Exercise 1. Let $\alpha$ be one of the following complex numbers:

$$j, \sqrt{2}, 1 + \sqrt{3}, \sqrt{2} + \sqrt{3}.$$ 

1. Prove that $\alpha$ is an algebraic number by finding a monic polynomial $P$ in $\mathbb{Q}[X]$ such that $P(\alpha) = 0$.

2. Determine the degree of $\alpha$ and its minimal polynomial.

3. Consider the set $\mathbb{Q}[\alpha]$ defined to be

$$\mathbb{Q}[\alpha] = \{T(\alpha), \ T \in \mathbb{Q}[X]\}.$$ 

4. Prove that $\mathbb{Q}[\alpha]$ is a subring of $(\mathbb{C}, +\times)$.

5. Prove that $\mathbb{Q}[\alpha]$ is a field (that is to say: any nonzero element in $\mathbb{Q}[\alpha]$ has an inverse for $\times$ in $\mathbb{Q}[\alpha]$).

6. Express the inverse of $\alpha$ as an element $T(\alpha)$ for $T \in \mathbb{Q}[X]$.

7. Same question for $\alpha + 1$ and $\alpha^2 + 3$.

This is a general result:

**Theorem (1).** — If $\alpha \in \mathbb{C}$ is an algebraic number, then

$$\mathbb{Q}[\alpha] = \{T(\alpha), \ T \in \mathbb{Q}[X]\}$$

is a subfield of $\mathbb{C}$.

And you have to be able to compute the inverse of an element in $\mathbb{Q}[\alpha]$...

8. Now denote by $d \geq 1$ the degree of $\alpha$. Prove that any element $x$ in $\mathbb{Q}[\alpha]$ can be written uniquely in the form

$$x = \nu_{d-1}\alpha^{d-1} + \nu_{d-2}\alpha^{d-2} + \ldots + \nu_0$$

where $\nu_i \in \mathbb{Q}$. This form is called a linear combination of $1, \alpha, \ldots, \alpha^{d-1}$.

9. Let $T = X^5 + 4X^3 + 2$. Express $T(\alpha)$ as a linear combination of $1, \alpha, \ldots, \alpha^{d-1}$.

This is a general result:
Theorem (2). — If $\alpha \in \mathbb{C}$ is an algebraic number, then

$$Q[\alpha] = \{ T(\alpha), \ T \in \mathbb{Q}[X] \}$$

is a $\mathbb{Q}$-vector space with dimension $d$ and basis $\{1, \alpha, \ldots, \alpha^{d-1}\}$.

And you have to be able to express $T(\alpha)$ as a linear combination of $1, \alpha, \ldots, \alpha^{d-1}$ for any $T \in \mathbb{Q}[X]$. 