Duality interfaces
in 3-dimensional theories

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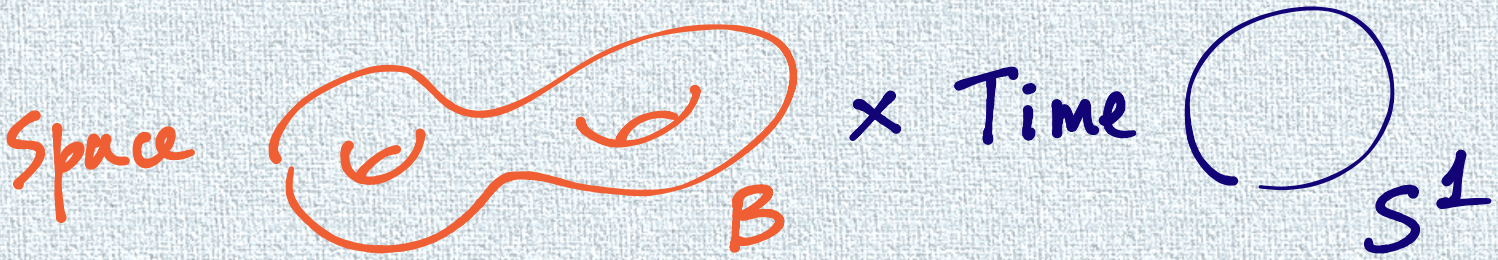
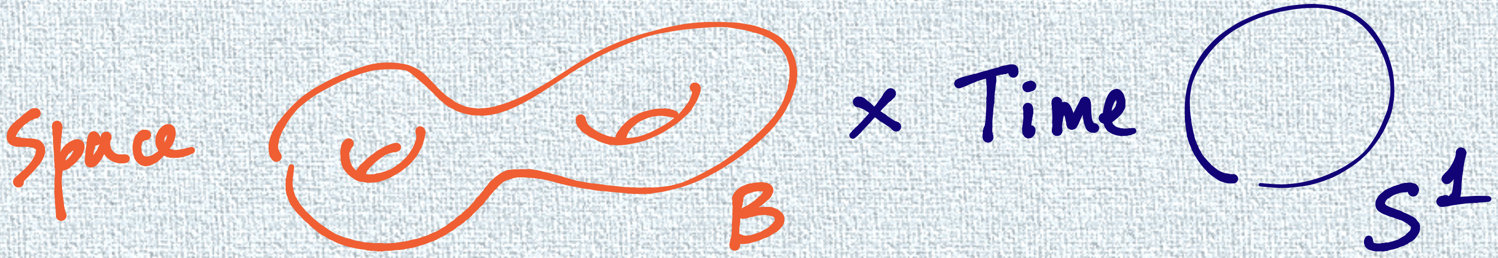
We study indices in SUSY theories in dim=2+1

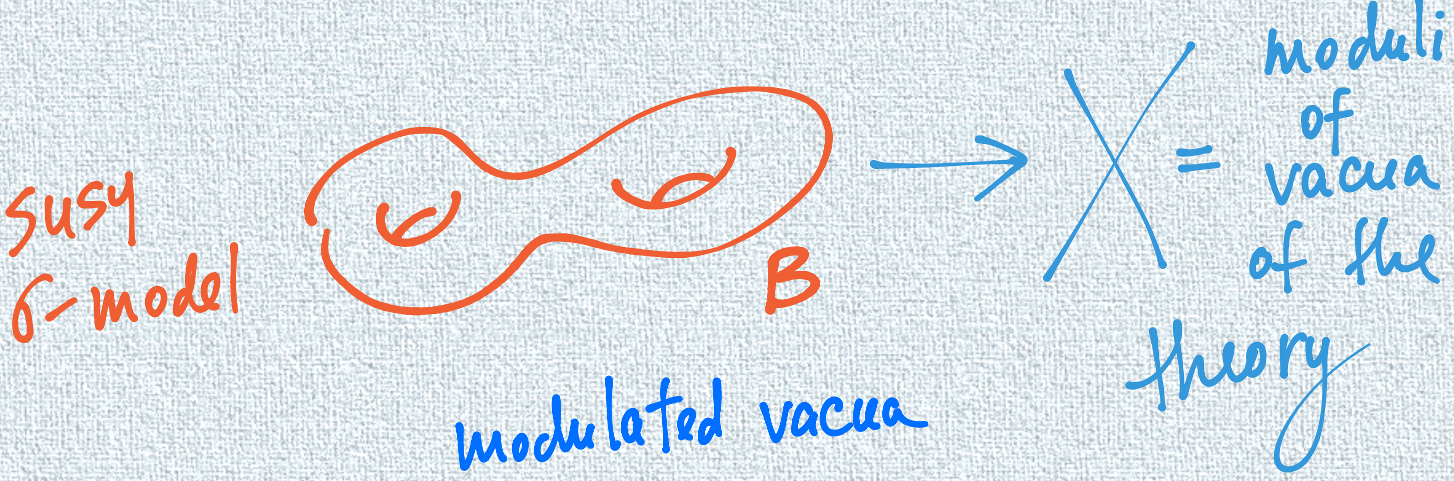


A lot of important work on these in recent years, in particular by D.Gaiotto, T.Dimofte, ... and their collaborators.

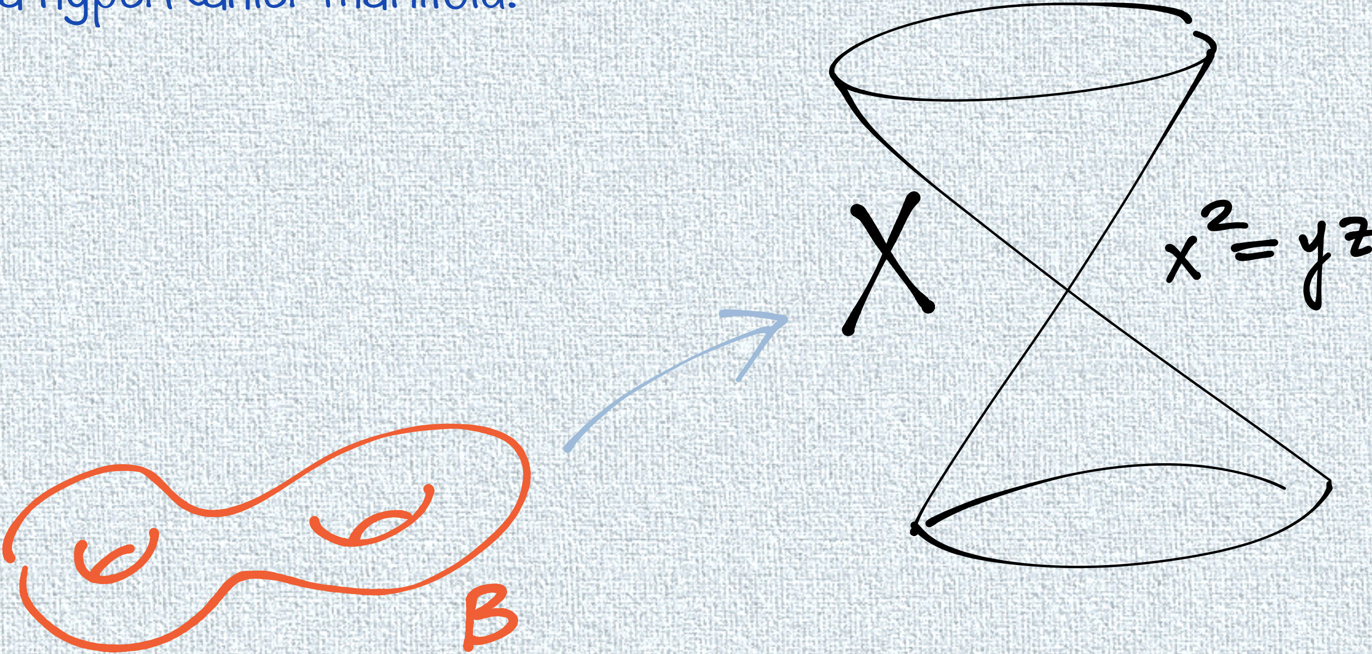
We study indices in SUSY theories in dim=2+1



Indices may be computed in the IR, that is, when the space B is very large. In that limit, we can replace the original theory by



We study theories with 8 supercharges, and for them X wants to be a hyperKähler manifold.



In reality, a singular algebraic symplectic variety.

In alg geom, nice moduli space of maps B ---> X, namely **quasimaps** (=vortex solutions), exist for SUSY gauge theories, described by

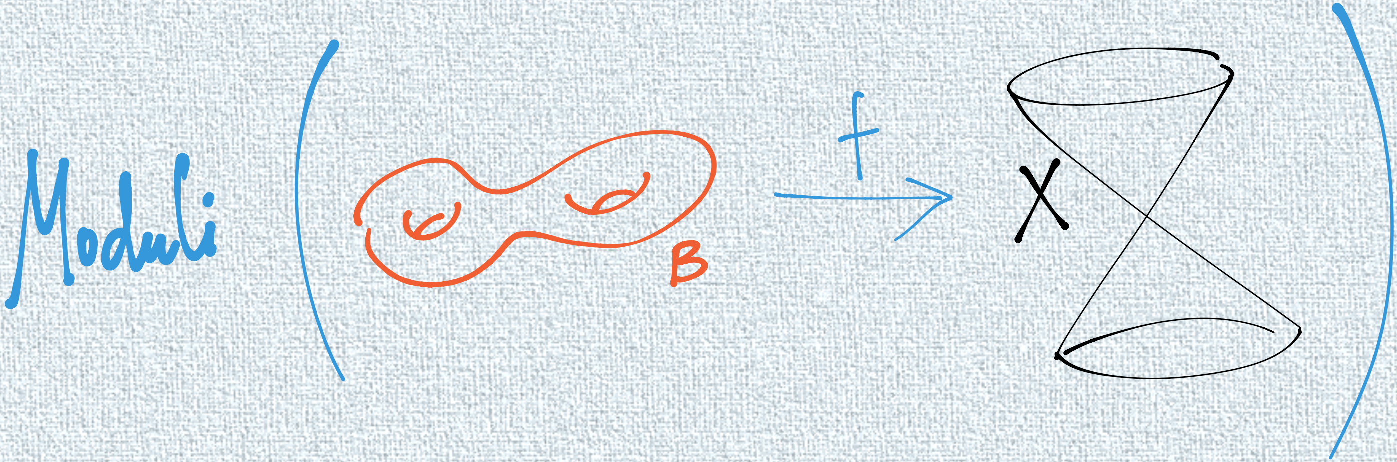
$$G = \text{complexified gauge group}$$

$$M = \text{symplectic representation of } G$$

$$X = \underset{\substack{\uparrow \\ \text{complex moment map}}}{\mu}{}^{-1}(0)\underset{\substack{\uparrow \\ \text{real moment} \\ \text{map} = \theta}}{\text{stable}} \Big/ G$$

It is a very interesting geometric and physical problem to figure out what is the right geometry for more general theories.

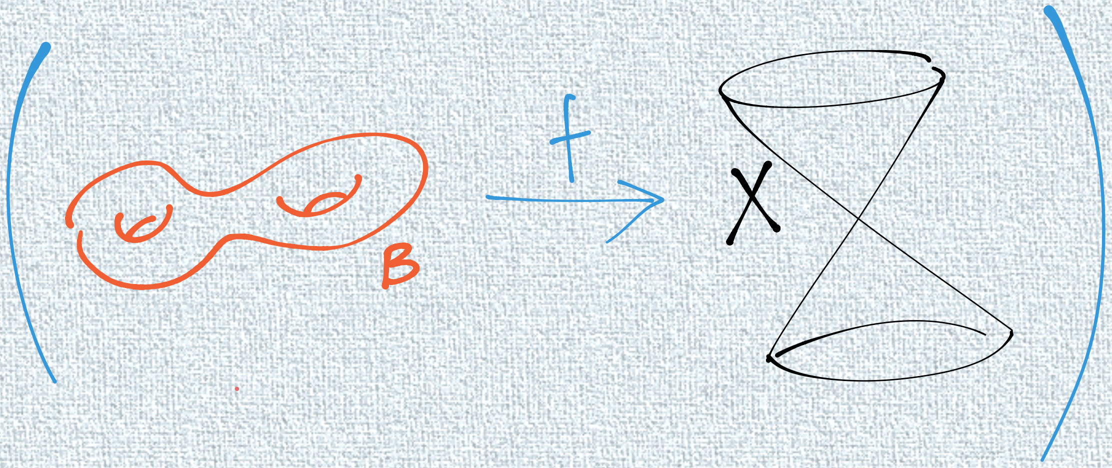
SUSY indices are defined in algebraic geometry as indices of virtual Dirac operator on the moduli space of maps:



In reality, the Euler characteristic of the virtual Â-genus, which may be defined as an element of K(Moduli) in very favorable circumstances (see above).

Indices for

Moduli



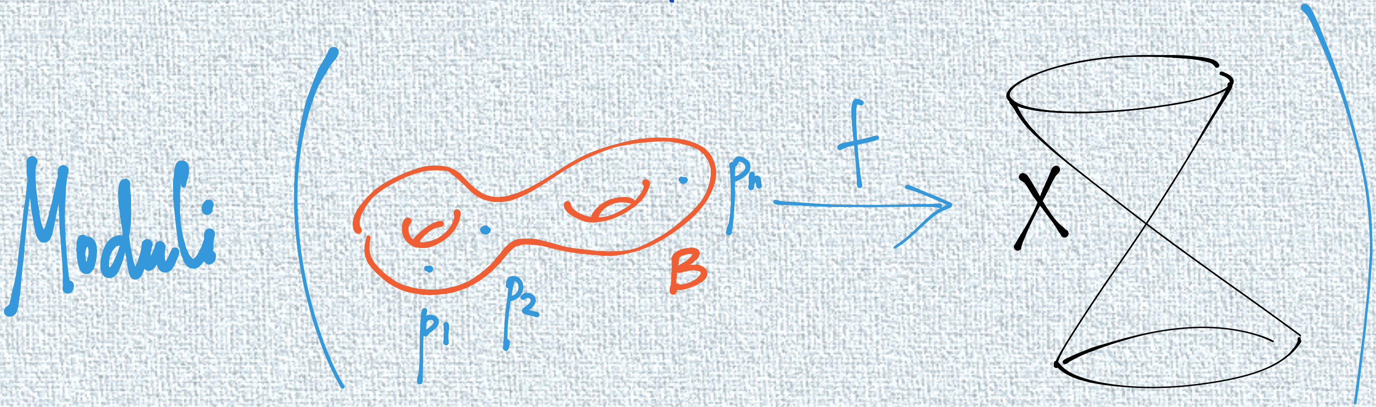
are graded by

- the action of Aut(X) x Aut(B)
- the topology of the map, recorded by $z^{\deg(f)}$

In other words, they are functions on Aut(X) x Aut(B) times

$$Z = \mathrm{Pic}(X) \otimes \mathbb{C}^{\times} \ni z$$

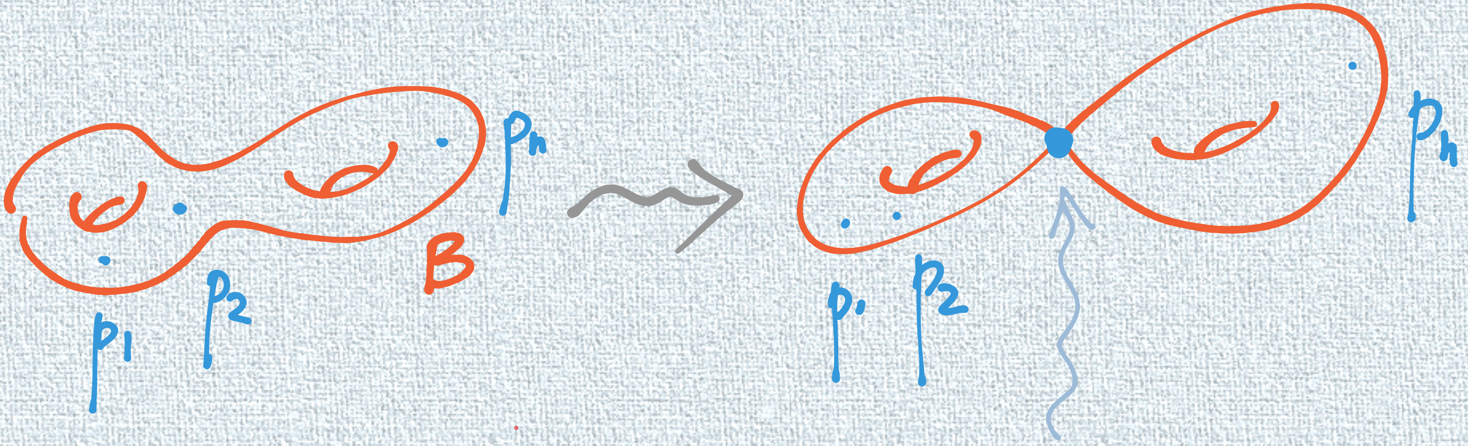
The surface B can have marked points



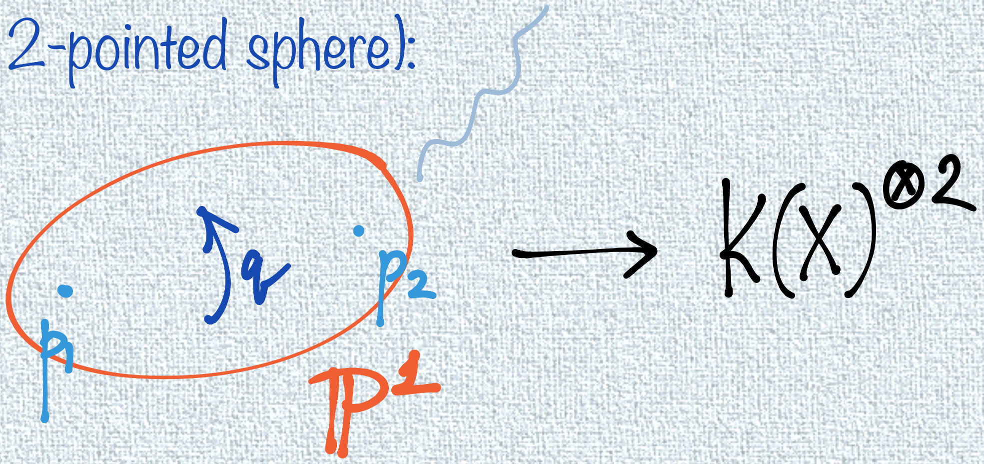
where one can put insertions (line operators in dim=2+1). The space of possible insertions is as big as the K-theory of X. If there are n marked points, the indices are functions of z and equivariant variables with values in

$$K(X)^{\otimes n}$$

This data forms K-theoretic analog of CohFT



With the extra feature that the metric depends on z (but not on the automorphisms q of the 2-pointed sphere):



Reconstruction theorems of Givental-Teleman kind are expected to hold in all examples of interest. In other words, one should be able to construct indices directly as virtual vector bundles on the moduli spaces of pointed curves starting from the data that will be discussed below.

This K-thereoretic-FT really is the complete description of indices. It may be taken as a substitute to curve-counting when no curve-counting may be set up with existing technology.

A key phenomenon in the field is that of 3-dimensional mirror symmetry. It goes back to [Intrilligator-Seiberg] and predicts certain pairs of KthFT's to be exactly equal with:

- the exchange of degree a.k.a. Kähler variables z and the equivariant variables a in a maximal torus A of $\mathrm{Aut}(X,\omega)$
- a suitable identification of $K_{\mathrm{equiv}}(X)[[z]]$ with $K_{\mathrm{equiv}}(\check{X})[[a]]$

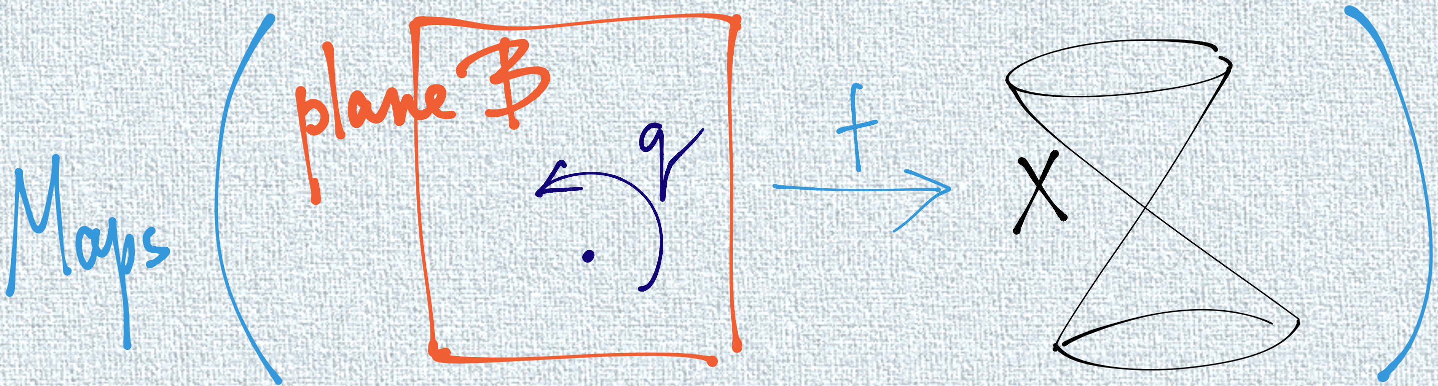
This is very powerful and also something we can actually prove when both curve counts are defined within present technology.

For instance, equivariant K-theoretic counts in algebraic geometry are group characters (in particular Laurent polynomials), whenever compact and rational functions otherwise (with location of the poles under excellent control).

This lends support to the basic expectation (known in many very challenging cases) that the indices are also rational functions of the Kähler variables z whenever B is compact. This includes e.g. the famous rationality in the boxcounting variable in the DT theory

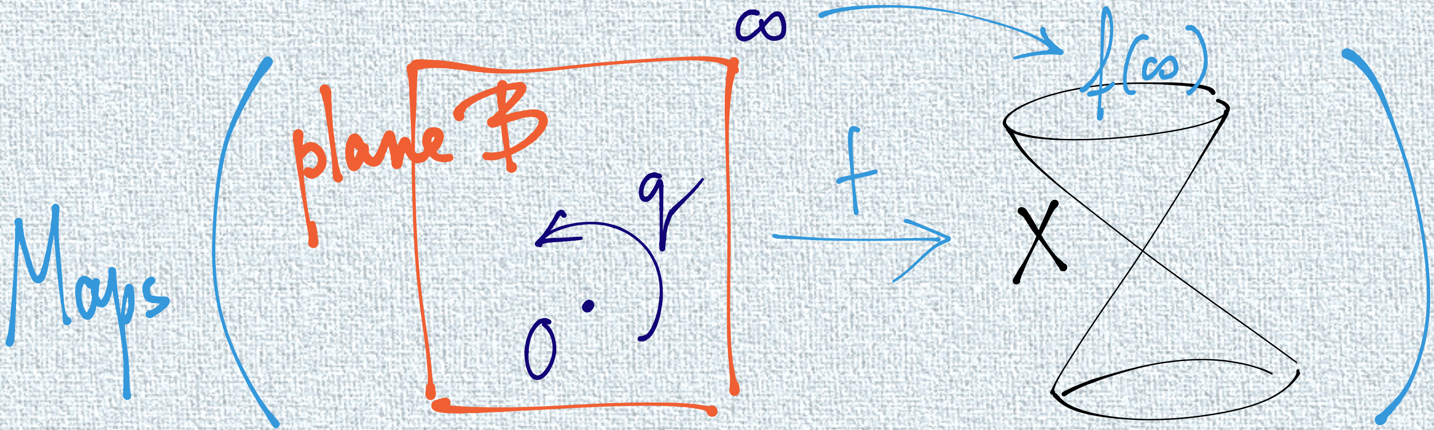
Rational functions are very simple and therefore not so easy to characterize. To get a better grip on the things, we will introduce progressively more and more complicated, and hence more rigid, functions and structures.

As a first step, we will take B which is noncompact. The simplest such geometry is the whole complex plane. We work equivariantly with respect to rotation q\in Aut(B), which is analogous to the definition of Nekrasov's function in 5d.



In math, this moduli space is defined as an open set inside maps from P^1 to X for which **f** is nonsingular at infinity (similarly to the use of Uhlenbeck compactification in Nekrasov theory).

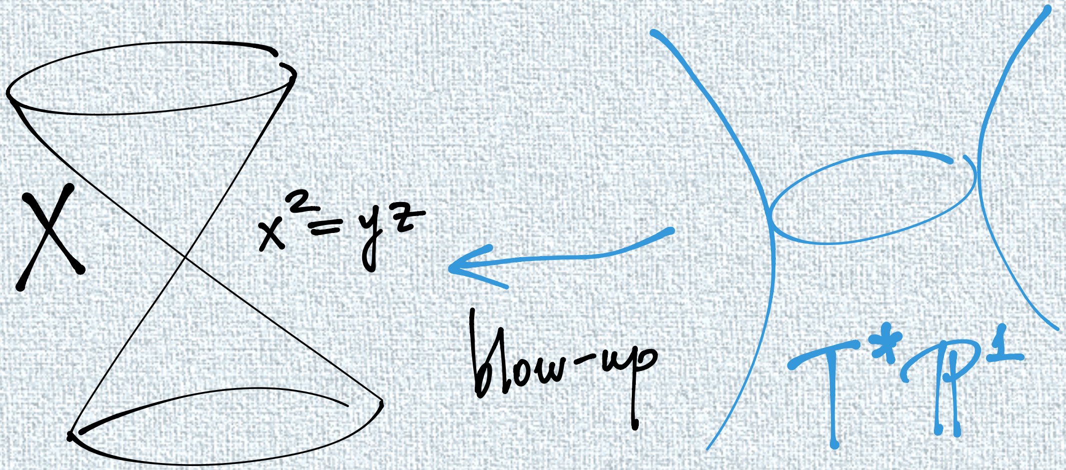
One name for these indices are vertex functions. They take an argument in K(X) via the evaluation f(∞) \in X.



They can take a further insertion at 0, in which case they look like operators in K(X)

These vertex functions are not rational functions. Instead, they solve q-difference equations in all other arguments with rational coefficients. (The matrices of these equations come from indices on P^1, which sort of explains why they are rational. The actual math argument goes in the opposite direction.)

Vertex functions are fancy generalization of q-hypergeometric functions, and become exactly q-hypergeometric functions for the simplest nontrivial X.



weight of the symplectic form ω

$$\text{Vertex} = \text{perturbative prefactor} \sum_{d>0} z^d \frac{(\hbar)_d(\hbar a)_d}{(q)_d(qa)_d}$$

$$\begin{pmatrix} 1 & \\ & a \end{pmatrix} \in PGL(2) = \mathrm{Aut}(X,\omega)$$

$$(x)_d = \prod_{i=0}^{d-1}(1-q^i x)$$

The XIX century charm of the q-hypergeometric function should not be misleading, in general these are not as huggable:

For instance, for X=Hilb(C^2,n) the vertex function is the 1-leg vertex in K-theoretic Donaldson-Thomas theory (whence the name). Vertex functions for X=Hilb(A_r,n) contain 2- and 3-leg vertices for K-theoretic DT counts.

Remarkably, the q-difference equations are known in some generality (in terms of certain powerful geometric representation theory [O., Smirnov-O.]). If the q-difference equation in just one of the variables is known, this constrains them all essentially uniquely.

E.g., for X=Hilb(C^2,n) the equation is the q-difference version of the quantum differential equation of [O.-Pandharipande]. It is what people may call the dynamical equation for quantum double loops into gl(1) (= q-difference version of the algebra from Davide's lecture)

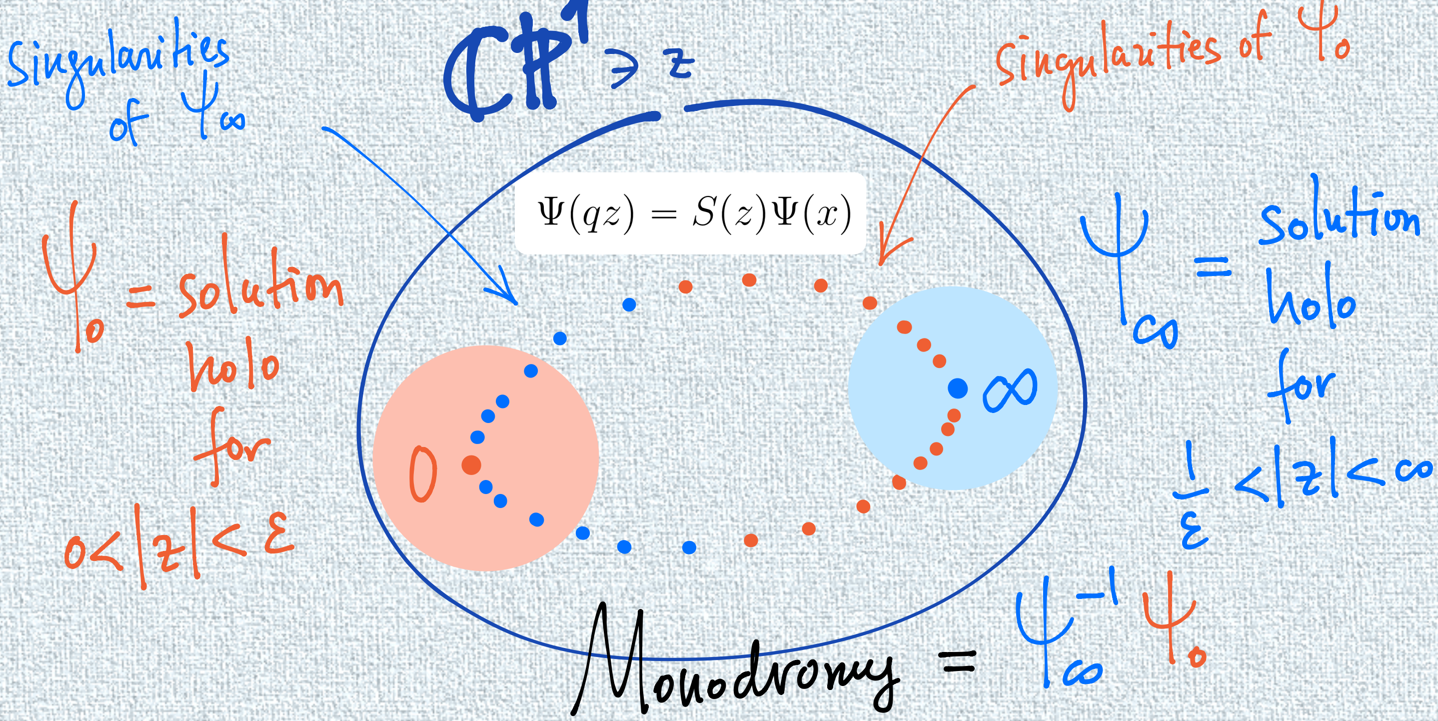
For the 3d mirror symmetry, X=Hilb(C^2,n) is self-mirror, so the equations in z and a \in SL(2) should be the same. True, but very nontrivial to see, a very indirect argument is needed.

The best handle we have on these equations goes through the identification of their monodromy.

The monodromy of a q-difference equation is the comparison of the fundamental solution near z=0, e.g.

$$\mathsf{Vertex}_{\pm} = \cdots \exp\left(\mp\frac{\ln \mathbf{z}\ \ln \mathbf{a}}{\ln q}\right)\sum_{d=0}^{\infty} \mathbf{z}^d \frac{(\hbar)_d(\hbar\mathbf{a}^{\pm 1})_d}{(q)_d(q\mathbf{a}^{\pm 1})_d}$$

with the solution of the same equation near z=∞
(which is given by the vertex functions of the flop of X). Given by a matrix of elliptic functions in z, a, and \hbar



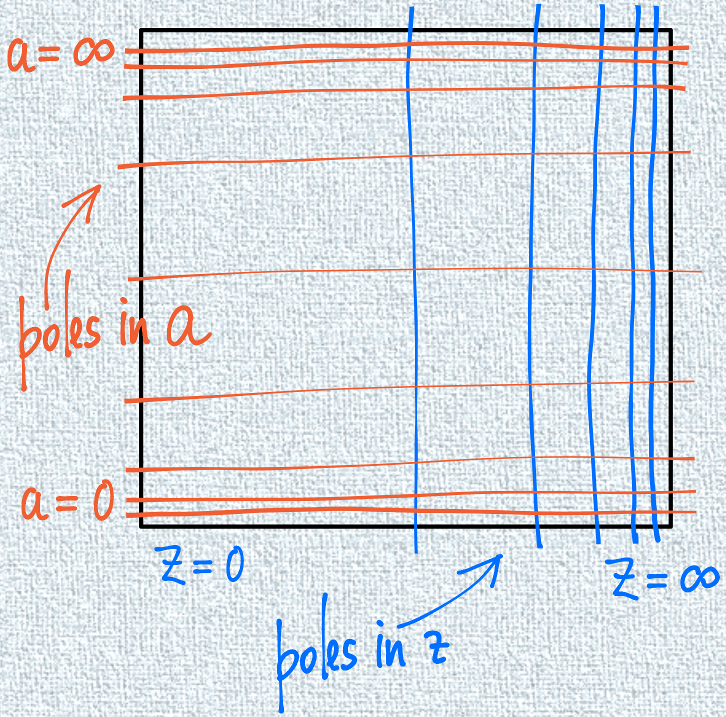
$$\Psi(qz) = S(z)\Psi(x)$$

Monodromy $= \psi_\infty^{-1}\psi_0$

Abstractly, the monodromy is a transcendental map between two algebraic varieties, like the exponential map in Lie theory. In our case, under excellent control due to the following physical/ geometric reason ...

Recall that for compact B, the 3d mirror symmetry equates counts of maps for B to X and $\check{X}$ with a change of variables. For the vertex functions (i.e. for B=complex plane), only the equations should be the same, not the solutions. Solutions pick the z-side or the a-side!

$$\sum_{d=0}^{\infty} \mathbf{z}^{d} \frac{(\hbar)_d(\hbar \mathbf{a}^{\pm 1})_d}{(q)_d(q\mathbf{a}^{\pm 1})_d}$$



It is an interesting feature of q-difference equations, that an equation can be separately regular in two groups of variables, but not jointly. This can never happen for differential equations !

The z-solutions and the a-solutions of the same difference equation must be connected by a matrix of elliptic function that exchanges poles in z-variables for the poles in a-variables. We call it the the **pole-subtraction matrix**.

Monodromy vs. pole subtraction:

- both elliptic
- one global/analytic another local/algorithmic
- pole subtraction contains the monodromy as follows

$$\begin{tikzcd}
\Psi_{z\to 0} \arrow[r, "\text{monodromy}"] \arrow[d, "\text{pole subtraction}"'] & \Psi_{z\to\infty} \arrow[d, "\text{pole subtraction}"] \\
\Psi_{a\to 0} \arrow[r] & \Psi_{a\to 0}
\end{tikzcd}$$

$\sum_n \psi_n(z)\, a^n$

where ψ_0 solves the $a=0$ equation

monodromy at $a=0$!

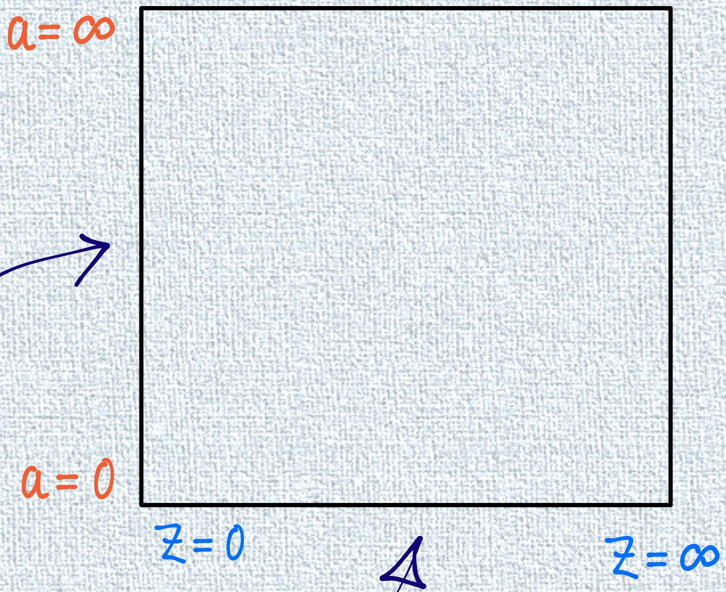
In our context

at $z=0$

there are no curves

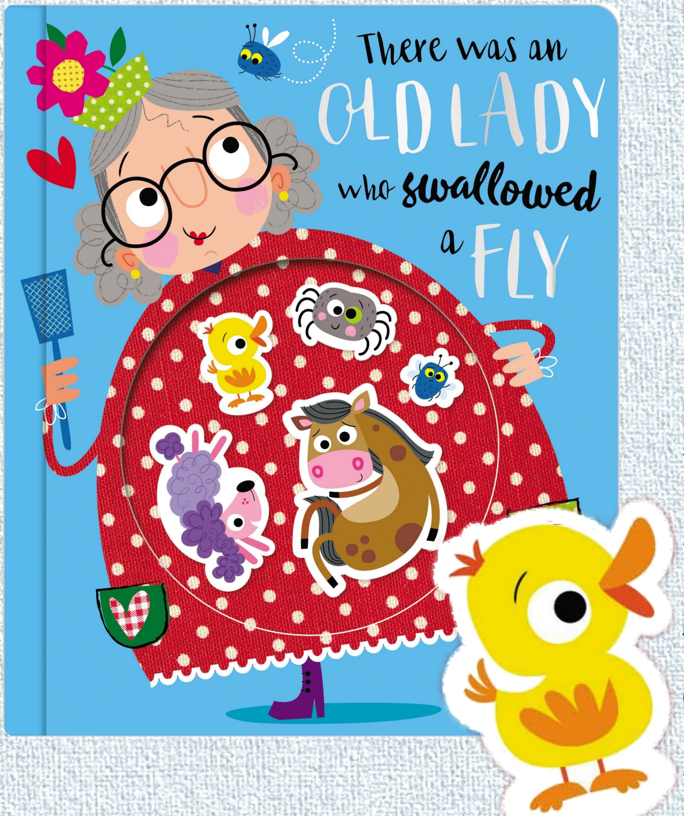
easy computation

in $K(X)$



at $a=0,\infty$ we count curves in

the fixed locus $X^a \subset X$



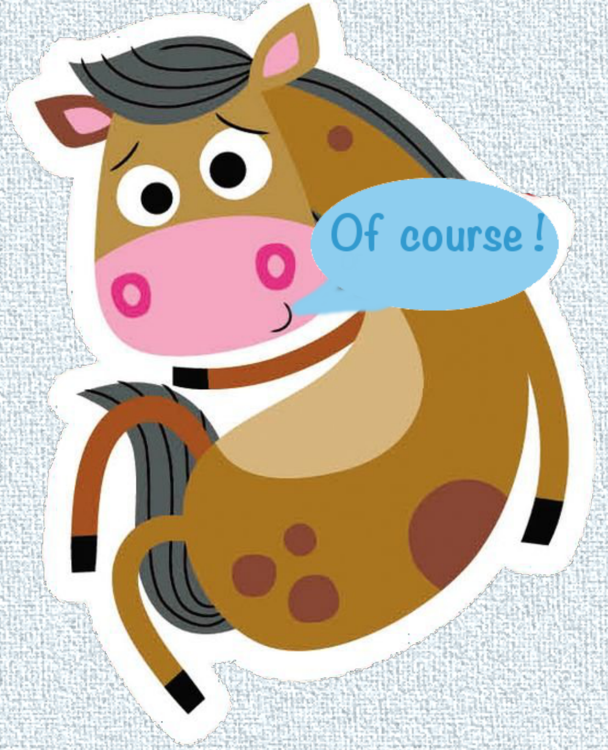
In case you feel lost:

At first, we wanted to compute some rational functions and check whether they are the same for X and $\check{X}$

To catch these rational functions, we brought in the vertex functions, which are not the same for X and $\check{X}$, but should solve the same difference equation, and so their monodromy should be the same.

To catch the monodromy we introduced the pole subtraction matrices, and now we want to prove those are the same for X and $\check{X}$

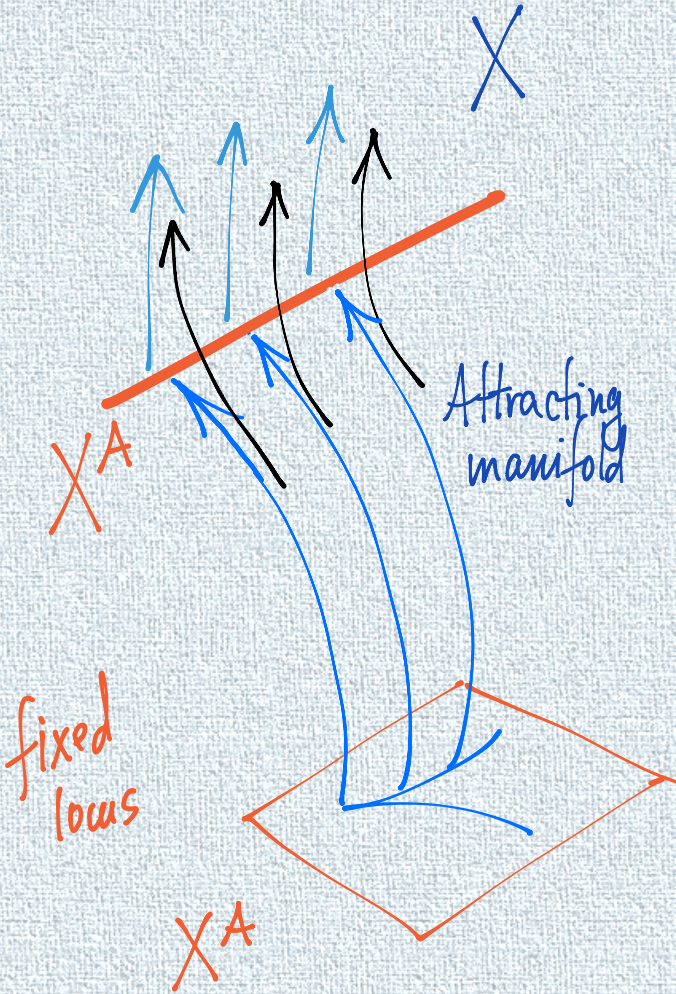
almost there, because the pole subtraction matrices are caught by the following



Theorem [Aganagic-O]
The pole subtraction matrices for X are the elliptic stable envelopes for the action of the symplectic torus A on X.

Stands to reason, as the pole subtraction matrix turns vertex functions for the fixed locus X^A into vertex functions for X

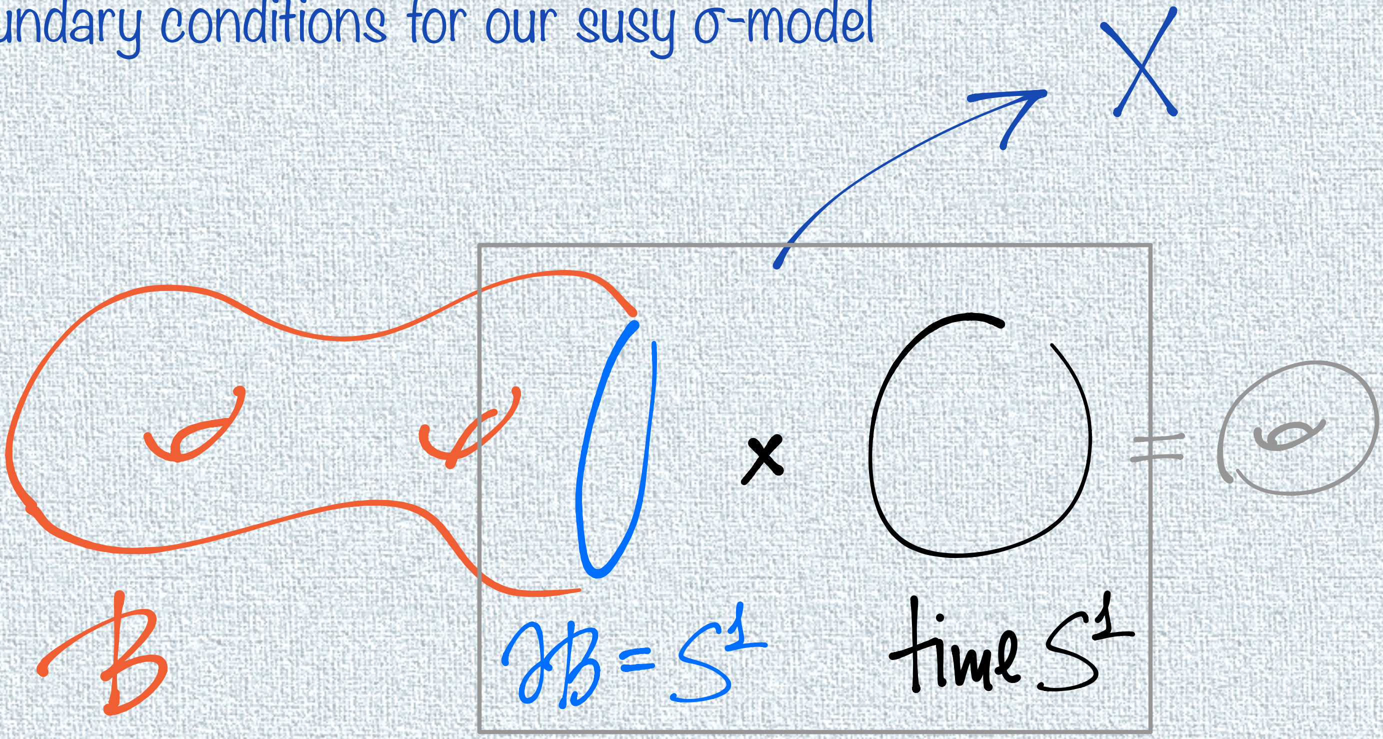
Elliptic stable envelopes of [M.A. & A.O.] are correct elliptic "classes" of attracting manifolds of the fixed loci X^A. Characterized as unique sections with given support of the right line bundles on the (spectrum of) elliptic cohomology of **X^A** × **X**. Very rigid. The line bundle is an elliptic function of the Kähler variables z.



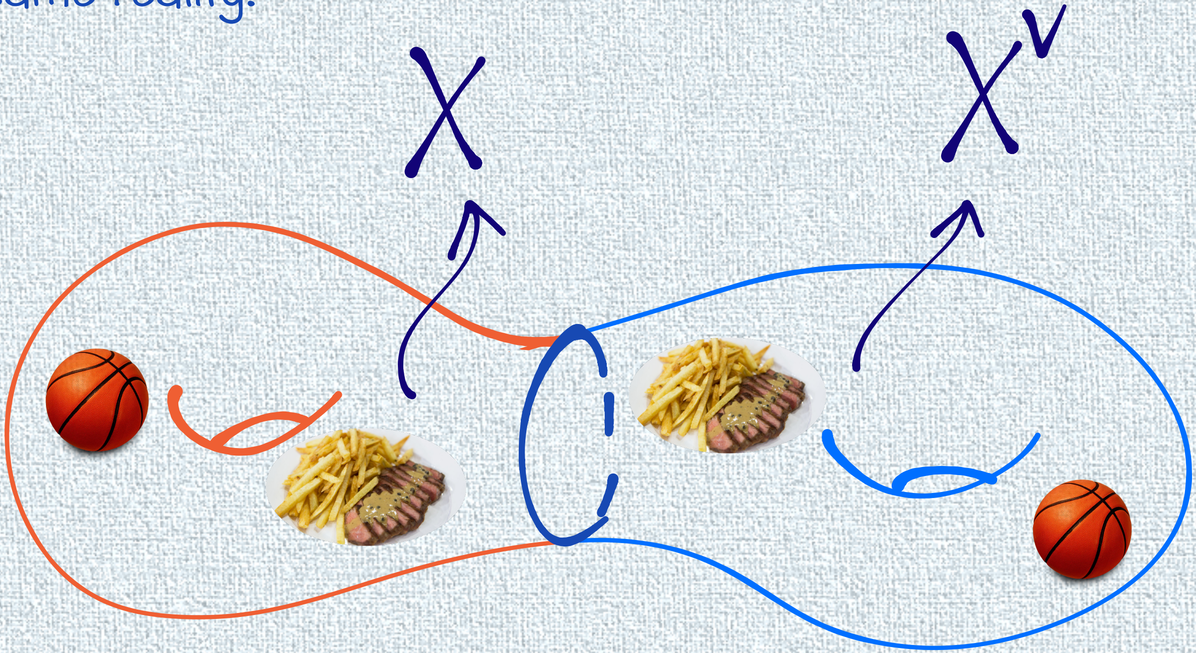
While it takes time and care to properly set up things in equivariant elliptic cohomology, it is quite remarkable that quantum, i.e. K-theoretic curve-counting computations for X may be reduced to classical computations in elliptic cohomology of X.

In the end, $\mathrm{Ell}_{\mathrm{equiv}}(X)$ is just a scheme (in good sense of this term), not too complicated, basically a union of abelian varieties, and we are asking for a section of a line bundle on it.

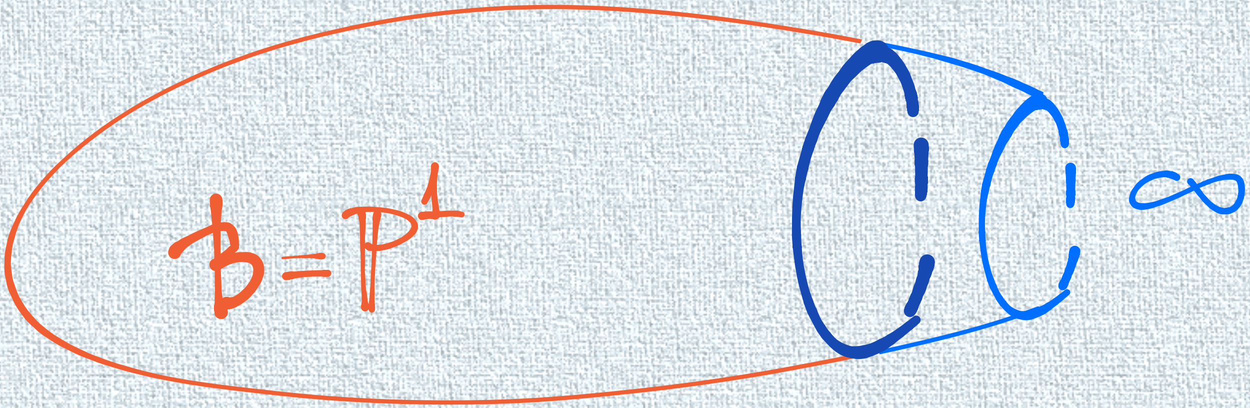
There is also a transparent physical interpretation of elliptic stable envelopes in terms of boundary conditions. Elliptic cohomology of X is one, maybe the only, version of the K-theory of the category of boundary conditions for our susy σ-model



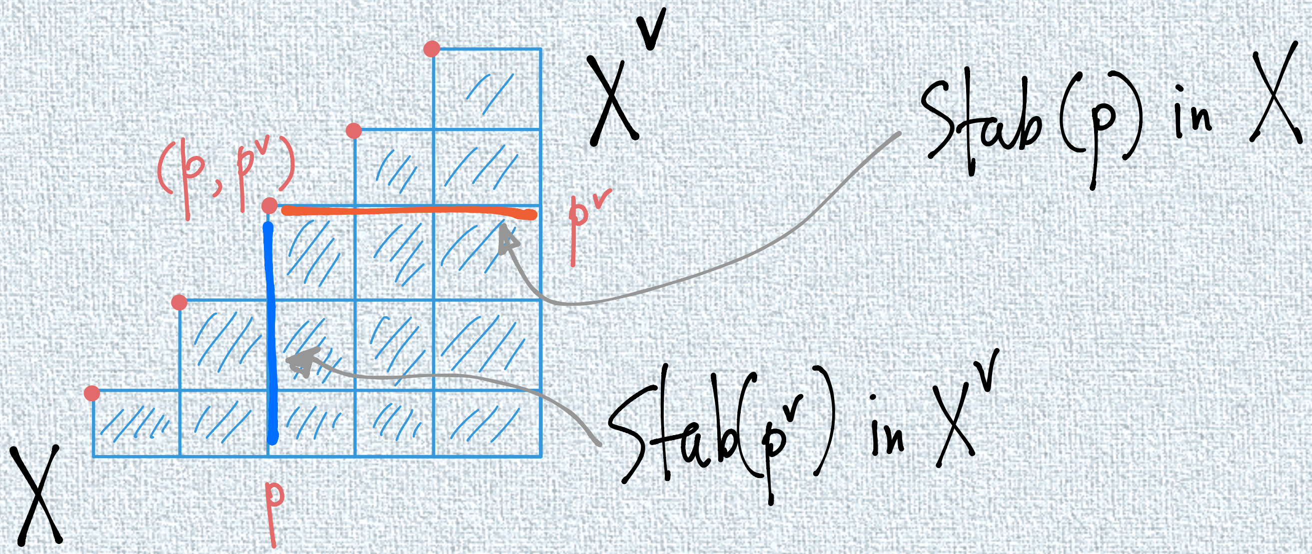
If two theories are equivalent, there is a special **duality interface** between the two bulk theories along which nothing changes, except the words that we use to describe the same reality.



In particular, the duality interface should turn the vertex functions for X into vertex functions for its mirror $X^\vee$, which are the a-solutions of the same q-difference equation, and vice versa.



Therefore, the duality interface produces an elliptic class **M** on $X \times X^\vee$, which gives the stable envelopes for X upon restriction to torus-fixed points in $X^\vee$ and vice versa. Recall both a and z enter the definition of stable envelopes for X, and **M** is a section of a line bundle that depends on a and z symmetrically.



In fact, just the data of two schemes $\mathscr{E}$ and $\mathscr{E}^\vee$ with certain maps that make them look like Ell(X) and Ell($X^\vee$) is enough to uniquely characterize the mother class **M**, the stable envelopes, the monodromies, the difference equations, and KthFTs … From this, or any perspective, the existence of **M** is highly nonobvious

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "\text{chern classes}"] \arrow[d] & \mathrm{Hom}\left(\mathrm{Pic}(X), \mathbb{C}^*/q\right) \\
\mathrm{Ell}_{\mathrm{equiv}}(pt) \supset A/q & \text{dual to } \mathbb{Z}/q \arrow[u]
\end{tikzcd}$$

For **gauge theories**, let Xˇ be the mirror of X in the following sense: the fixed points are isolated on both sides and the fixed points p in X are mirror to the tangent spaces to fixed points in Xˇ. This means that certain 1 x 1 difference equations coincide, which is reasonably straightforward to check.

Theorem* (Aganagic-O) With these hypothesis, the class M exists and the two KthFTs are equivalent.

Example: Abelian theories. These come from a dual pair of exact sequences of tori

$$1 \to T_1 \longrightarrow \overset{\cong (\mathbb{C}^\times)^n}{T} \longrightarrow T_2 \longrightarrow 1 .$$

$$X = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/ T_1 \qquad X^\vee = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/ T_2^\vee$$

$$M = \prod_{i=1}^{n} \vartheta(t_i t_i^\vee)$$

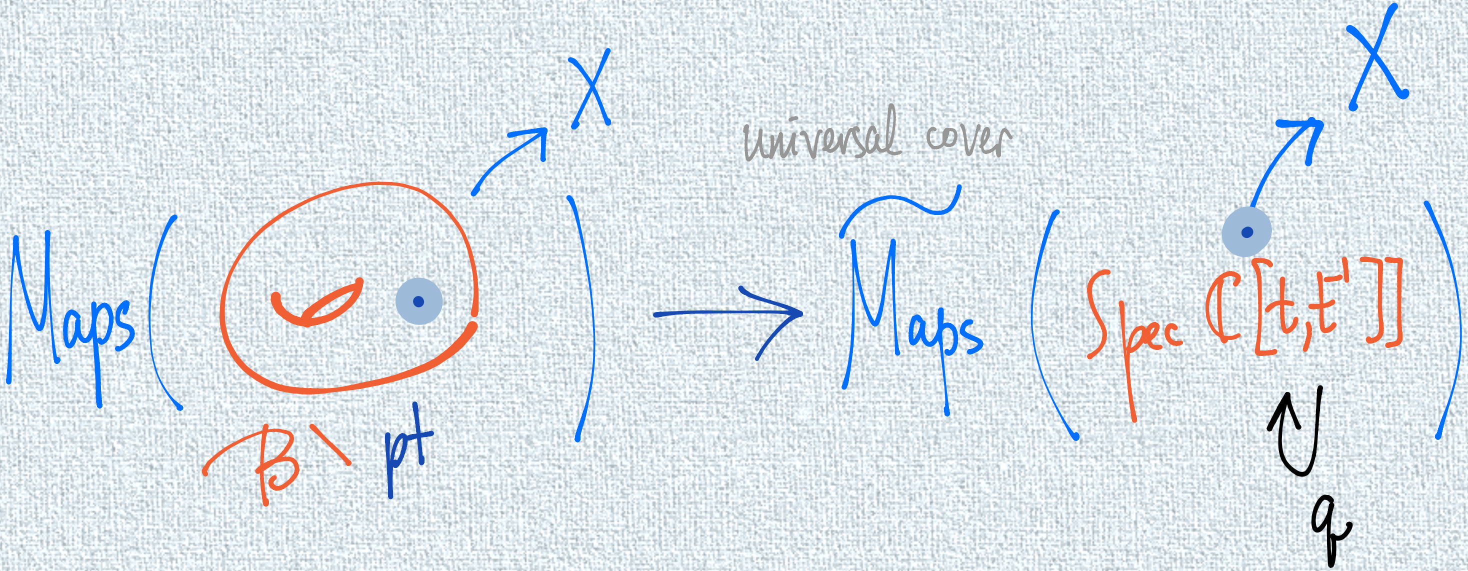
Coordinates on T

dual coordinates

Explicit formulas for some nonabelian theories have been investigated by R.Rimányi, A.Smirnov, A.Varchenko, Z.Zhou

Categorification

Alg geom version of boundary values is the natural map



therefore the (Tate version of the) category of boundary conditions should be certain equivariant quasicoherent sheaves on LX with, roughly, half-dimensional support

On LX itself, there is an action of

- q by loop rotations
- Cochar(A) \subset Loops(A)
- Char(Z) = pi_1=connected components of Loops(Gauge)

On equivariant sheaves on LX, there is further action of

- Char(A) by equivariant twists
- Cochar(Z) twists by line bundles

We expect the mother class M to categorify to an equivalence intertwining these actions.