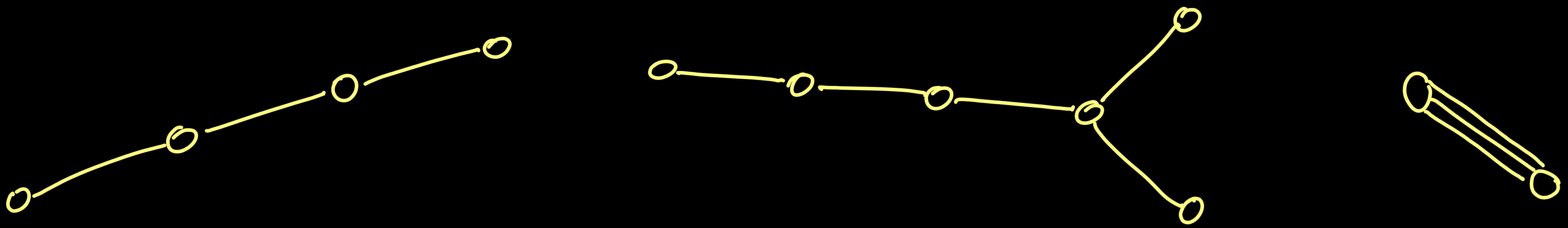
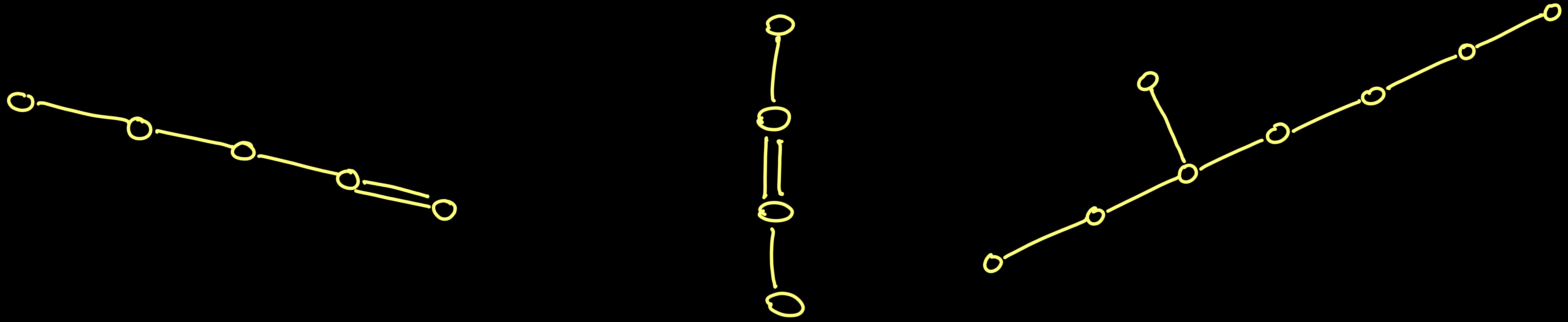

New worlds for Lie Theory
math.columbia.edu/~okounkov/icm.pdf

Lie groups, continuous symmetries, etc. are among the main building blocks of mathematics and mathematical physics

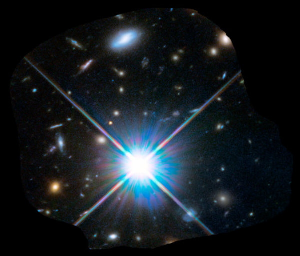
Since its birth, Lie theory has been constantly expanding its scope and its range of applications.



Simple finite-dimensional Lie groups have been classified by the 1890s. Their elegant structure and representation theory in many ways shaped the development of mathematical physics in the XX century



In return, very concrete questions prompted many fruitful directions of research in Lie theory



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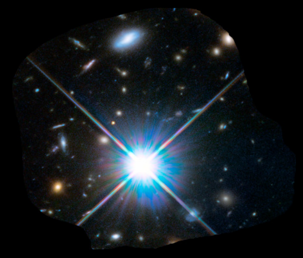

The Virasoro Lie algebra, Affine Lie algebras (a special case of Kac-Moody Lie algebras), and their relatives, play a key role in the study of 2-dimensional critical phenomena described by the Conformal Field Theories

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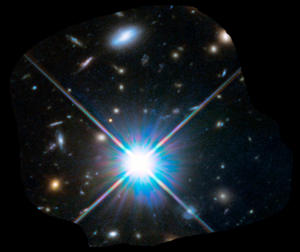

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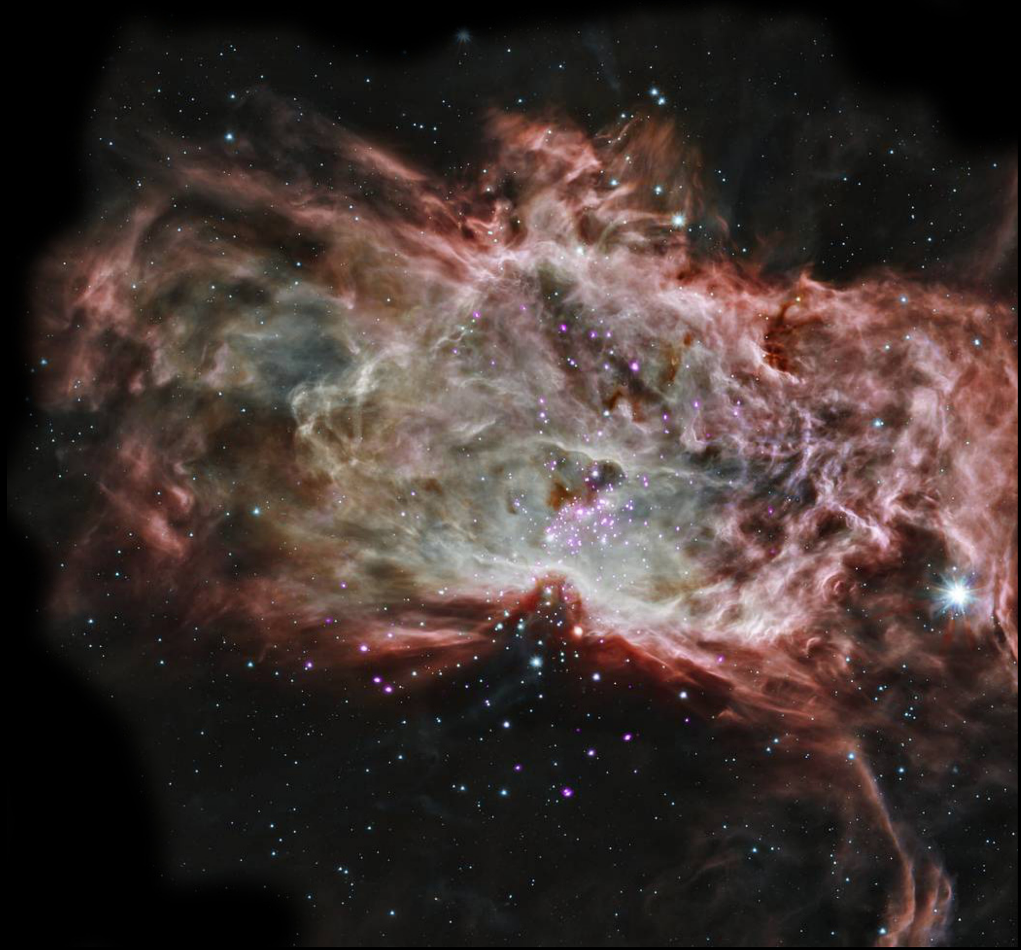
Their quantum group analogs underlie integrable lattice discretizations of CFT

Today, I want to talk about a more recent set of ideas that links mathematical physics with Lie theory in a new, much expanded sense. It originates in the study of supersymmetric QFT, in particular, susy gauge theories in <4 (especially 3) space-time dimensions



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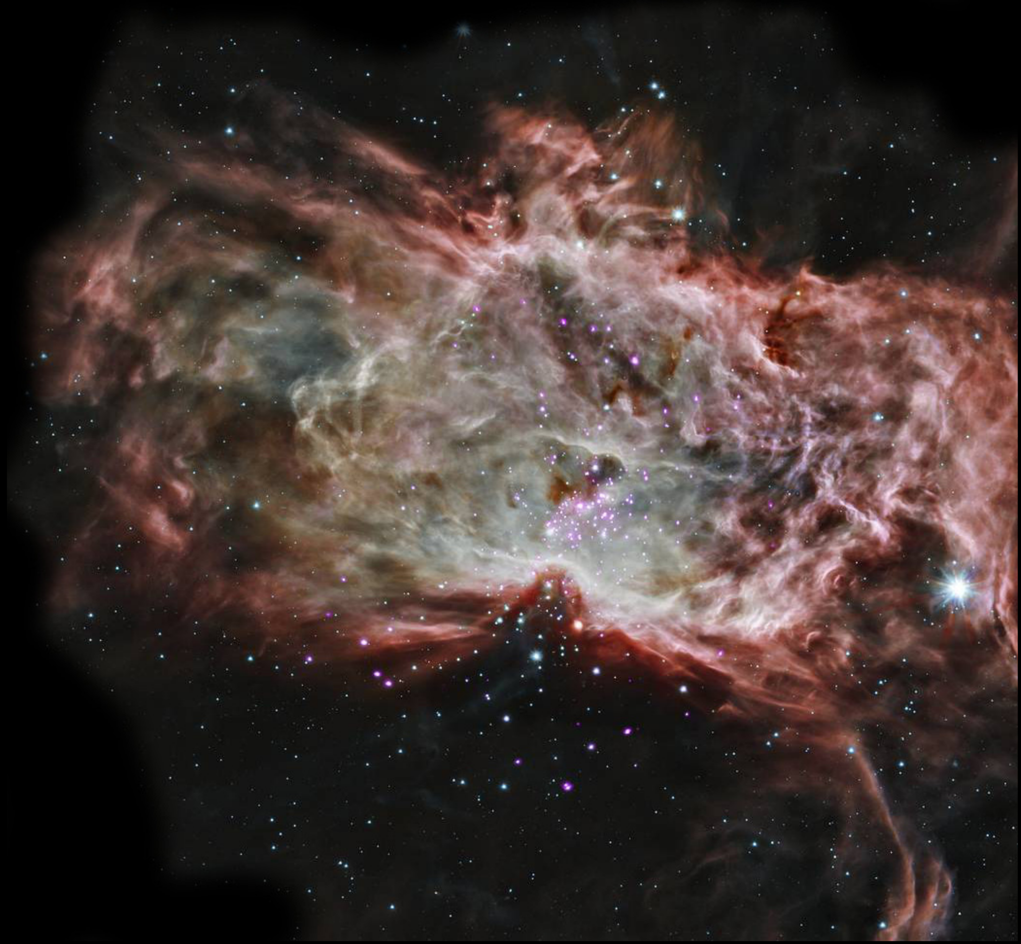
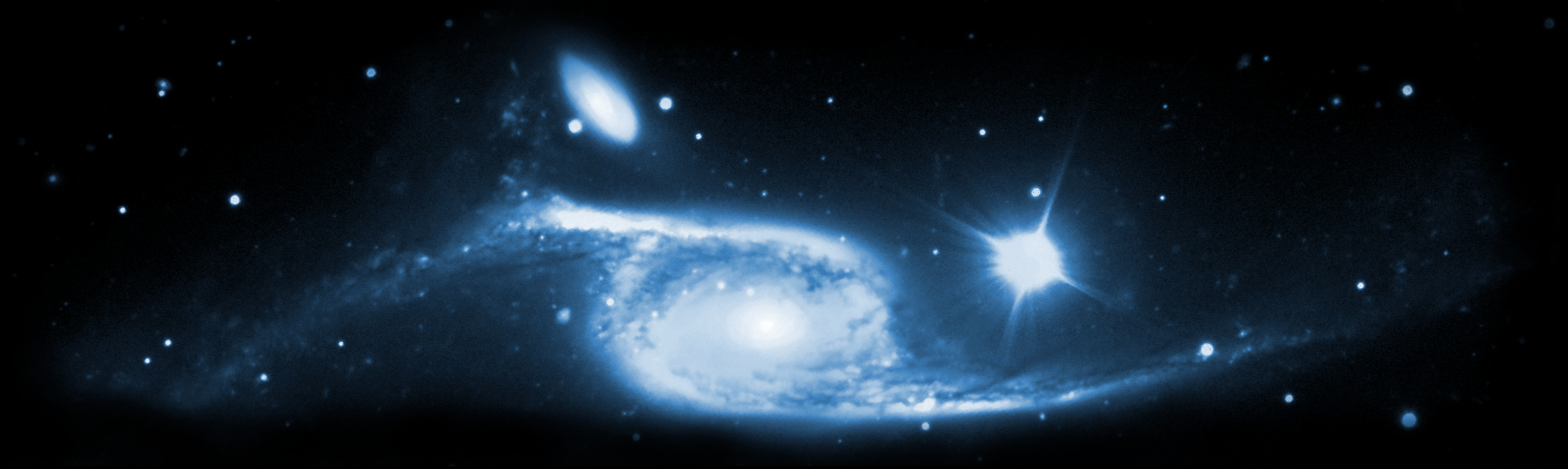
I want to share with you my excitement about a subject that is still forming. We don't see yet its true logical boundaries and our definitions, technical foundations, etc. are improving in real time. I will try to stick to what we know for certain and not try to be too visionary.



What I understand about the subject owes a great deal to Nikita Nekrasov and Samson Shatashvili, as well as to Mina Aganagic, Roman Bezrukavnikov, Hiraku Nakajima, Davesh Maulik, and many others



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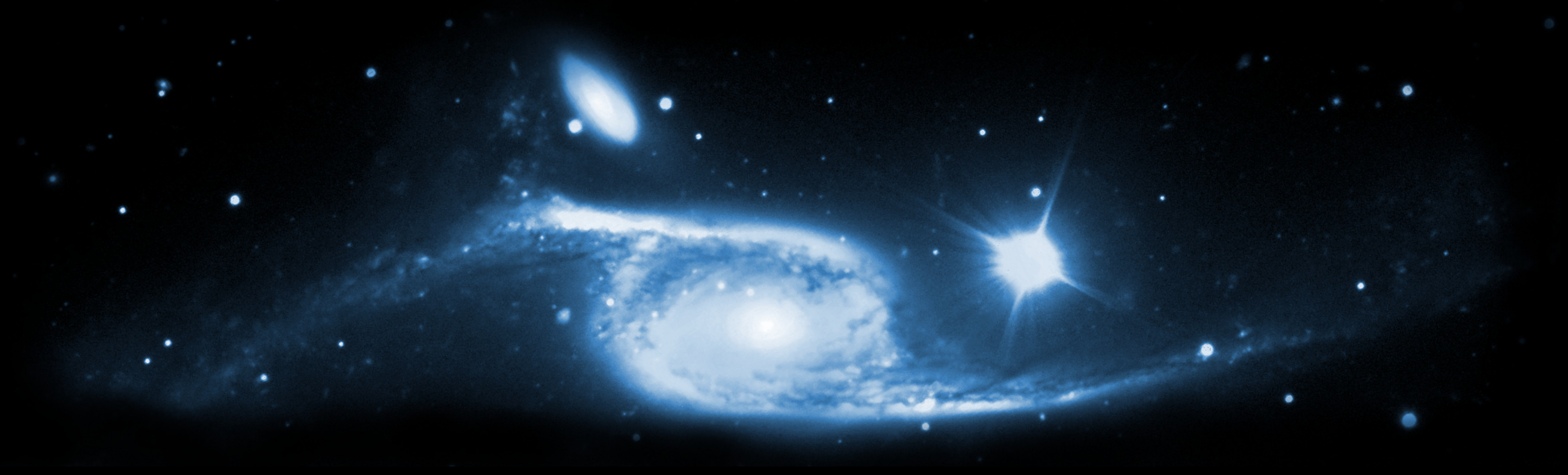
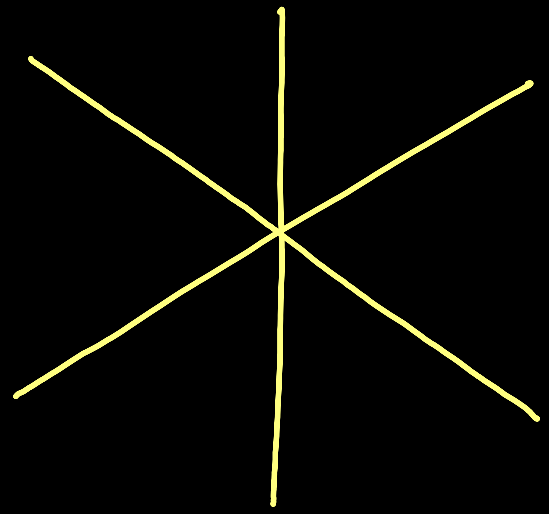


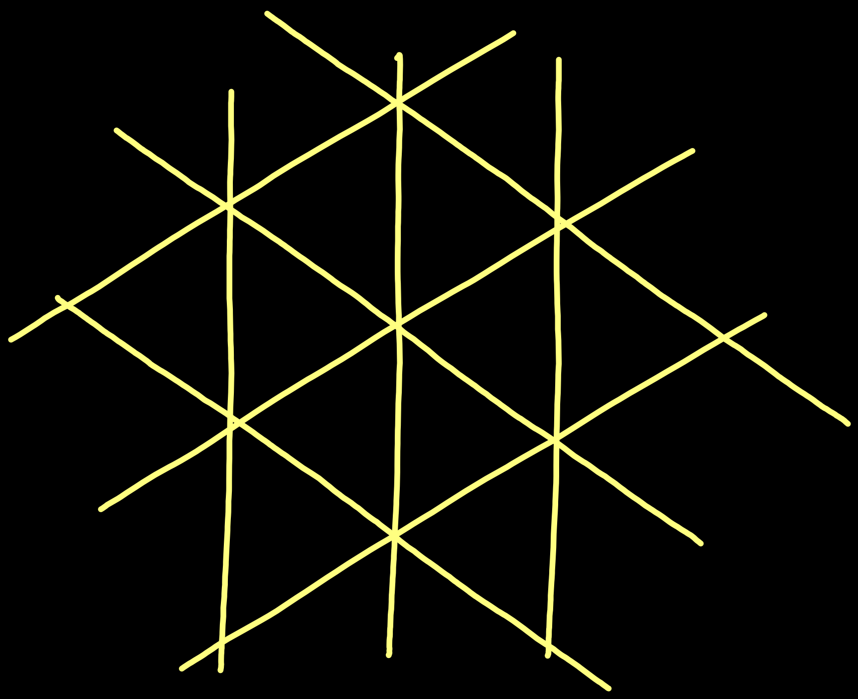
One of the guiding stars in the subject has been a certain powerful duality that generalizes Langlands duality to this more general setting. It goes back to Intrilligator and Seiberg, and has been studied by many teams of researchers, in particular, by Davide Gaiotto, Hiraku Nakajima, Ben Webster, and their collaborators

It may be easier to explain what is new by explaining which highlights of the late XX century Lie theory are being generalized

Before, it may be helpful to remind ourselves what is a Weyl group, Hecke algebra, etc. as generalizations of these objects will be essential in what follows

Finite and discrete reflection groups W of a Euclidean space $\mathbb{R}^n$ appear in Lie theory as finite and affine Weyl groups and play a central role in classification and representation theory





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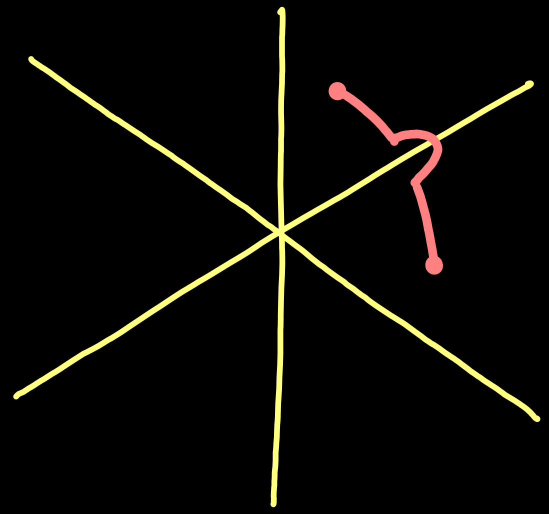
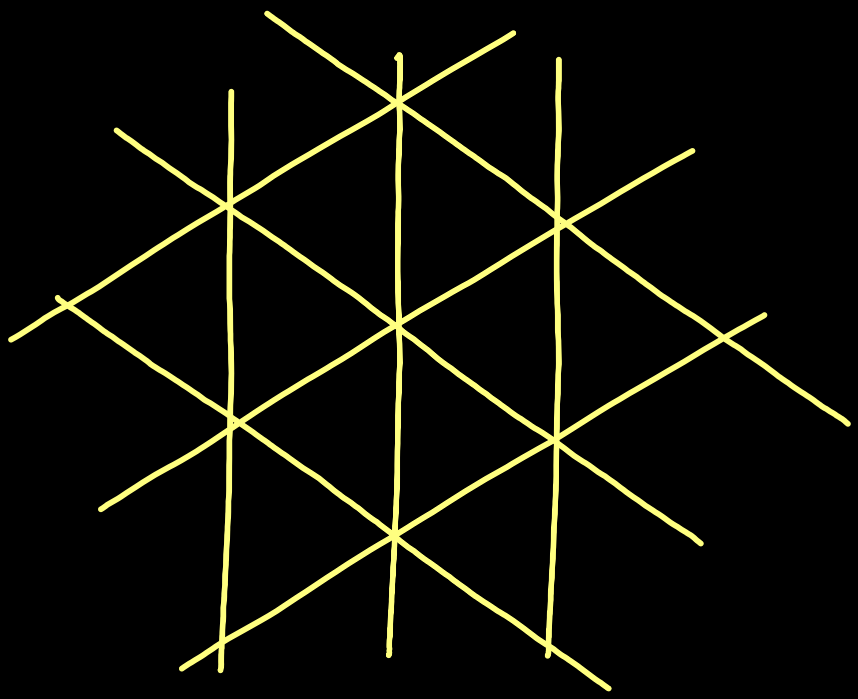




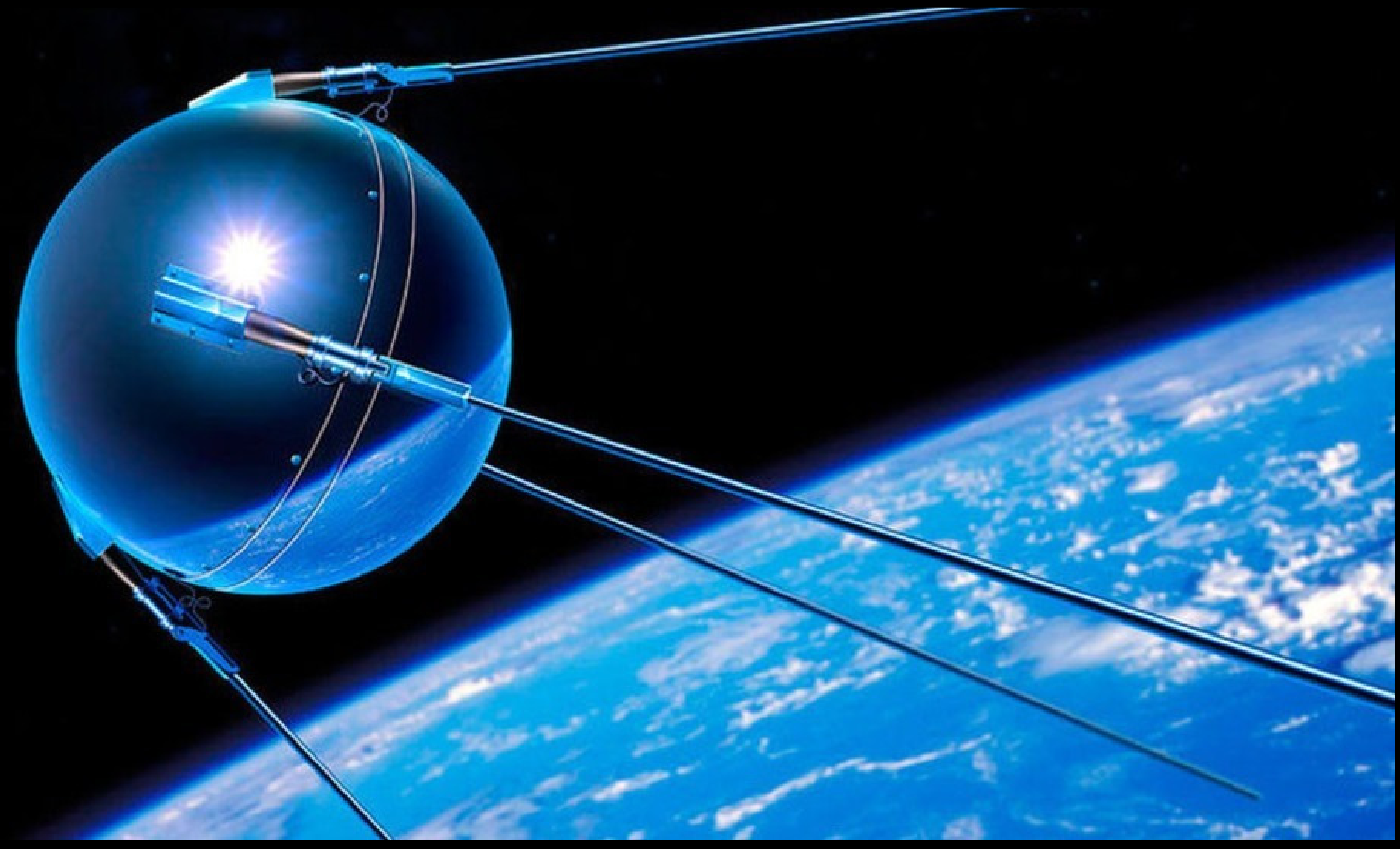
To every W one can associate a braid group

$$\mathcal{B} = \pi_1\left(\mathbb{C}^n_{reg}/W\right)$$

and a Hecke algebra, in which the generators satisfy a generalization of $S^2=1$

Back to

... It may be easier to explain what is new by explaining which highlights of the late XX century Lie theory are being generalized



Macdonald-Cherednik theory

Irreducible Lie group characters and, more generally, spherical functions, are eigenfunctions of invariant differential operators, that is, solutions to certain linear differential equations. In MC theory, these are generalized to certain q-difference equations associated to root systems and involving additional parameters. Solutions of these equations are remarkable multivariate generalizations of q-hypergeometric functions, whose terminating cases are known as the Macdonald polynomials.

Numerous applications of those in combinatorics, number theory, probability theory, algebraic geometry etc. have been found.

Macdonald–Cherednik theory

The algebraic backbone of the theory is a certain double version of the affine Hecke algebra constructed by Cherednik. A fundamental symmetry of this doubling yields an amazing label-argument symmetry in Macdonald polynomials, of which

$$P_n(q^m) = P_m(q^n), \qquad P_n(x) = x^n,$$

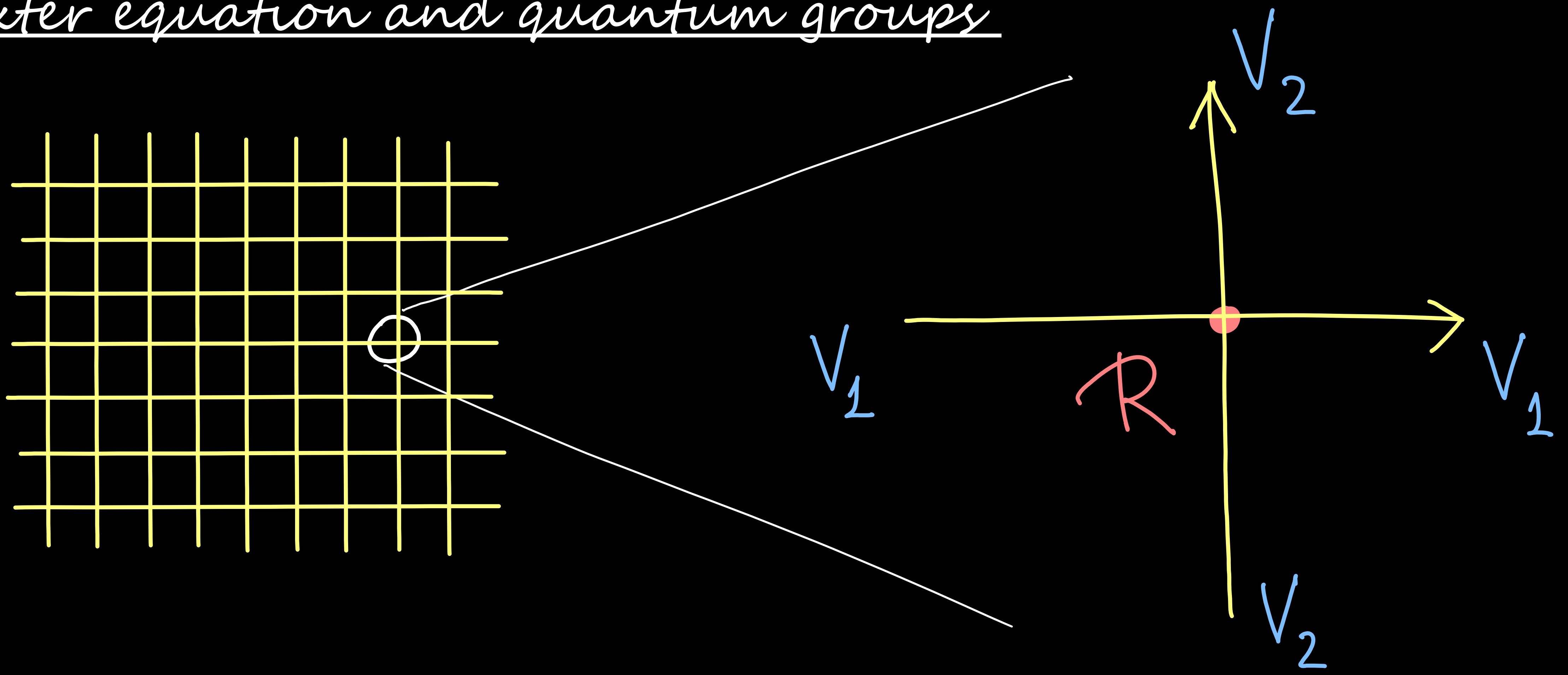
is a kindergarten example. It plays a key role in applications and is a preview of the general duality statements.

Kazhdan–Lusztig theory

in its simplest form, describes the characters of irreducible highest weight modules over a Lie algebra in terms of the combinatorics of the associated finite Hecke algebra. The proof of the original KL conjectures by Beilinson–Bernstein and Brylinski–Kashiwara is, perhaps, one of highest achievements in all of Lie theory, with further contributions by Ginzburg, Soergel, Bezrukavnikov, Williamson, and many, many others. In characteristic $p \gg 0$, there is a version with affine Hecke algebra.

$\Longrightarrow$ the talk by Geordie Williamson

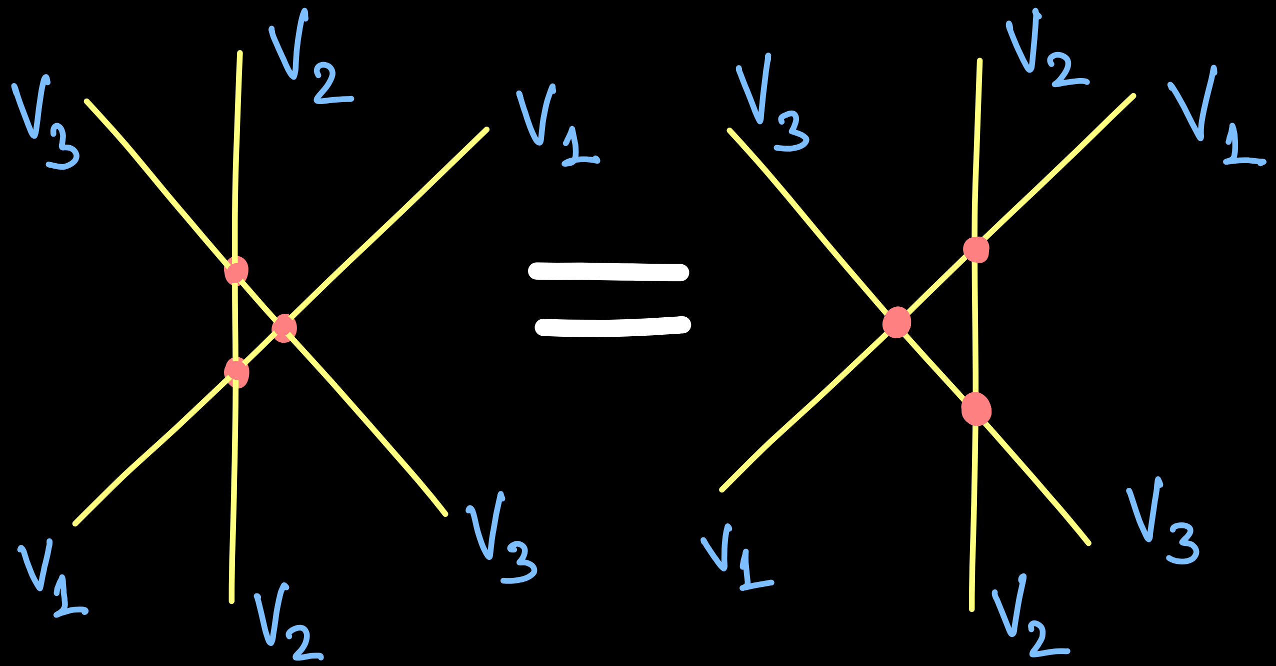
Yang-Baxter equation and quantum groups



In vertex models of 2D statistical mechanics, the degrees of freedom live in vector spaces V_i attached to edges of a grid and their interaction is described by a matrix R of weights attached to each vertex

Yang-Baxter equation and quantum groups

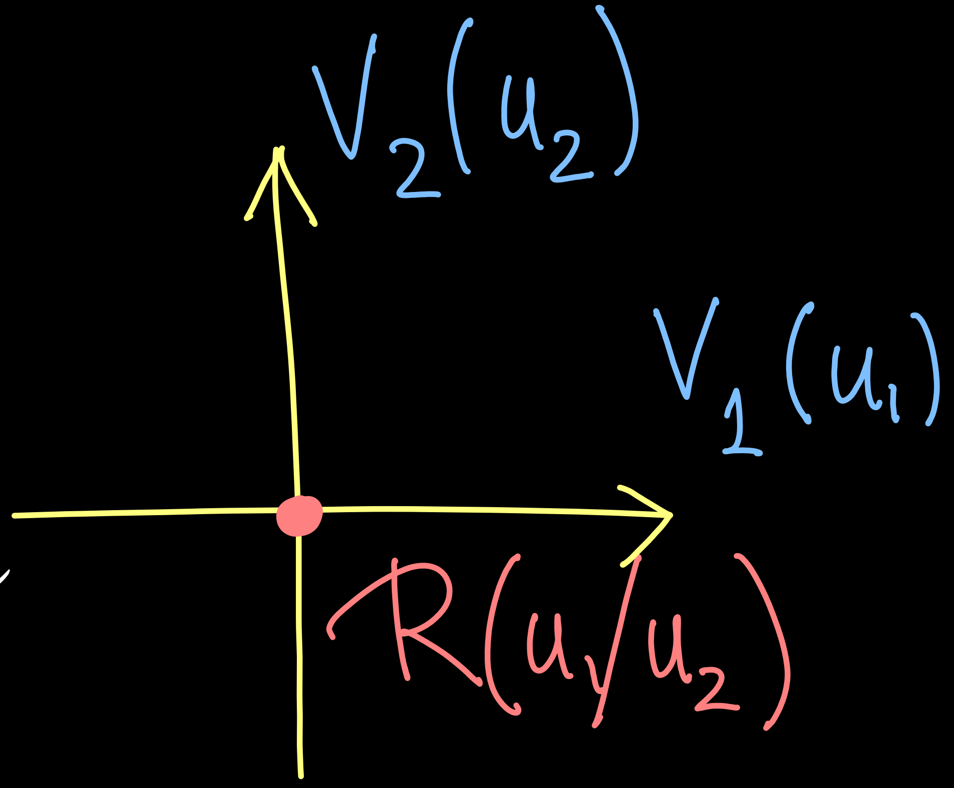
Baxter noted the importance of the YB equation



for exact solvability, with further important insights by the Faddeev and Jimbo-Miwa-Kashiwara schools. This gives rise to the whole theory of quantum groups (Drinfeld, ...), associated knot invariants, et cetera, et cetera

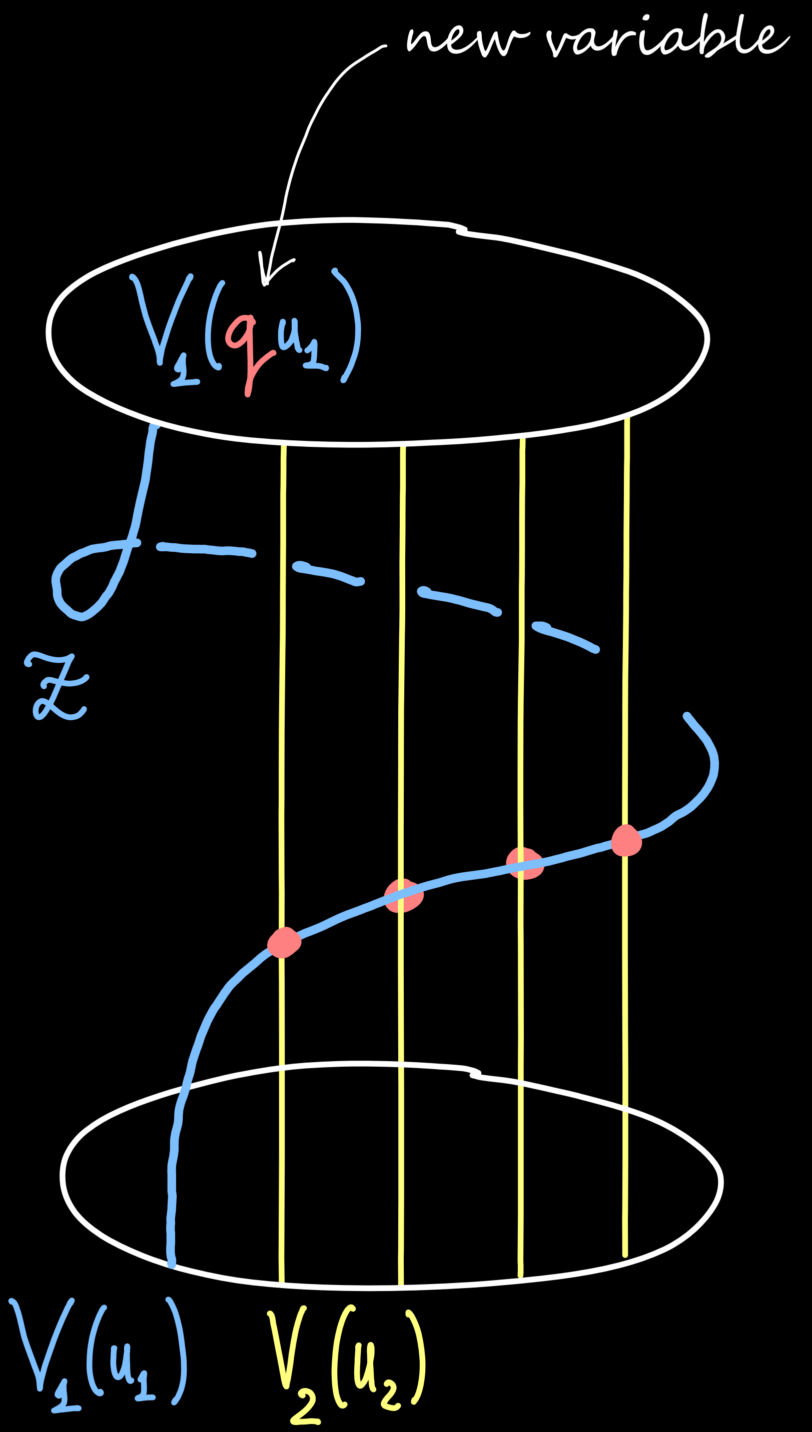
Note by Reshetikhin et al the quantum group may be reconstructed from matrix elements of the R-matrix, or as the algebra behind the braided tensor category constructed from R.

Particularly important are R-matrices with a spectral parameter that correspond to quantum loop groups. By Baxter, these contain large commutative subalgebras that become quantum integrals of motion in vertex models and associated quantum spin chains



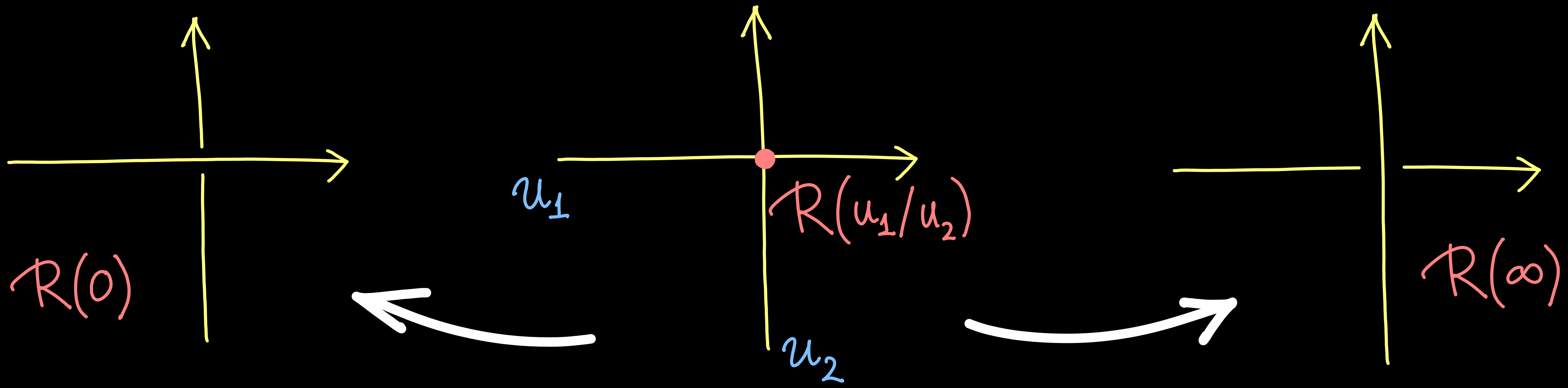
Many brilliant minds worked on diagonalization of these algebras, a problem known as the "Bethe Ansatz"

More generally, R-matrices with a spectral parameter define an action of an affine Weyl group of type A by q-difference operators, the lattice part of which are the quantum Knizhnik-Zamoldchikov equations of Frenkel and Reshetikhin. These are among the most important linear equations in mathematical physics; solving them generalizes the Bethe Ansatz problem

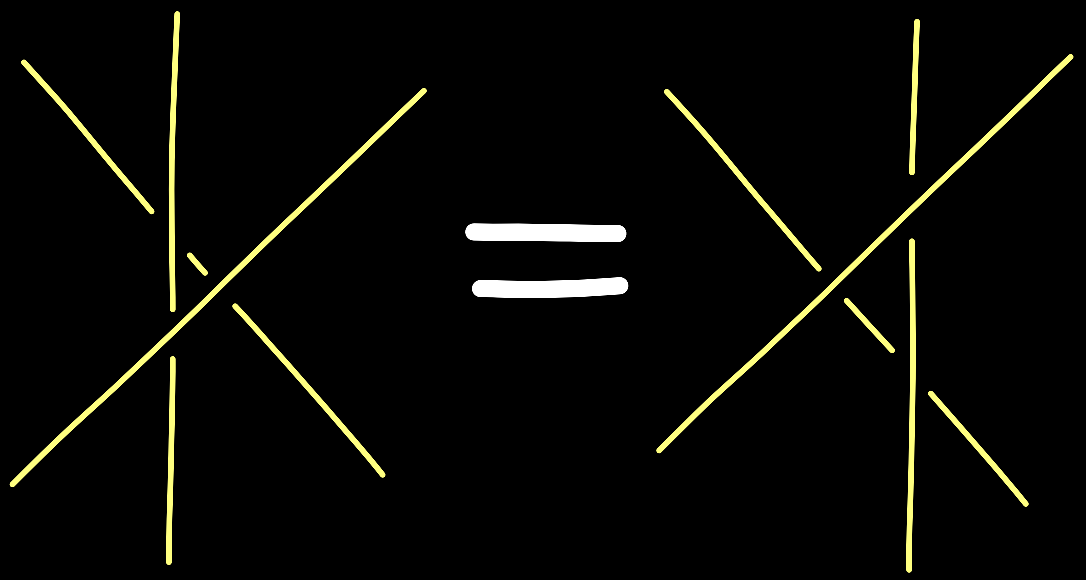


$$[Z \otimes Z, R] = 0$$

For knot theory and other topological applications, limits of $\mathbf{R}(u)$ are important



YB equation becomes a Reidemeister move

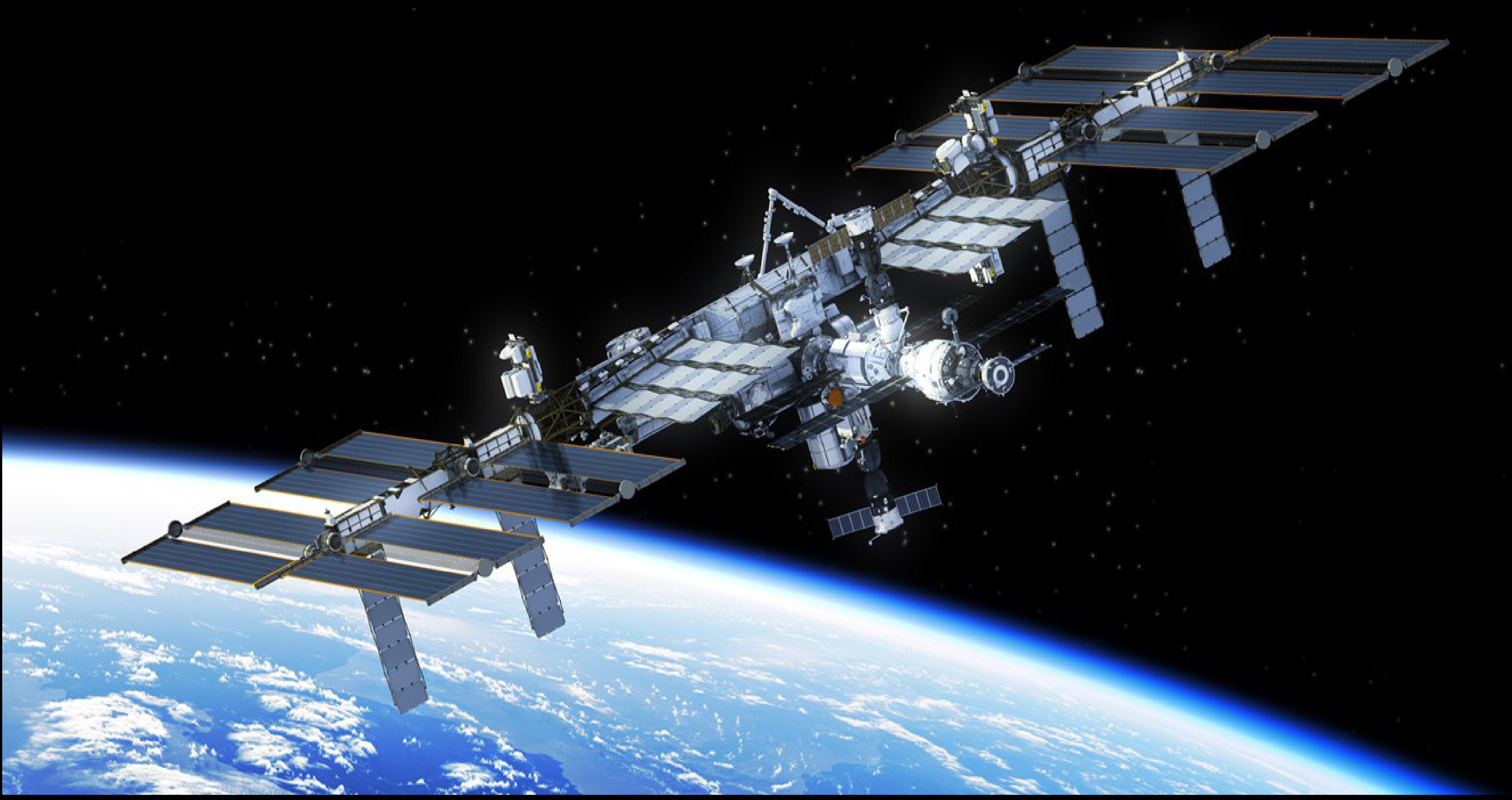


Example: $\mathfrak{gl}_2$ in $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$, $u = u_1/u_2$

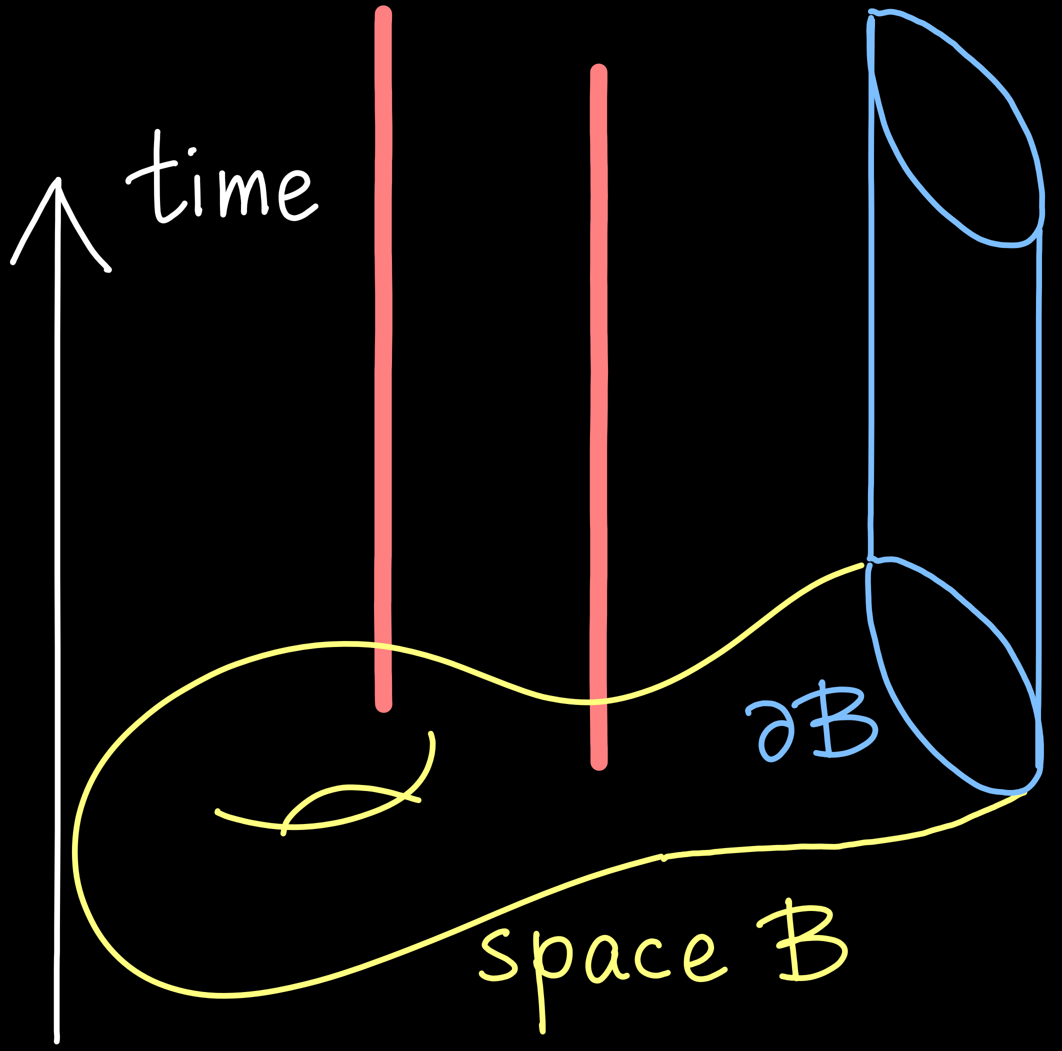
$$R(u) = \begin{pmatrix} 1 & & & \\ & \hbar^{1/2}\dfrac{1-u}{\hbar-u} & u\,\dfrac{1-\hbar}{u-\hbar} & \\ & \dfrac{1-\hbar}{u-\hbar} & \hbar^{1/2}\dfrac{1-u}{\hbar-u} & \\ & & & 1 \end{pmatrix}$$

associated to $T^*\mathbb{P}1$, classical hypergeometry, etc. Self-dual!

modern formulas are more suitable downloads than slides



A 3-dimensional supersymmetric Quantum Field Theory is a lot of data, of which we will be using only a very small piece – the susy states in the Hilbert space associated to a given time slice, a Riemann surface B with, maybe, boundary and marked points.



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Index = Even fermion number − Odd

as a virtual representation of all symmetries and as a virtual vector bundle over the moduli of B, as in the talk by Rahul Pandharipande.

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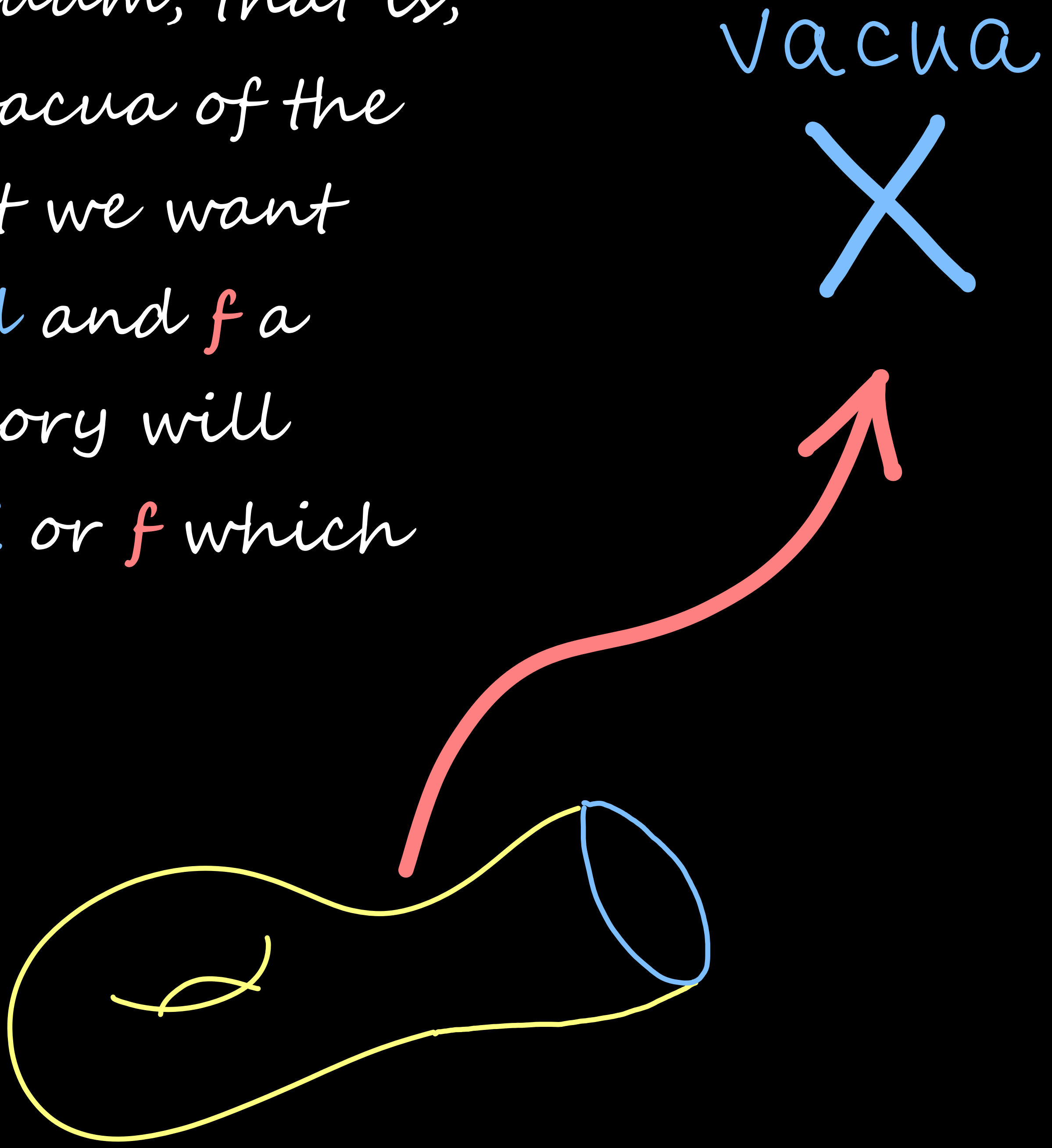
This (Witten) index being deformation invariant, it can be studied using any of the different description of the QFT in various corners of its parameter space.

At lowest energies (that is, for very large B), the states of a QFT may be described as modulated vacuum, that is, a map f from B to the moduli space X of vacua of the theory. The amount of supersymmetry that we want makes X, ideally, a hyperkähler manifold and f a holomorphic map. Finer details of the theory will become important at the singularities of X or f which are, in general, unavoidable.



Mathematically, this becomes a problem in the spirit of enumerative geometry. Susy states are holomorphic maps f from B to X, which is a symplectic algebraic variety, or stack, or ... The index is the Euler characteristic of a certain coherent sheaf (a virtual $\hat{A}$-genus, like for the index of a Dirac operator) on the moduli space of such. This index is graded by the action of $\mathrm{Aut}(X)$. The additional grading on this index by the degree of the map may be viewed as a character of the Kahler torus

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To make the index nontrivial, we require that the symplectic form ω_X is scaled by $\mathrm{Aut}(X)$ with a nontrivial weight $\hbar$

For example, susy gauge theories contain gauge fields for a compact form of a Lie group G, matter fields in a symplectic representation M of G, and their superpartners. In this case

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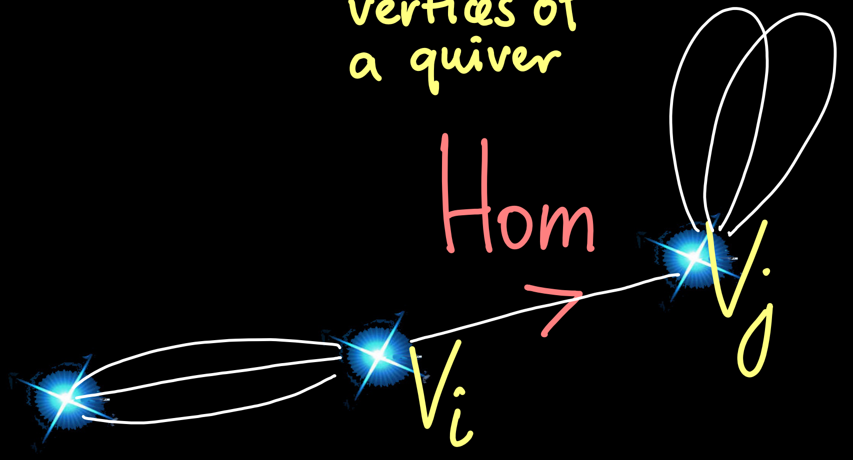
$$G = \prod GL(V_i) \qquad M = \bigoplus \mathrm{Hom}(V_i, V_j) \oplus V_i\text{'s} \oplus \text{duals}$$

vertices of a quiver (under $\prod GL(V_i)$)

edges of a quiver (under $\bigoplus$)

many (under V_i's)



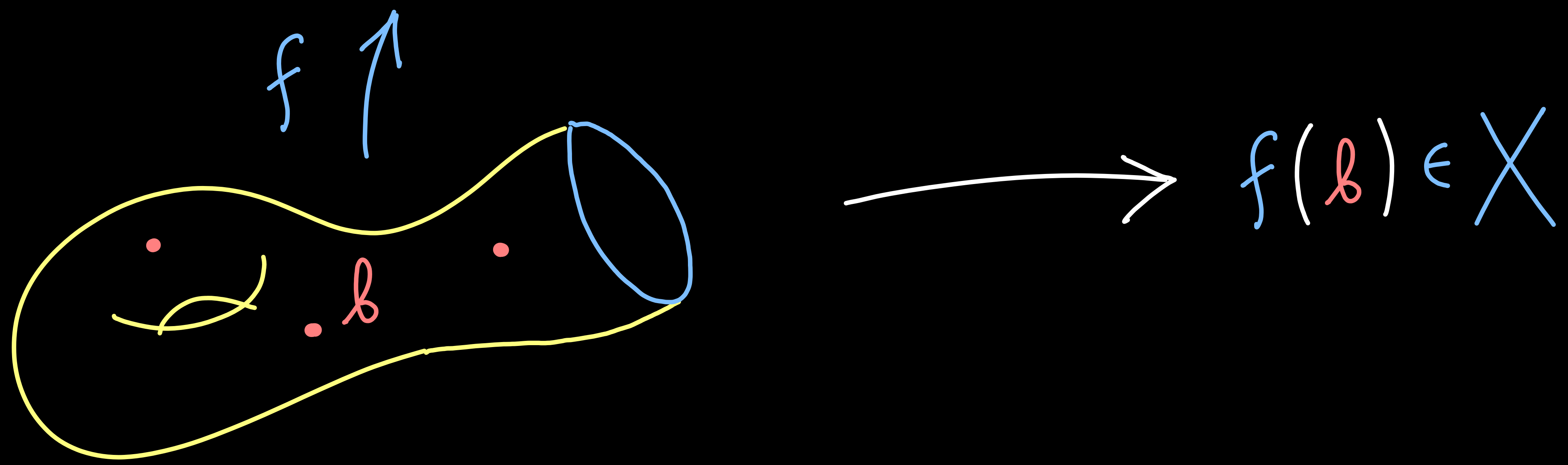
then X is a Nakajima quiver variety. The quiver is the generalization of the Dynkin diagram from before

For instance, for this quiver this theory is:

(1) the K-theoretic Donaldson-Thomas theory of Y3=rank 2 bundle over B, which together with its sister theories eventually determines the K-theoretic DT counts in all threefolds (not just CY). These capture deeper information than the cohomological DT and Gromov-Witten counts

(2, conjecturally) the theory on the worldsheet of the M2 brane of M-theory

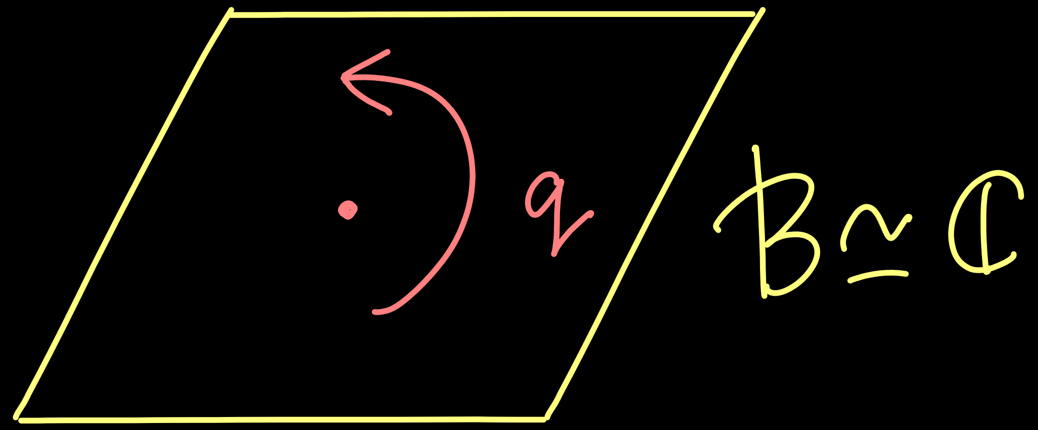
The physical diversity of operator insertions and boundary conditions translates into different flavors of evaluation maps from such moduli spaces to X, or ... As function of B, these define a K-theoretic analog of CohFT with a state space $K(X)$.



Further enriched by the data of an arbitrary $\mathrm{Aut}(X)$-bundle over B.

Of paramount importance are the vertex functions, that is, counts for B= complex plane, with boundary conditions imposed at infinity (this is formalized as maps from P1 nonsingular at ∞). Like for Nekrasov counts of instantons on R4, these make sense equivariantly for the action of

$$q \in \mathbb{C}^\times \subset \mathrm{Aut}$$



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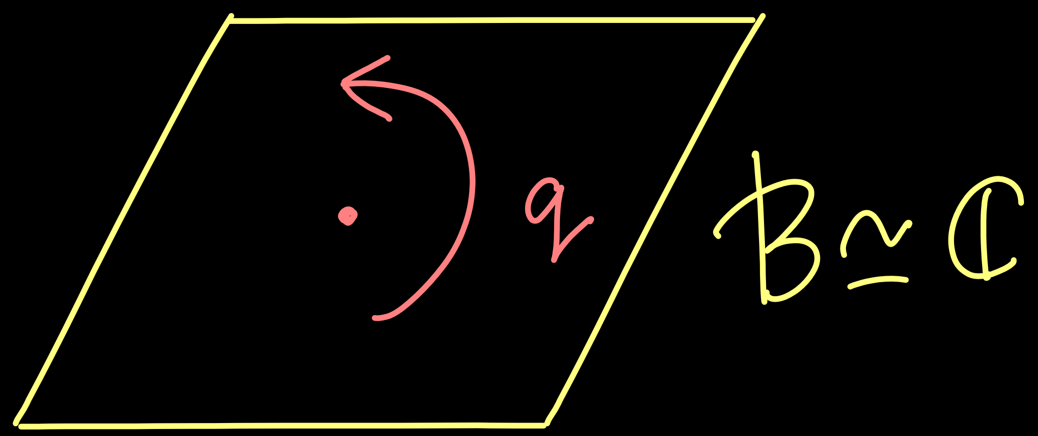

A fundamental feature of the theory are linear q-difference equations in **all variables**, Kahler z or equivariant $T \subset \mathrm{Aut}(X)$, satisfied by the vertex functions. The operators in these equations are certain counts for $B=\mathbb{P}^1$ with insertions at both 0 and ∞

Main point:

★ These q-difference equations generalize what we have seen before,

or if one prefers abstract statements to special functions, then

★ The whole enumerative theory may be described using certain new geometric representation theory

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In general, it will involve algebras that are not Hopf, but for Nakajima quiver varieties we get new quantum loops groups and their entire package

[Maulik-O,12] gave a geometric construction of solutions of the YB and related equations using their theory of stable envelopes. This associates a new quantum loop group $U_\hbar\hat{\mathfrak{g}}$ to any quiver so that $K(X)$ is a weight space in a $U_\hbar\hat{\mathfrak{g}}$-module. The corresponding Lie algebra $\mathfrak{g}$ is a generalization of the Kac-Moody Lie algebra constructed geometrically by Nakajima

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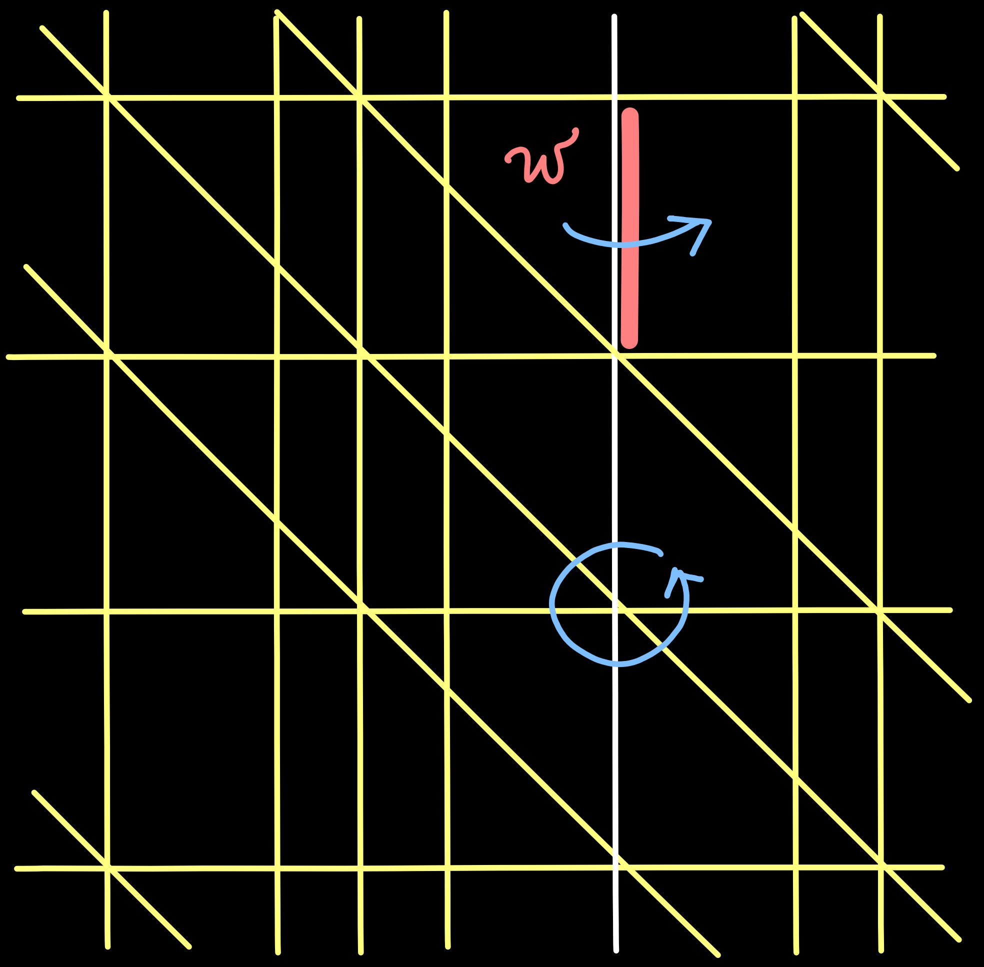
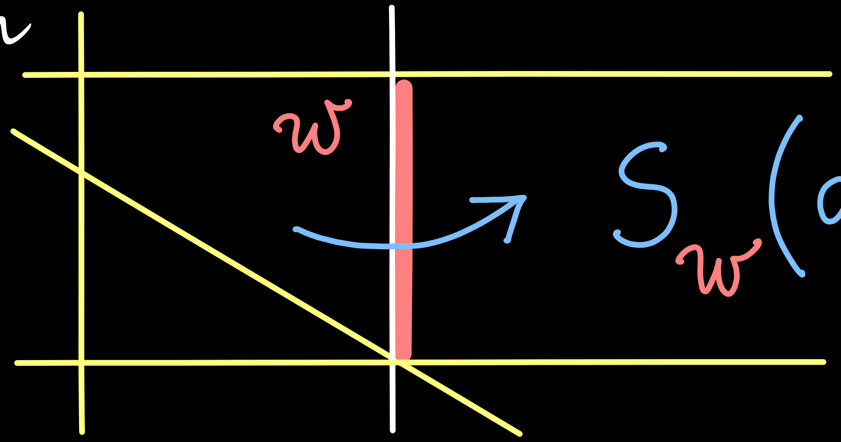
Last 2 statements generalize what was proven in cohomology in [MO]



$$\langle \ln z, \alpha \rangle + m = 0$$

a *dynamical groupoid* is a collection of operators of the form



$$S_w(q^m z^\alpha)$$

for every wall w of a periodic hyperplane arrangement. Must satisfy

$$\prod_{\circlearrowright} S_w = 1$$

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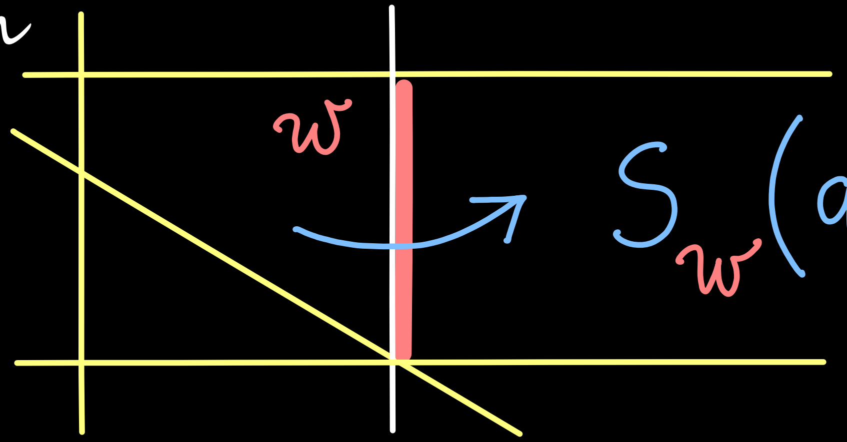

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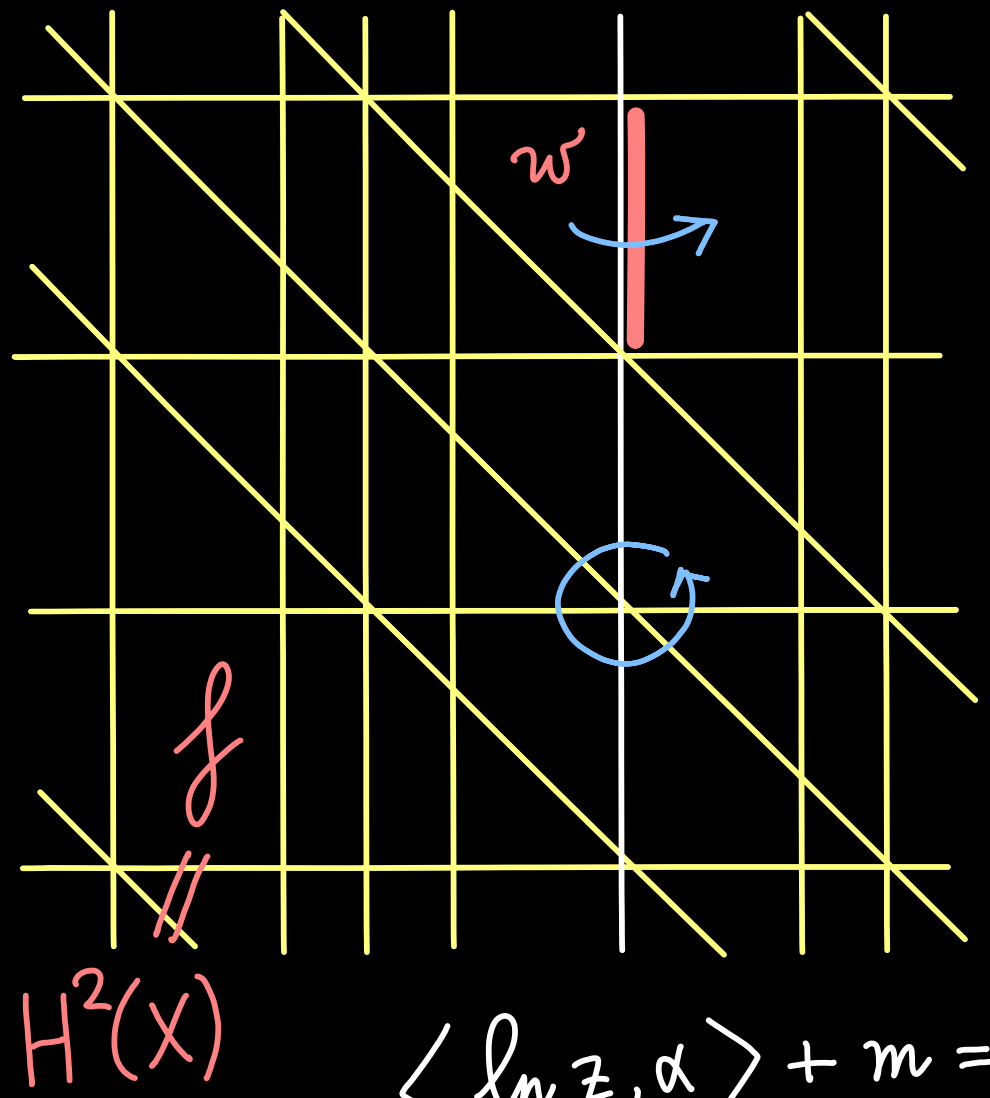

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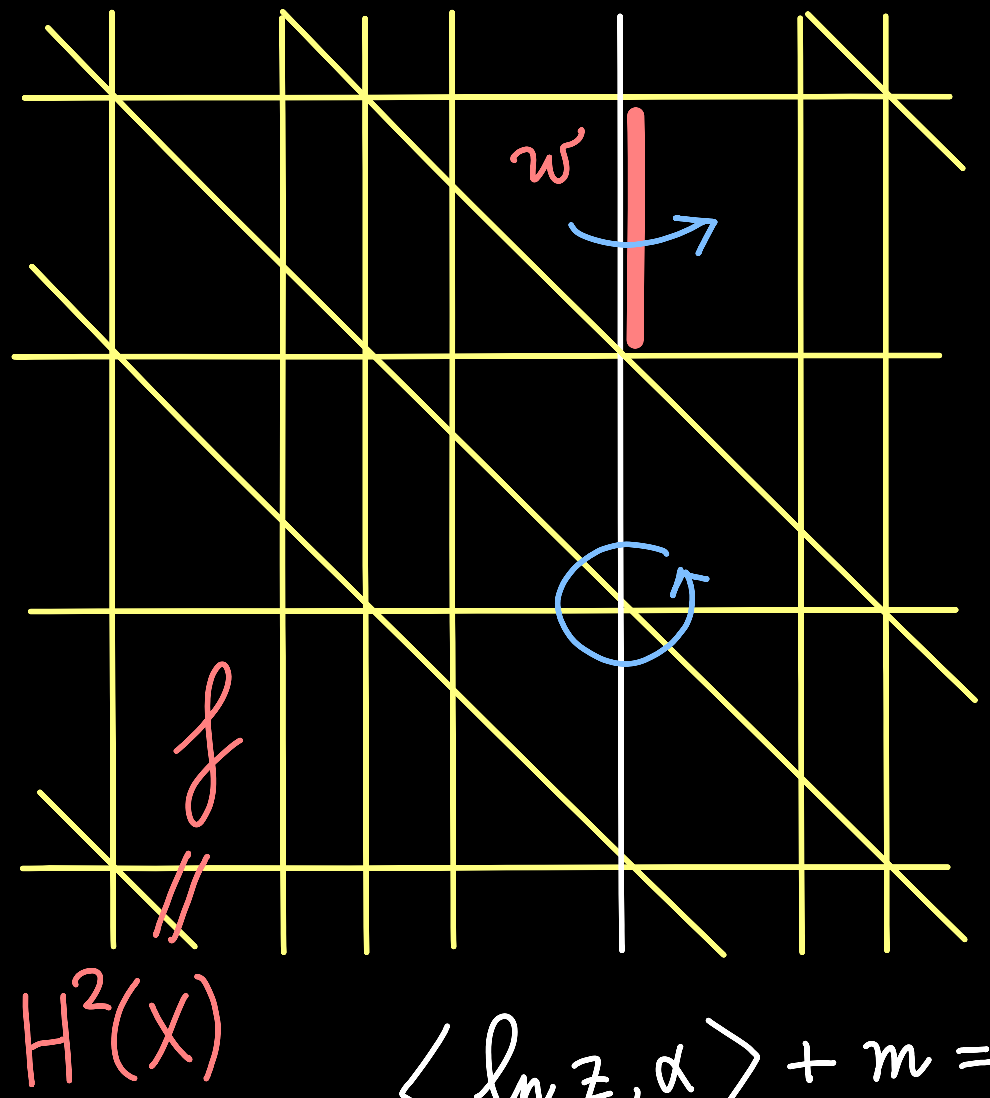
↑ root of $\mathfrak{g} \in H_2(X, \mathbb{Z})$



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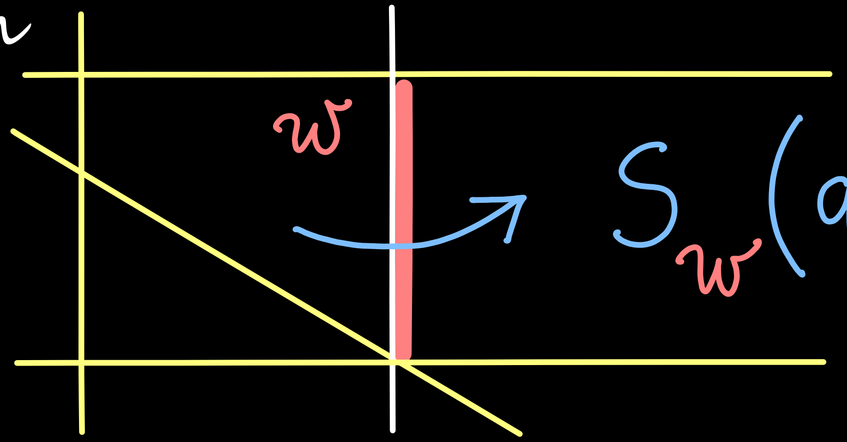

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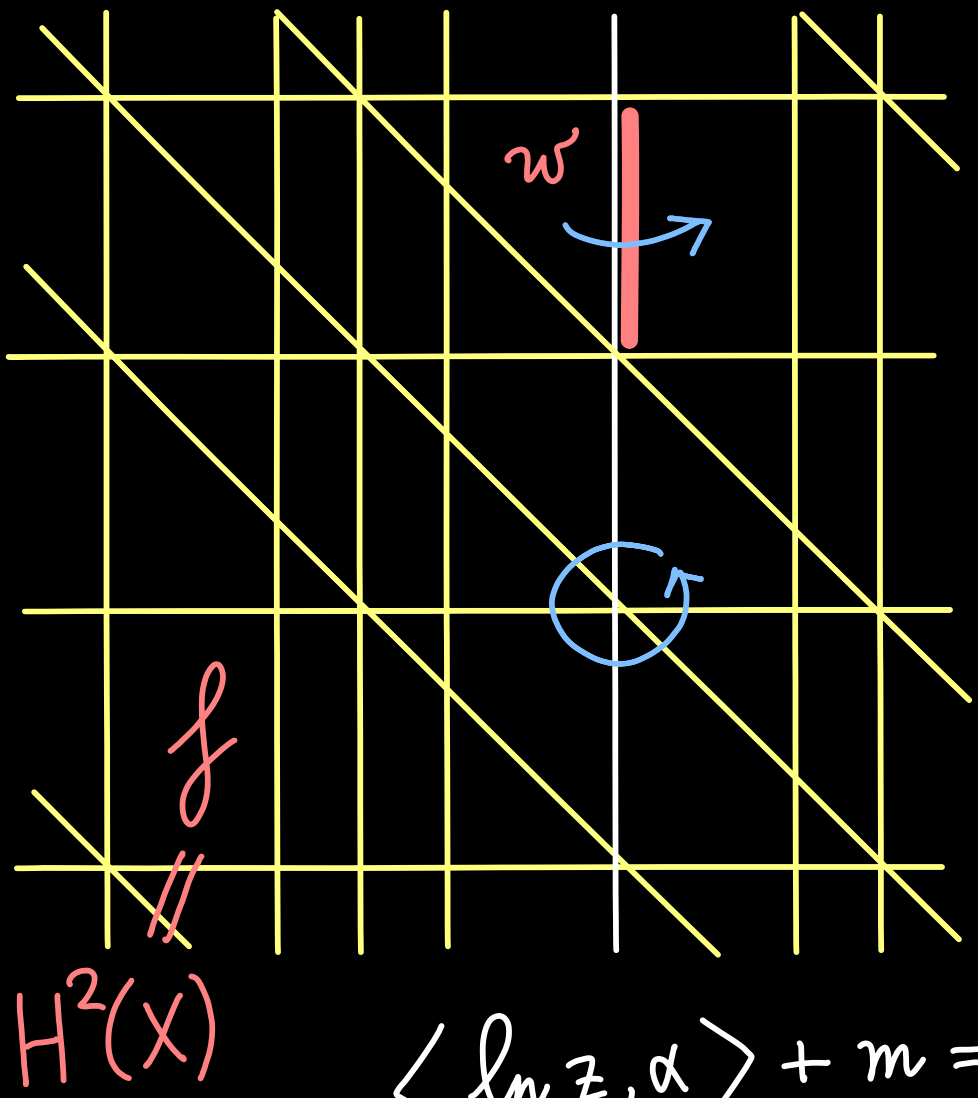
Generalize YB equation, braid groups, and give flat q-difference connections.

Constructed for every $U_\hbar \hat{\mathfrak{g}}$ in [OS]



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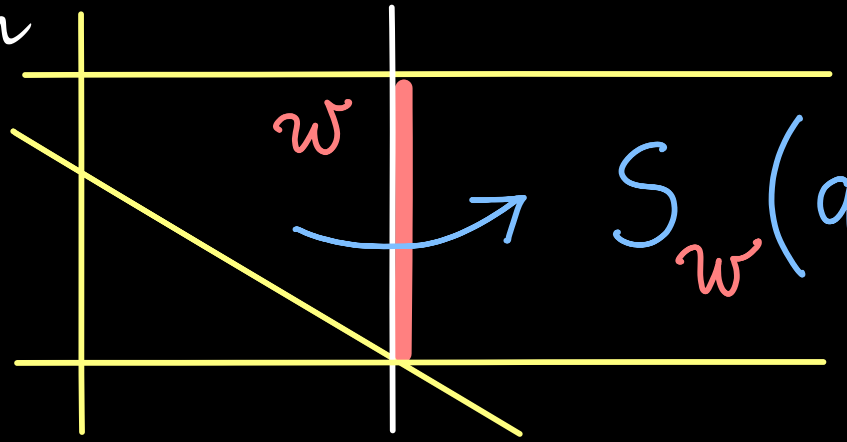

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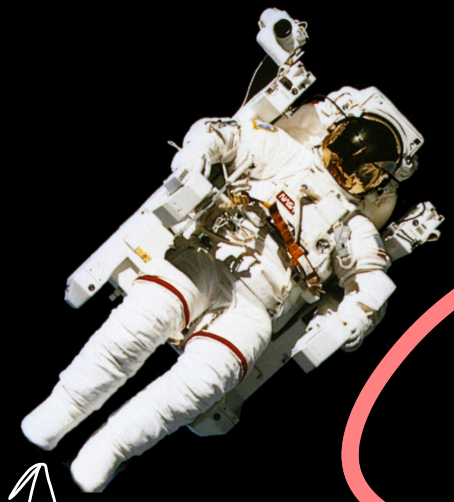

Bezrukavnikov

no Weyl group!

This gives a complete control over quantities of prime enumerative interest, such as those from K-theoretic DT theory of threefolds. Conversely, with geometric tools one can really control the analytic theory of these equations

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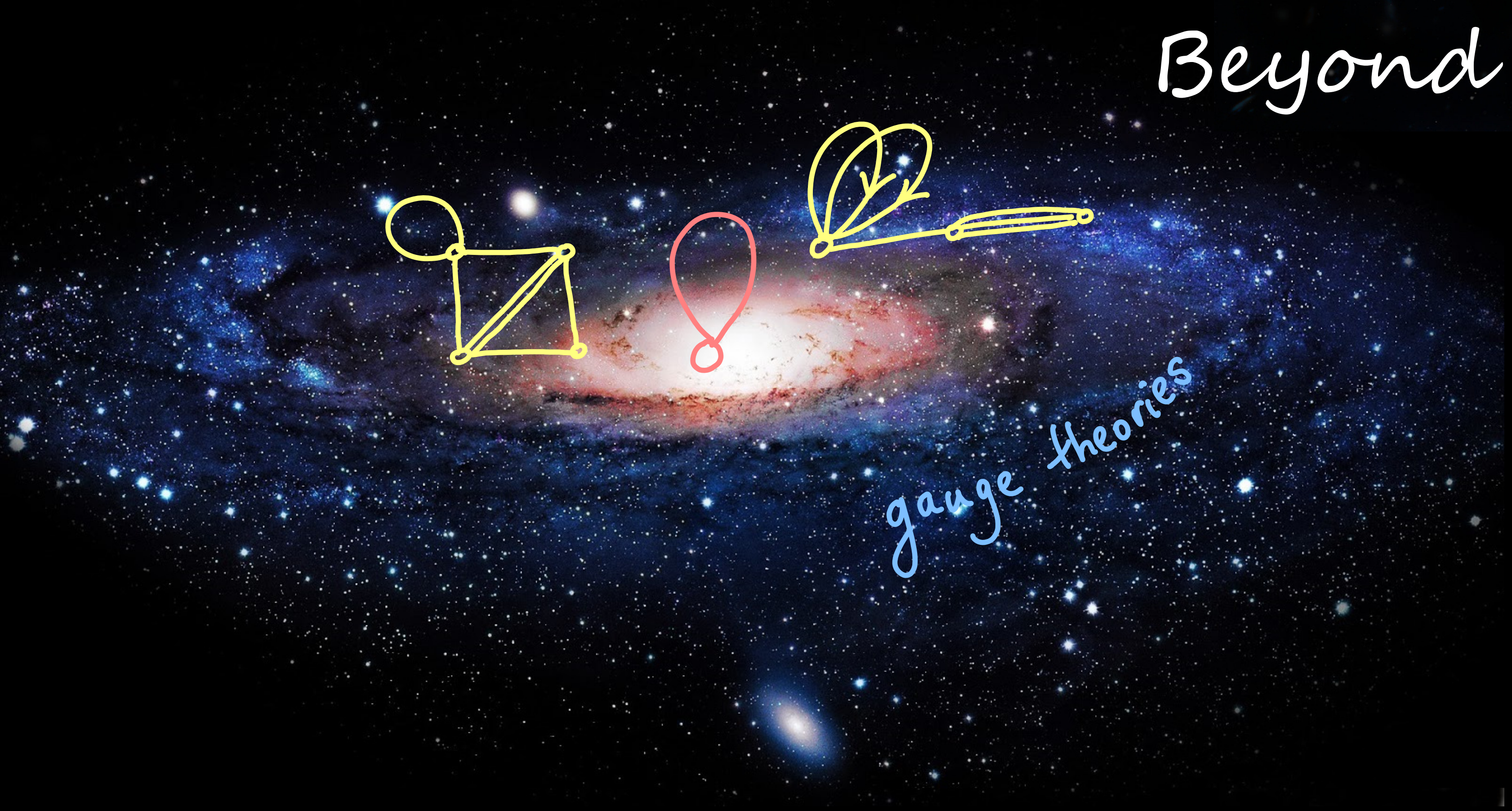
[Aganagic-O., 17] Integral representation of solutions which, in particular, solves the corresponding generalization of the Bethe Ansatz problem in the q->1 limit

Beyond Nakajima varieties



q-diff equations exist for abstract reasons

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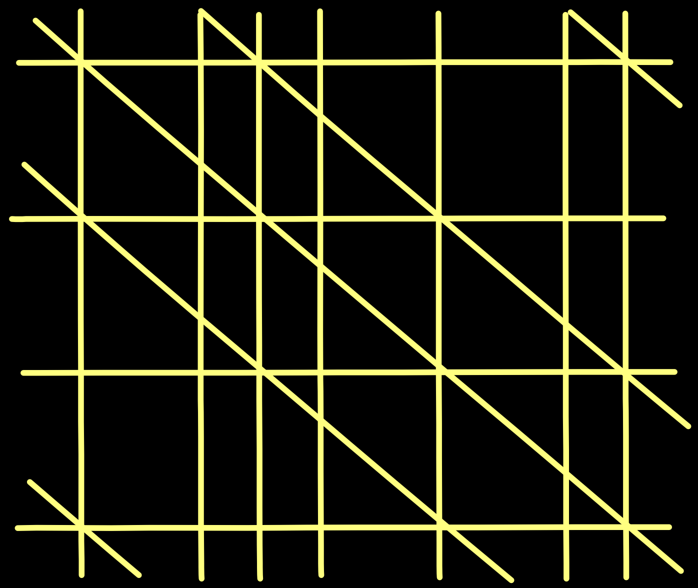

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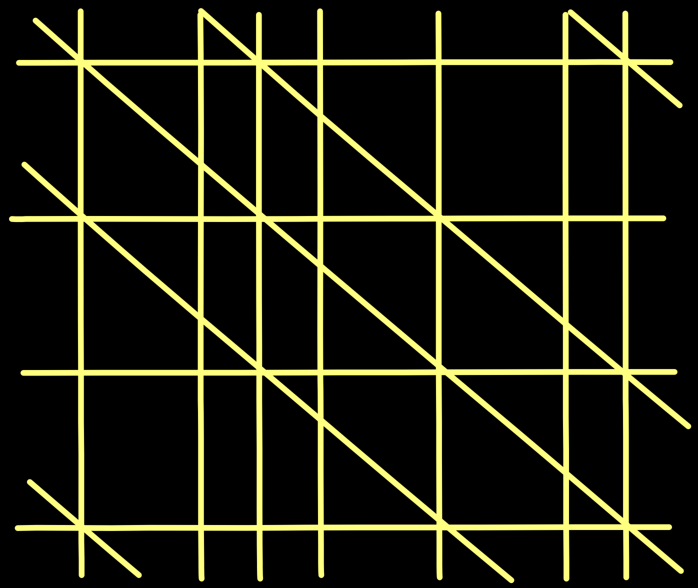

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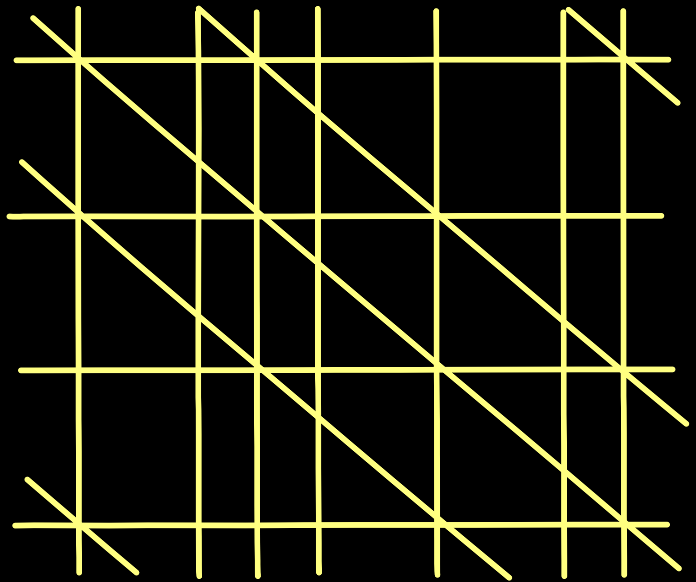

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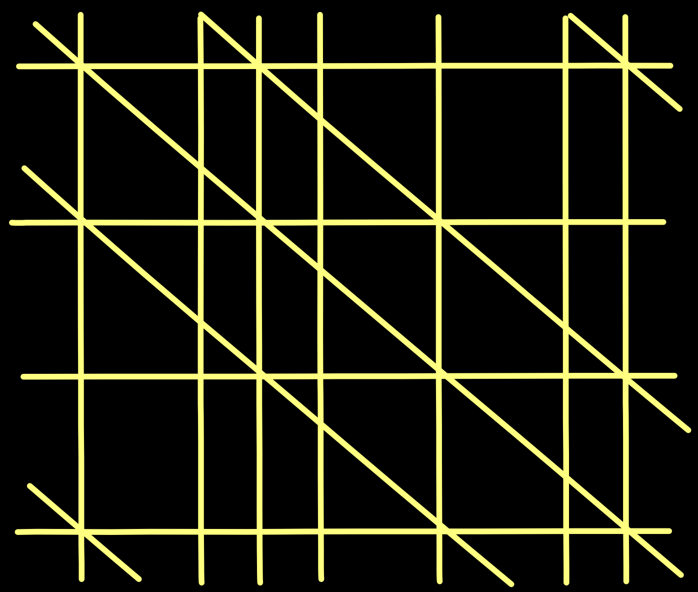

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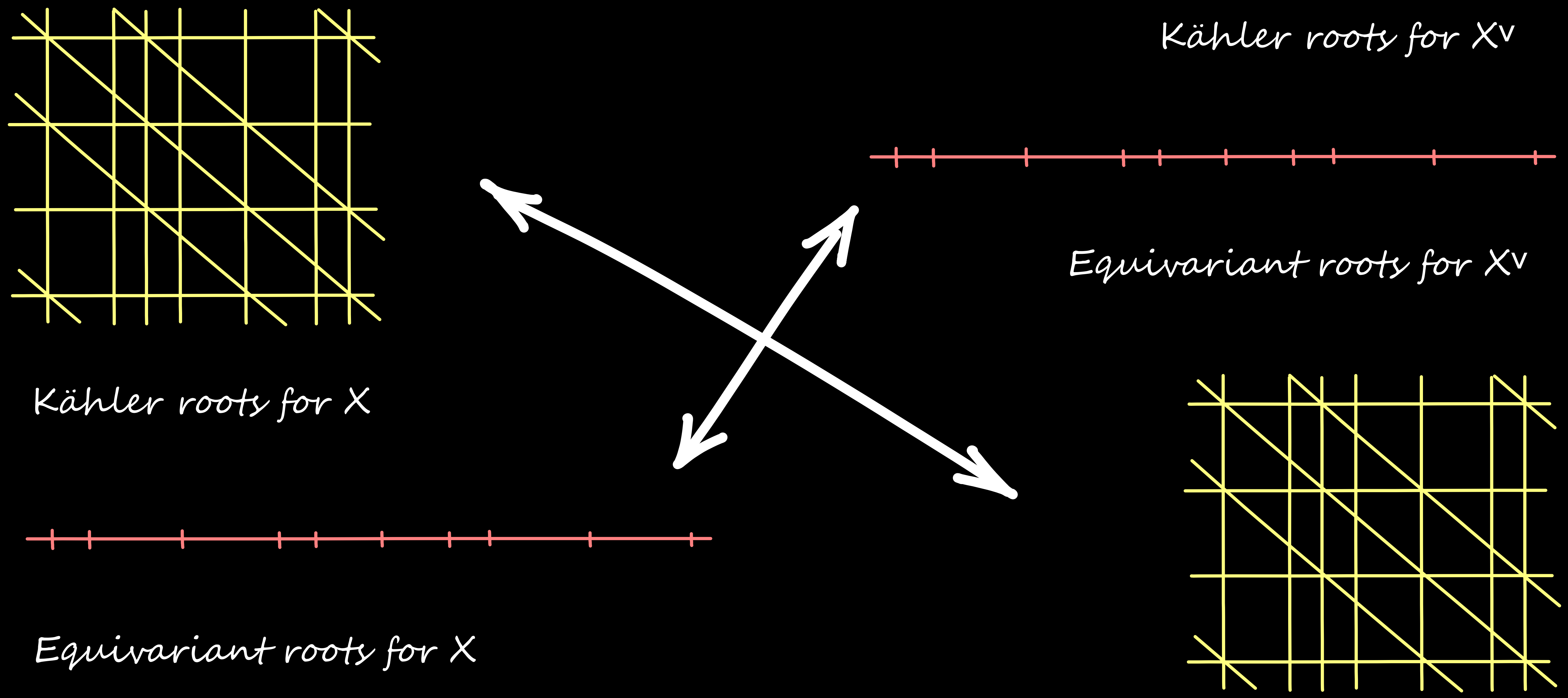
The duality should exchange the Kahler torus Z and A, and the two groupoids. In some limited generality, this is indeed shown in a work in progress with M. Aganagic

Unlike classical Langlands duality, the Kähler and equivariant roots live in spaces of apriori different dimension, making the duality more dramatic



Where is the Kazhdan-Lusztig theory ?

The braid group limit $S_W(0)$, $S_W(\infty)$ of the Kähler groupoid gives the right analog of the Hecke algebra for quantizations of X over a field of characteristic $p \gg 0$ as shown by [Bezrukavnikov-O] for a list of theories that includes all Nakajima varieties.

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Better still than such limit, one should study the full elliptic theory of [Aganagic-O]. It controls the roots of unity analogs of characteristic $p \gg 0$ quantization questions for finite p. It categorifies to equivalences between different descriptions of the category of boundary conditions in the QFT, which is where the different roads of categorification in Lie theory should converge.

Any formulas today ?

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I haven't put up any beyond the **X=T*P1** example, but these new worlds are full of e.g. remarkable q-diff equations whose solutions contain a treasure of geometric, representation-theoretic, combinatorial, and no doubt number-theoretic information.

Explicit formulas for e.g. stable envelopes [Smirnov], Bethe eigen-functions, etc. contain, as a special case, answers to many old questions.

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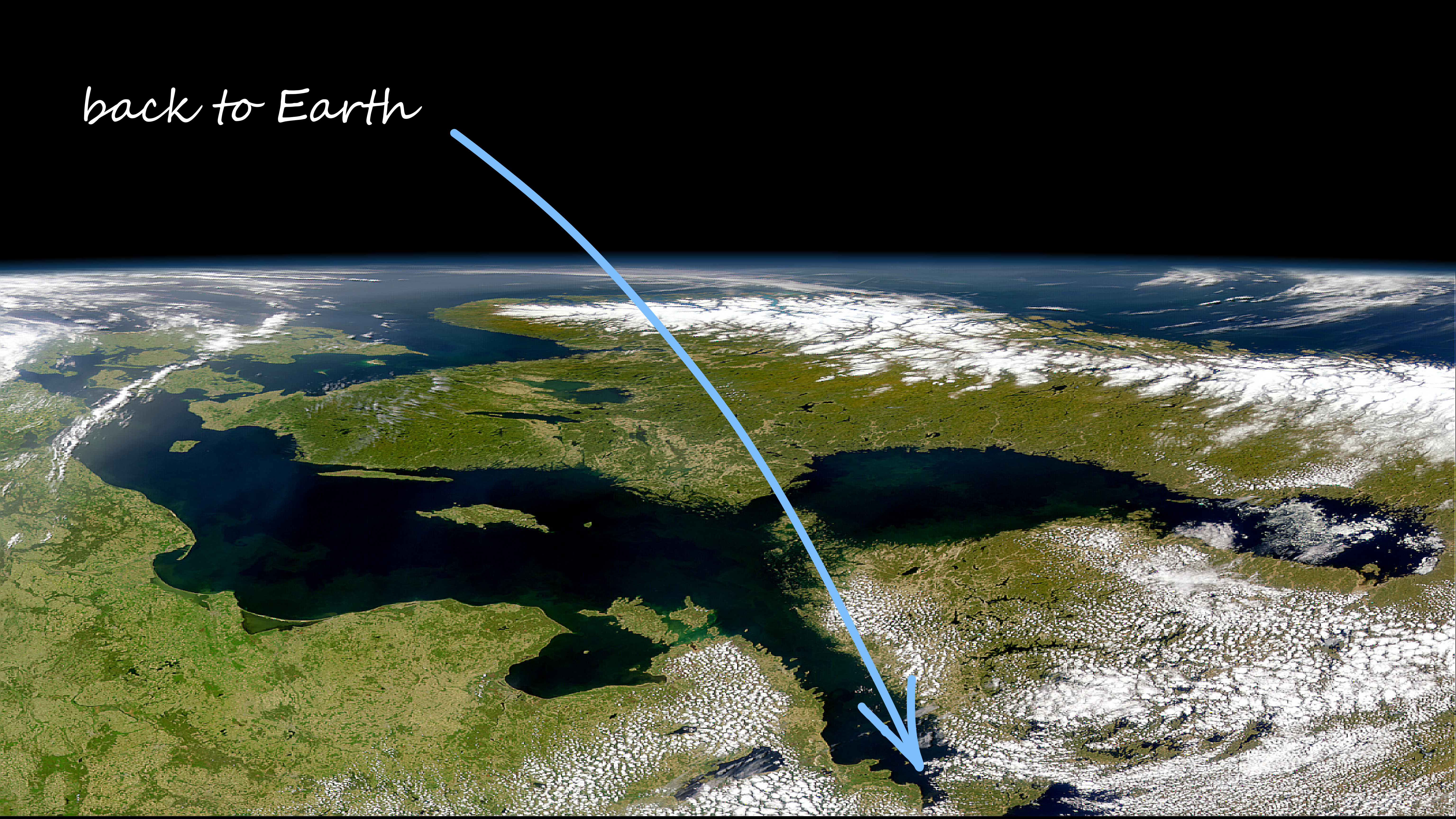
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