 1)]}{6}
\]
\[
= \frac{(N_0 + 1)(2N_0^2 + 7N_0 + 6)}{6}
\]
\[
= \frac{(N_0 + 1)(2N_0 + 3)(N_0 + 2)}{6}
\]

and we recognize here \( P(N_0 + 1) \). So we have proved that, provided (1) is true at rank \( N_0 \), then it is true at rank \( N_0 + 1 \).

* The first part of the proof finds the only possible polynomial \( P \) with degree 3 that satisfies the requirement. The second part proves (by induction) that this \( P \) indeed satisfies the requirement.