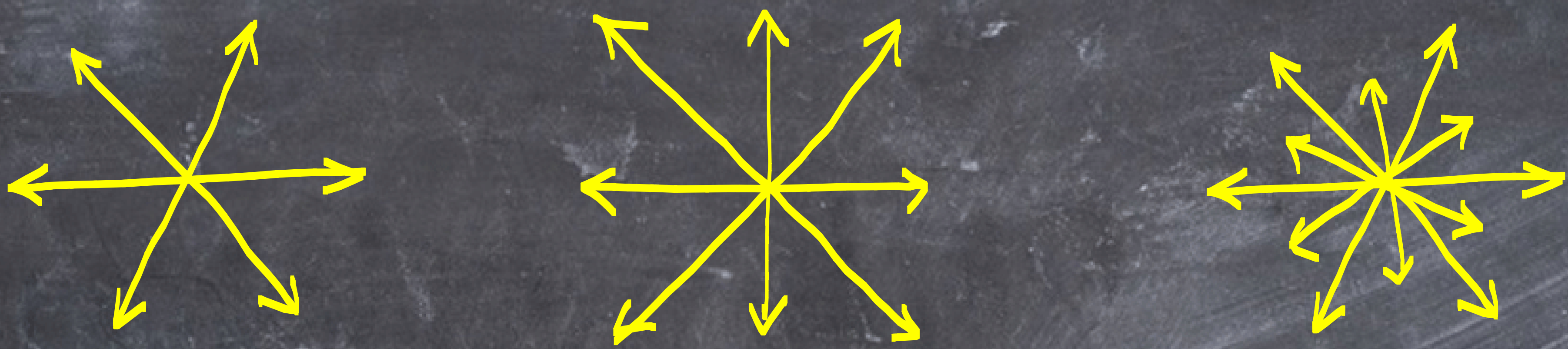
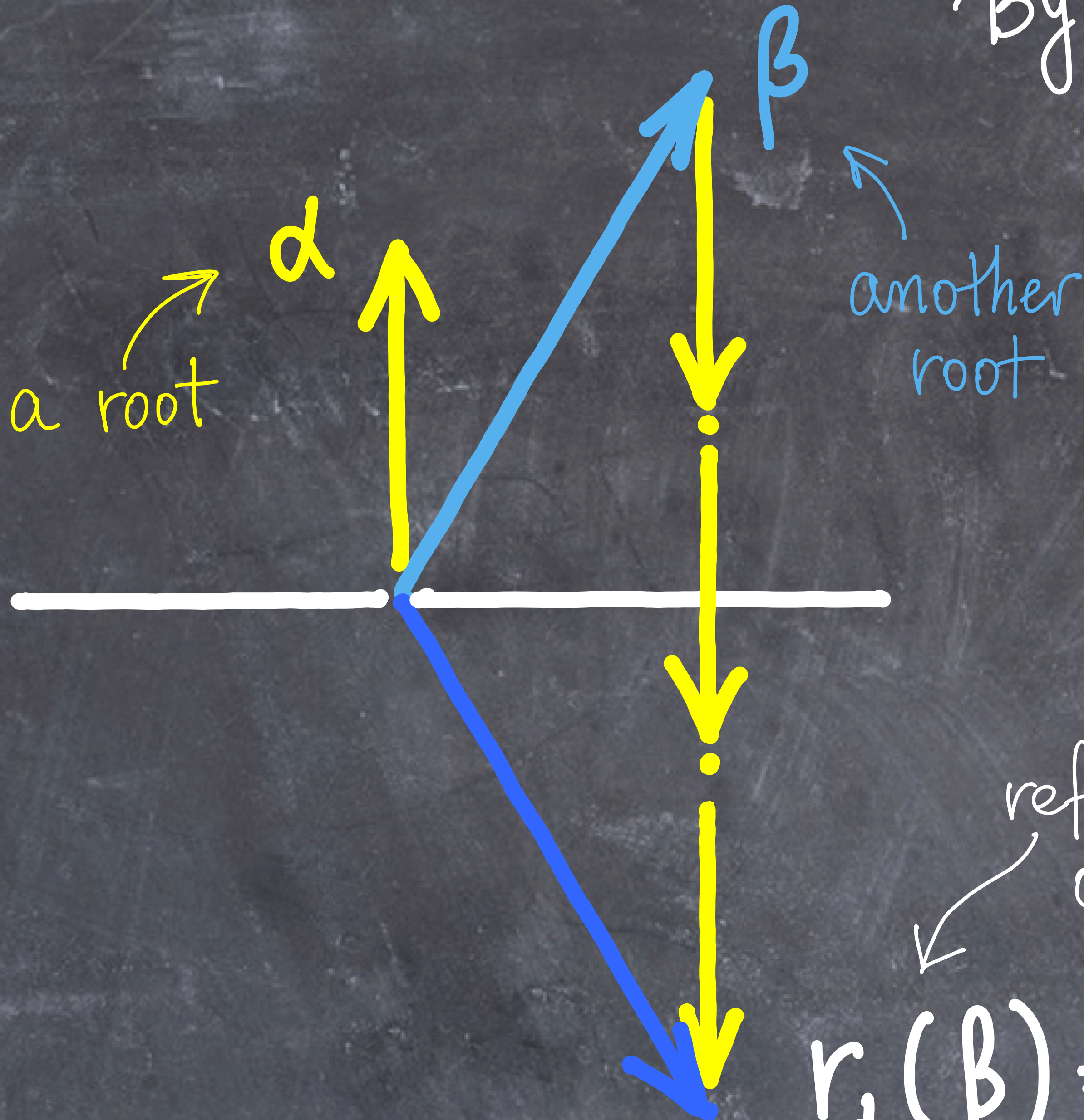
New roots of Lie theory



Root systems are beautiful combinatorial and geometric objects which are central to everything that has to do with Lie groups

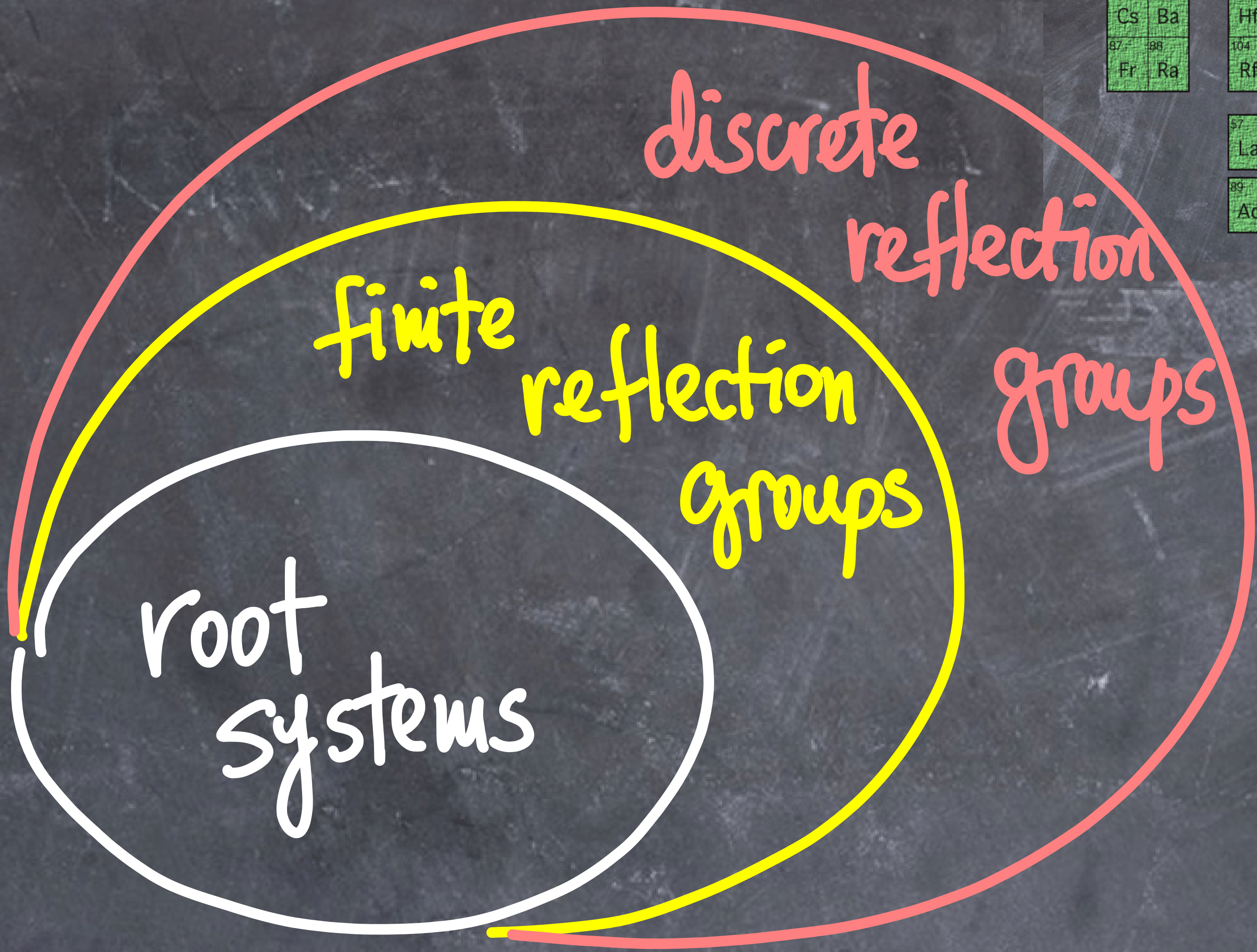


By definition, these are finite collections of vectors in Euclidean space such that this number (here, = 3) is an integer

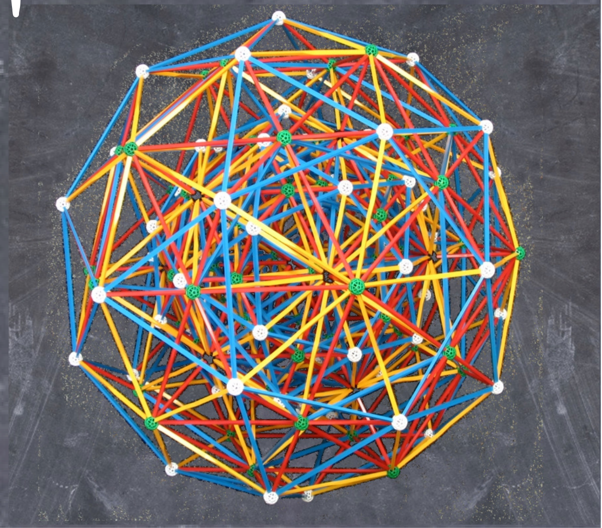
reflection of β

$$r_\alpha(\beta) = \beta - \overbrace{2\frac{(\alpha,\beta)}{(\alpha,\alpha)}}\,\alpha$$

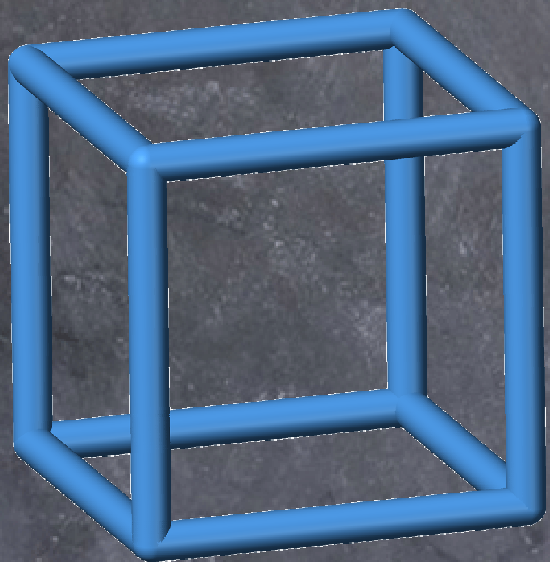
all these can be classified

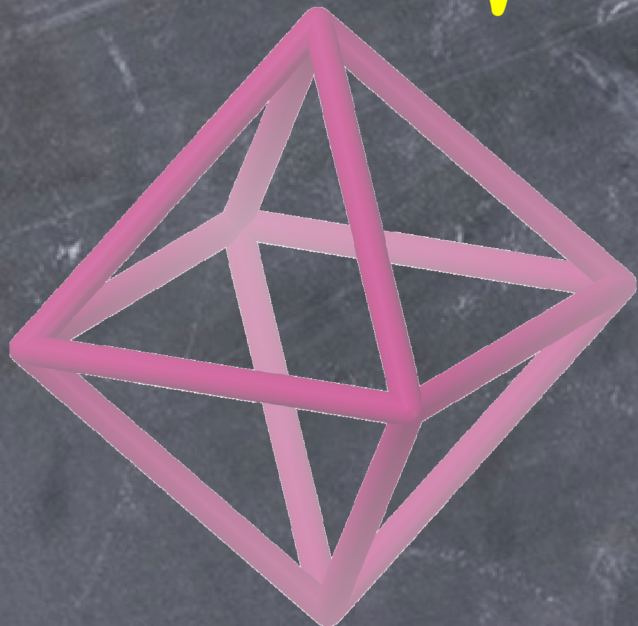


a few ∞ series
and a few
sporadic cases

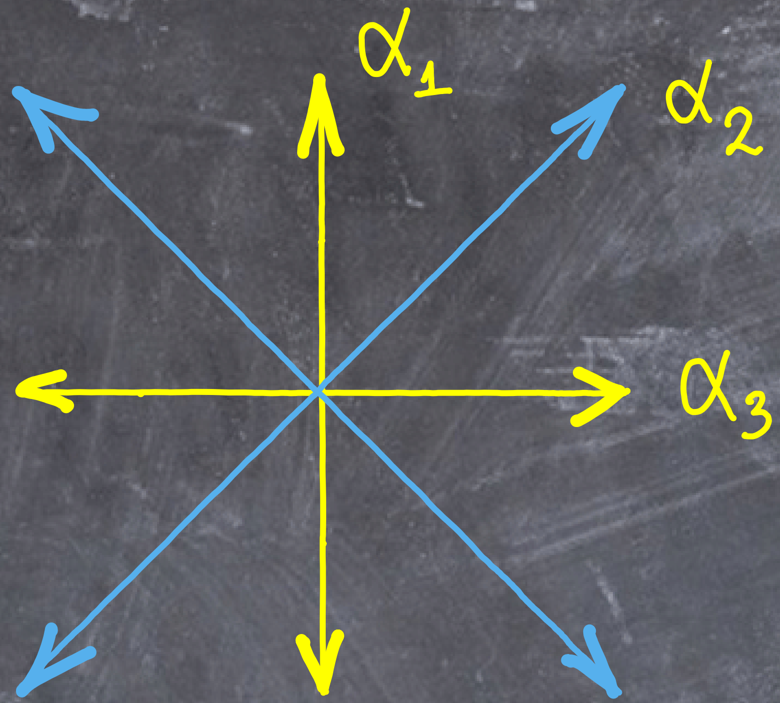


There are 2 important operations on root systems



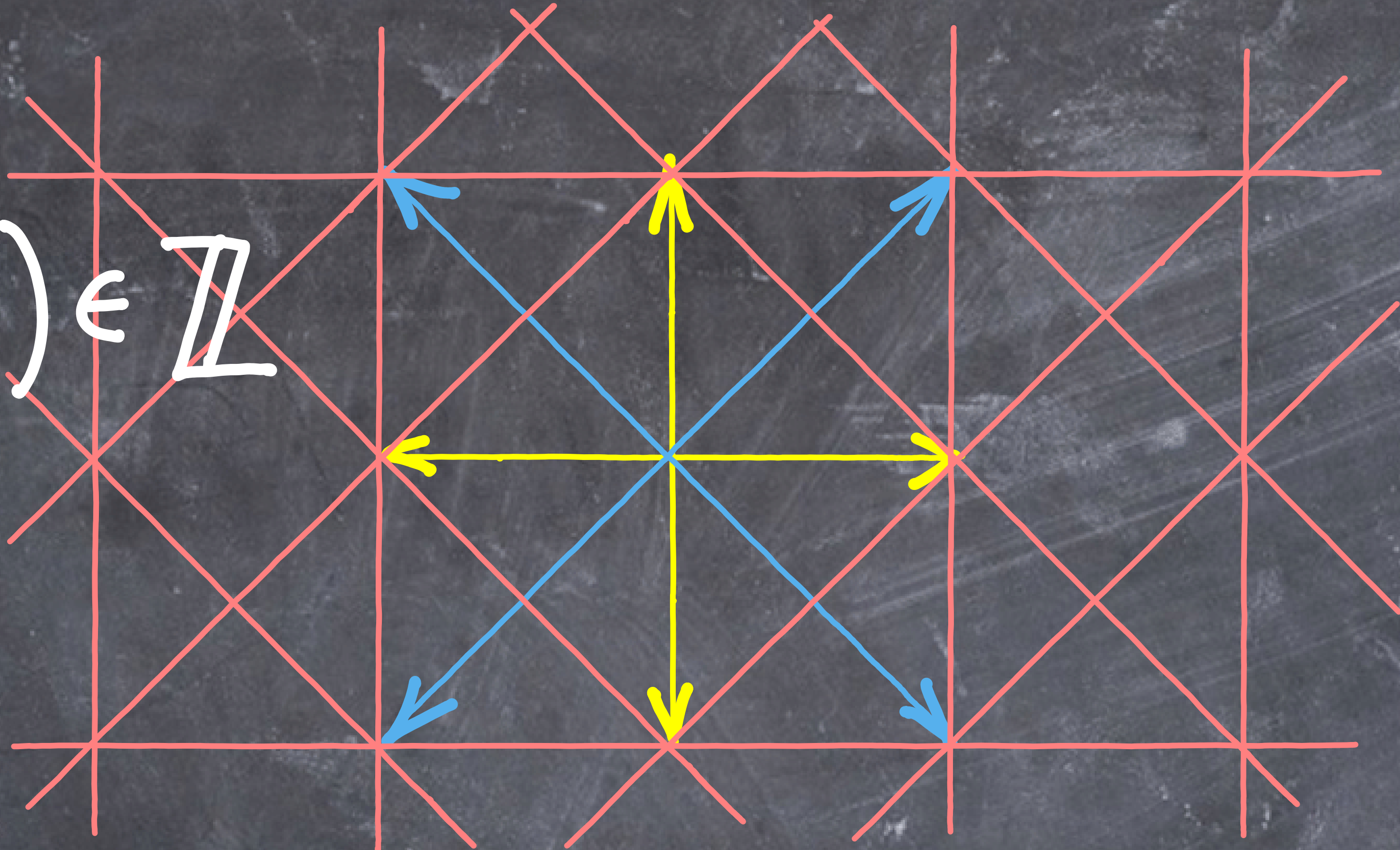


The other is going from a root system ....



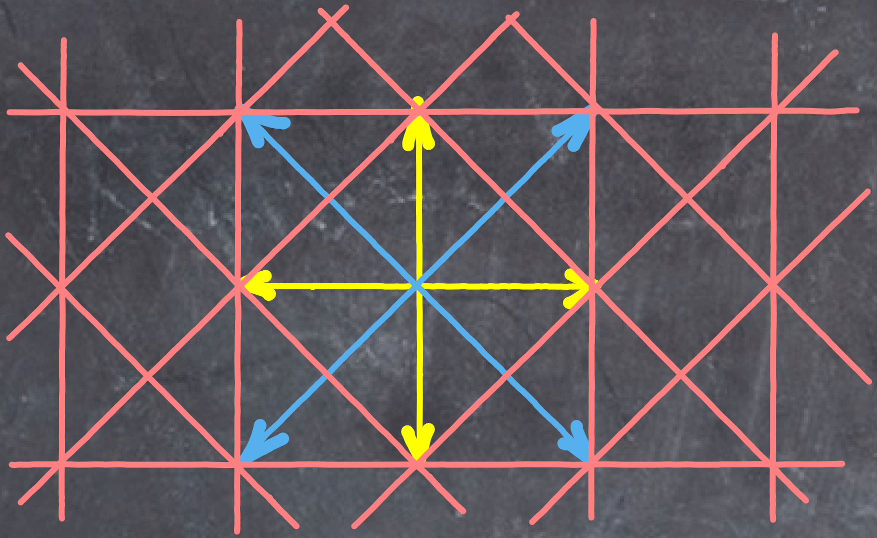
The other is going from a root system ....

$(x, \alpha) \in \mathbb{Z}$



to an affine hyperplane arrangement $(x, \alpha) \in \mathbb{Z}$

It remembers the length of roots and is very important



$$\langle \alpha, x \rangle \in \mathbb{Z}$$

$$x_i \in \mathbb{Z}, \quad \pm x_i \pm x_j \in \mathbb{Z}$$

$\Longleftrightarrow$

arrangement of subtori in $(\mathbb{C}^*)^n$

$$\bigcup_\alpha \{ z^\alpha = 1 \}$$

$$z_i = 1, \quad z_i z_j^{\pm 1} = 1$$

Root systems code the structure of

Compact and reductive Lie groups

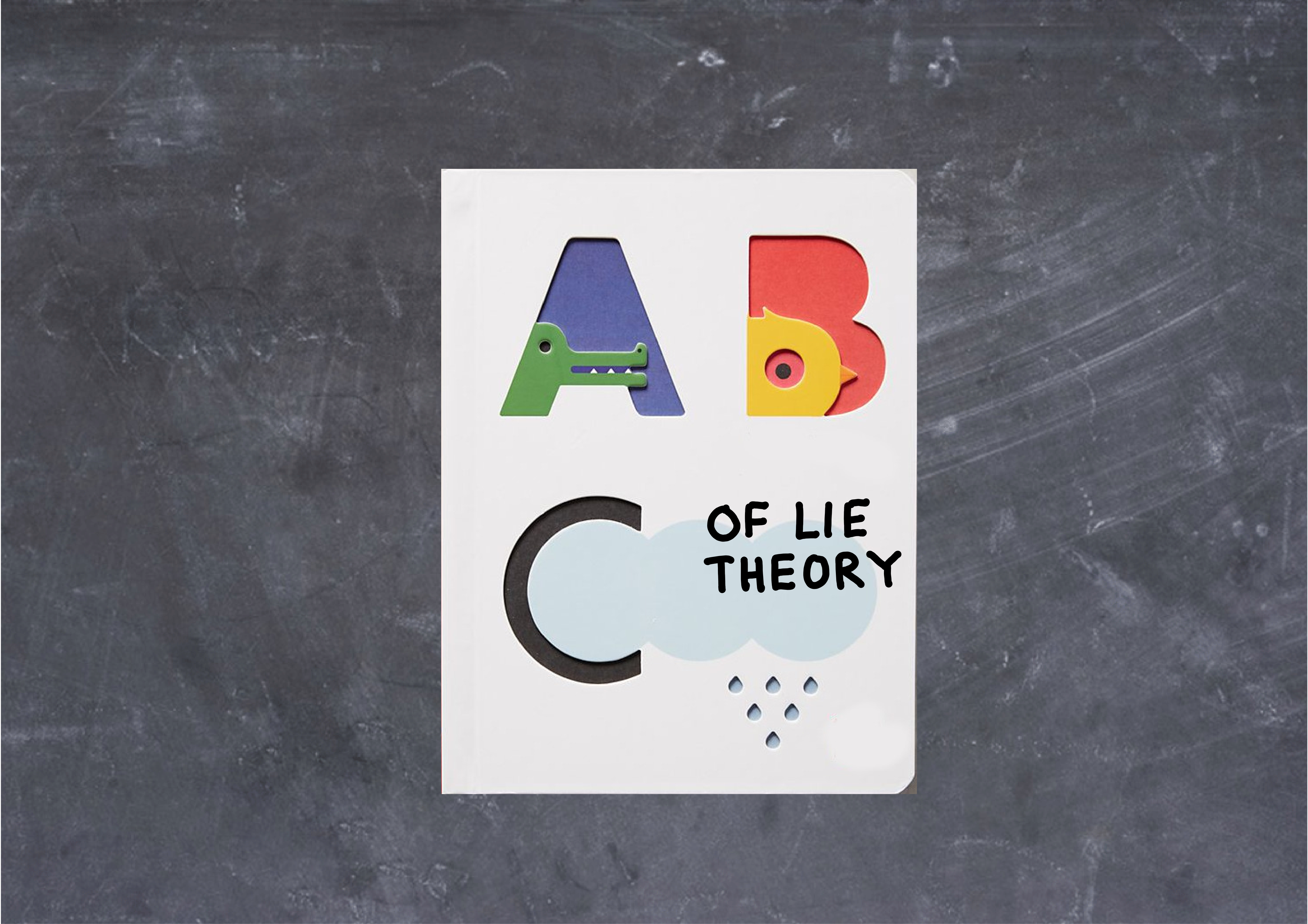
e.g. $U(n)$ = unitary $n \times n$ matrices

e.g. $GL(n, \mathbb{C})$ = all invertible $n \times n$ matrices

and of the corresponding Lie algebras

$\mathfrak{u}(n)$ = skew-Hermitian $n \times n$ matrices

$\mathfrak{gl}(n, \mathbb{C})$ = all $n \times n$ matrices


A B
C
OF LIE
THEORY

maximal Abelian

$$\begin{pmatrix} * & & & 0 \\ & * & & \\ & & * & \\ 0 & & & * \end{pmatrix}$$

maximal solvaBle

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ 0 & & & * \end{pmatrix}$$

maximal "kompact" subgroup, unitary $U(n)$

← a reductive Lie Group e.g. $GL(n)$

These are assembled according to the roots from

$$\left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right\} = A \qquad B \qquad \begin{pmatrix} * & * \\ & * \end{pmatrix}$$

$$SU(2) = K \qquad G = SL(2)$$

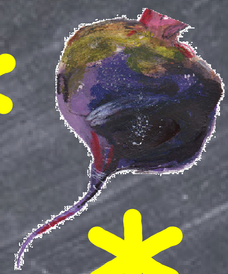
$$= \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}\right\} \qquad = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right\}$$

$$|a|^2 + |b|^2 = 1 \qquad ad - bc = 1$$

These are assembled according to the roots from

$$\left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right\} = $$

$$\begin{pmatrix} * & \\ & * \end{pmatrix}$$



$$SU(2) = K \qquad G = SL(2)$$

$$= \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}\right\} \qquad = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right\}$$

$$|a|^2 + |b|^2 = 1 \qquad ad - bc = 1$$

maximal Abelian

$$\begin{pmatrix} * & & & 0 \\ & * & & \\ & & * & \\ 0 & & & * \end{pmatrix}$$

maximal solvaBle

maximal "kompact" subgroup,
unitary $U(n)$

← a reductive Lie Group
e.g. $GL(n)$

The role of roots can be explained geometrically in tems of certain remarkable algebraic varieties associated to G

The conjugacy classes in K or $\mathrm{Lie}(K)$ are, in fact, naturally compact complex manifolds, e.g.

$$\left\{ \begin{array}{l} \text{Hermitian or} \\ \text{unitary} \\ \text{matrices} \end{array} \sim \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{pmatrix} \right\} = \begin{array}{c} \text{Grassmannian} \\ Gr(2,5,\mathbb{C}) \end{array}$$

The conjugacy classes in K or $\mathrm{Lie}(K)$ are, in fact, naturally compact complex manifolds, e.g.

$$\left\{\begin{array}{l}\text{Hermitian or} \\ \text{unitary} \\ \text{matrices}\end{array} \sim \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{pmatrix}\right\} = \begin{array}{c}\text{Grassmannian} \\ Gr(2,5,\mathbb{C})\end{array}$$

for instance, for $K = SU(2)$ we get

$$Gr(1,2,\mathbb{C}) \simeq \mathbb{CP}^1 \simeq$$

Riemann sphere

The conjugacy classes in K or $\mathrm{Lie}(K)$ are, in fact, naturally compact complex manifolds, e.g.

$$\left\{\begin{array}{l}\text{Hermitian or}\\ \text{unitary}\\ \text{matrices}\end{array} \sim \begin{pmatrix}\lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5\end{pmatrix}\right\} = \begin{array}{c}\text{flag manifold}\\ GL(5)/\mathbb{B}\end{array}$$

$$\Vert$$

$$\{V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 = \mathbb{C}^5\}$$

Span of eigenvector v_{λ_1}

$\mathrm{Span}(v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3})$

The conjugacy classes in K or $\mathrm{Lie}(K)$ are, in fact, naturally compact complex manifolds, e.g.

$$\left\{\begin{array}{l}\text{Hermitian or} \\ \text{unitary} \\ \text{matrices}\end{array}\right\} \sim \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5 \end{pmatrix} = \begin{array}{c}\text{flag manifold} \\ GL(5)/\mathcal{B}\end{array}$$

$$\| $$

$$\left\{ V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 = \mathbb{C}^5 \right\}$$

Span of eigenvector v_{λ_1}

$\mathrm{Span}(v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3})$

skip some steps of the flag when $\lambda_i = \lambda_{i+1}$

The conjugacy classes in G or $\mathrm{Lie}(G)$ are complex symplectic manifolds, and the generic one is diffeomorphic to

$$X = T^*G/B$$

The isomorphism class of X as algebraic symplectic variety depends on eigenvalues $\in A$ (abstractly, deformations are parametrized by $H^2(X,\mathbb{C}) \approx \mathrm{Lie}\, A$)

when eigenvalues collide, get a singular union of orbits e.g. of

$$\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \begin{pmatrix} \lambda & 1 & \\ & \lambda & \\ & & \lambda \end{pmatrix} \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}.$$

Recall that roots can be viewed as hypertori in A.

In terms of X they can be described in two a priori different ways

On the one hand,

$$
\begin{tikzcd}
A \arrow[r] & \mathrm{Deformations}(X) \\
\text{root subtori} \arrow[u, hook] \arrow[r] & \text{Discriminant} \arrow[u, hook]
\end{tikzcd}
$$

as alg symplectic variety

locus of singularities

$$\bigcup \{ \lambda_i / \lambda_j = 1 \}$$

On the one hand, A acts on X by conjugation and

as alg. sympl variety

$$
\begin{tikzcd}
A \arrow[r] & \mathrm{Aut}(X, \omega) \\
\text{root subtori} \arrow[u, hook] \arrow[r] & \{ a \mid X^a \neq X^A \} \arrow[u, hook]
\end{tikzcd}
$$

$$\bigcup \{ \lambda_i / \lambda_j = 1 \}$$

locus of automorphisms with extra fixed pts

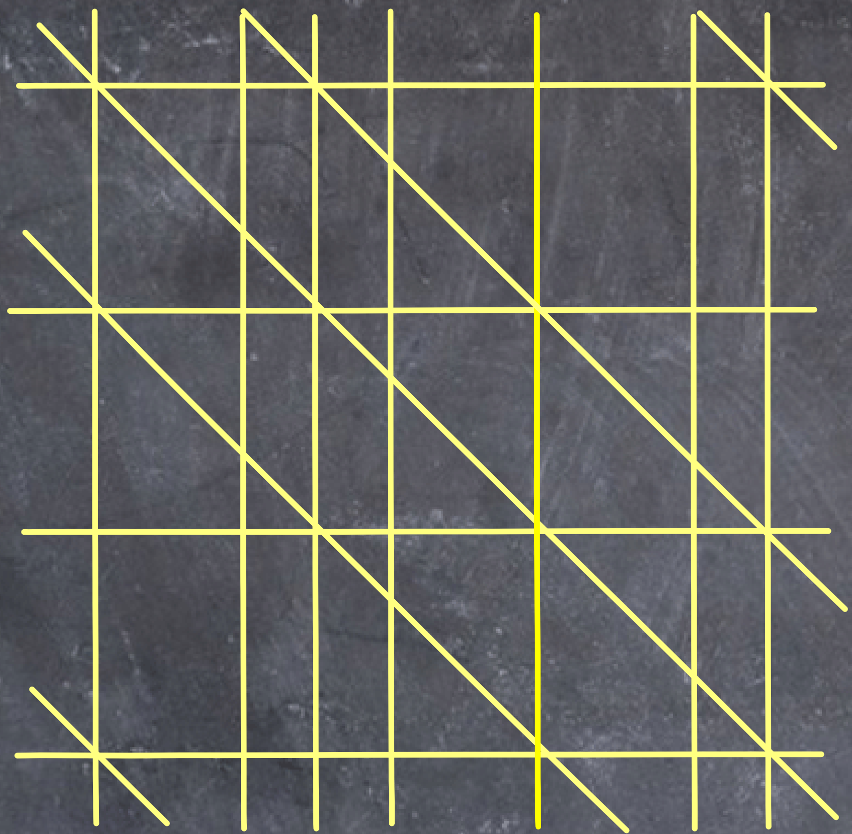
Main message: there is a vast extension of the classical Lie theory under construction in which

$X \rightsquigarrow$ a much wider class of algebraic varieties than T^*G/B

then: 1) Deformation roots $\subset H_2(X, \mathbb{Z})$ and Action roots $\subset$ weights of $\mathrm{Aut}(X, \omega)$ have nothing in common (e.g. different ranks)

2) Permuted by a rather dramatic Langlands-style duality: $X \longleftrightarrow X^\vee$

also, while there may be a certain Weyl group symmetry in the problem, its role is minor and it does not explain most of the roots



in other words, one works with arrangements that are <u>not</u> invariant w.r.t. reflections in hyperplanes the class of X that we want may have little to do with Lie theory at the first sight, however

- quantizations $\widehat{X}$ of X, that is, noncommutative deformations of functions on X are just as good as

$$\mathcal{U}(\mathfrak{g}) \,/\, \text{central character} = \widehat{T^*G/B}$$

including Kazhdan-Lusztig theory both in char $=0$ and char $= p \gg 0$ a la Bezrukavnikov et al

- natural special functions associated to X generalize character, spherical functions, Macdonald polys .....

etc etc

The spaces X that we want appear as

moduli spaces of vacua

steady states
Gibbs states etc.

in certan supersymmetric QFTs

The spaces X that we want appear as

moduli spaces of vacua ← steady states

Gibbs states etc.

in certan supersymmetric QFTs

cut out by equations like

$$p = \text{const}\ \rho\, T$$

in the space of all averages like

$$T = \langle \text{kinetic energy} \rangle$$

The spaces X that we want appear as

moduli spaces of vacua

steady states

Gibbs states etc.

in certan supersymmetric QFTs

often makes moduli
spaces of susy states
interesting complex
algebraic varieties

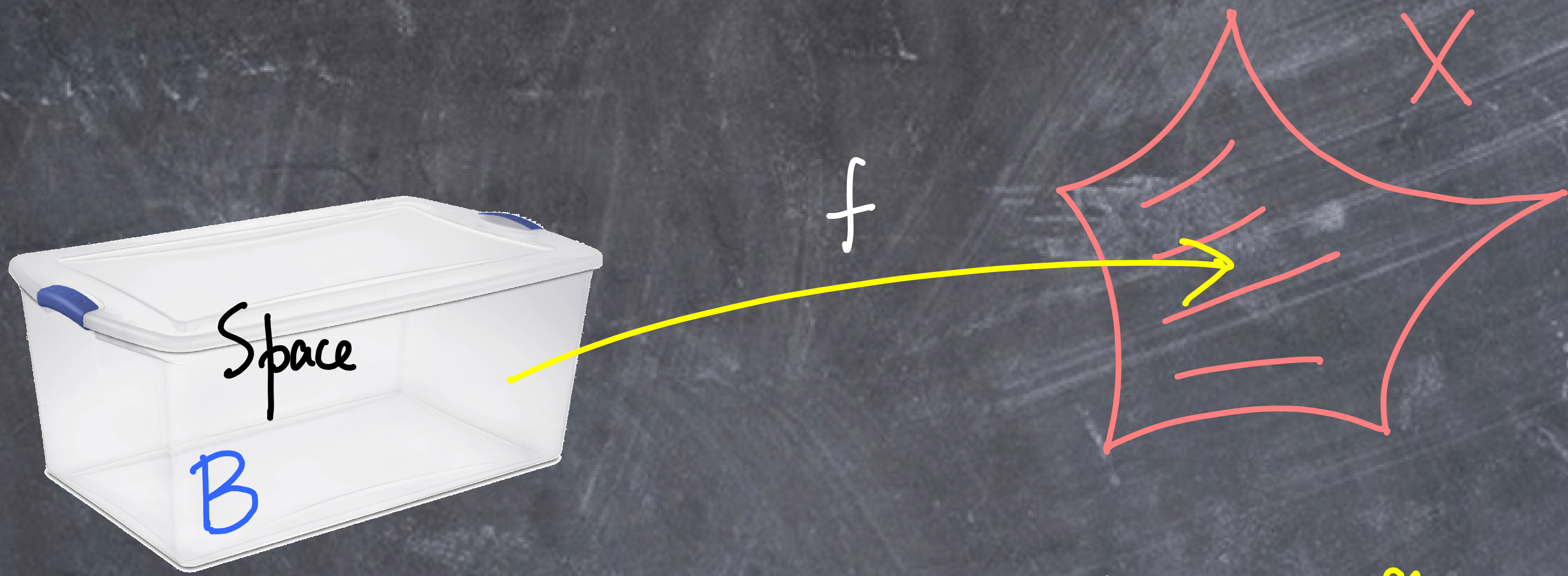
cut out by equations like

$$p = \text{const}\ \rho\, T$$

in the space of all averages like

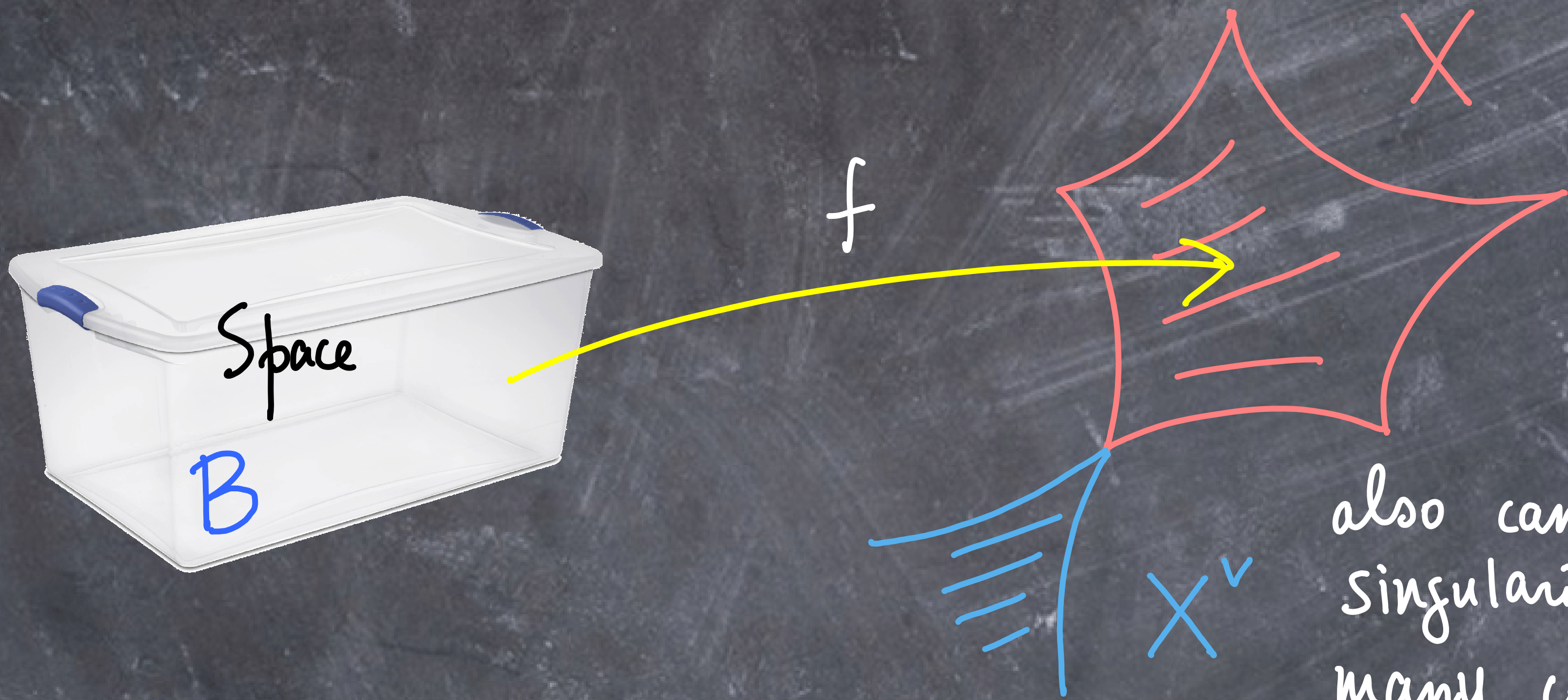
$$T = \langle \text{kinetic energy} \rangle$$

at low energies (or large distances) we can think of all states of the QFT as modulated vacua, i.e. we can think of the QFT as the theory of maps



- Maps can have different topology
- Susy imposes constraints like holomorphy on f

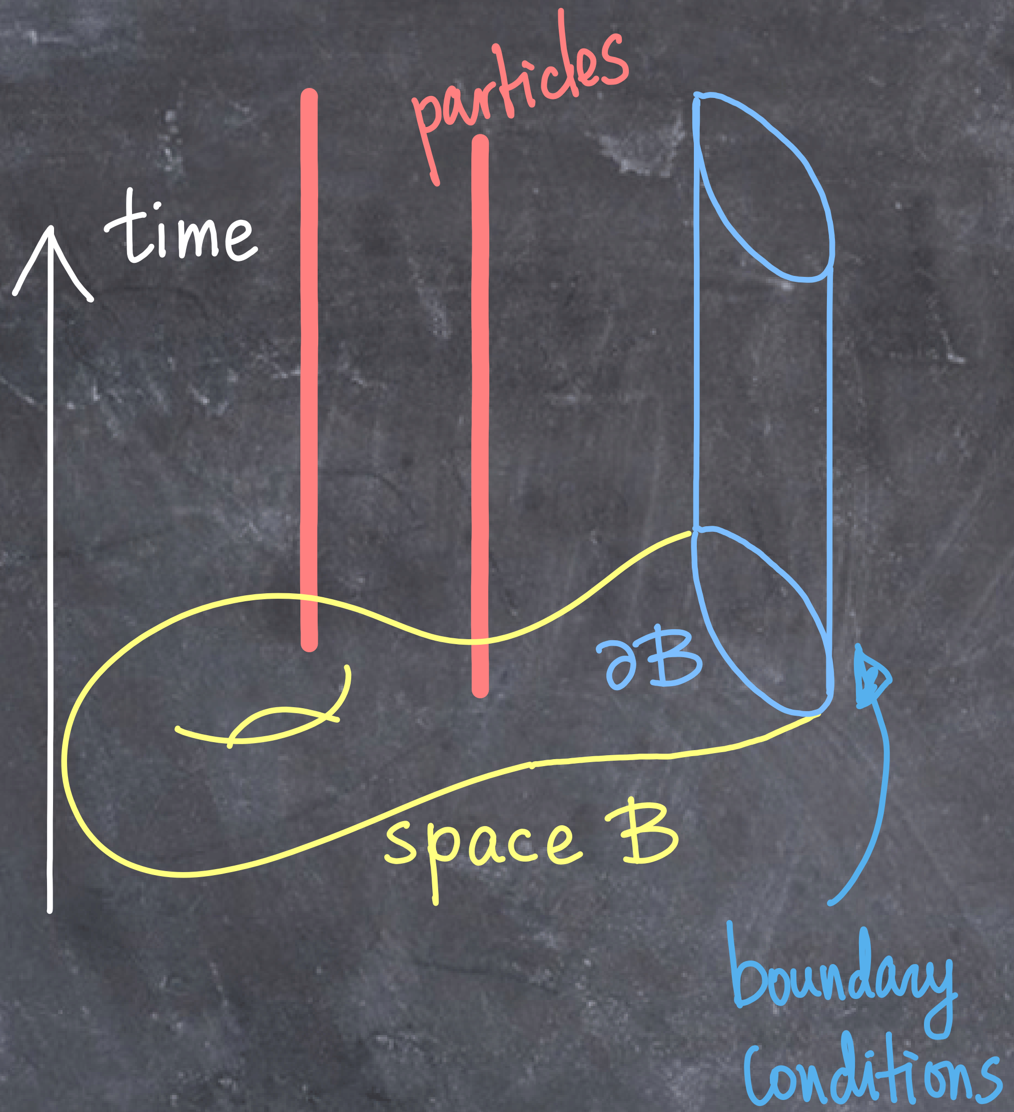
at low energies (or large distances) we can think of all states of the QFT as modulated vacua, i.e. we can think of the QFT as the theory of maps



also can have singularities and many components

genuine complications

Currently, a lot of interest in computing



index $= \mathcal{H}_{even} - \mathcal{H}_{odd}$

viewed as a virtual representation of

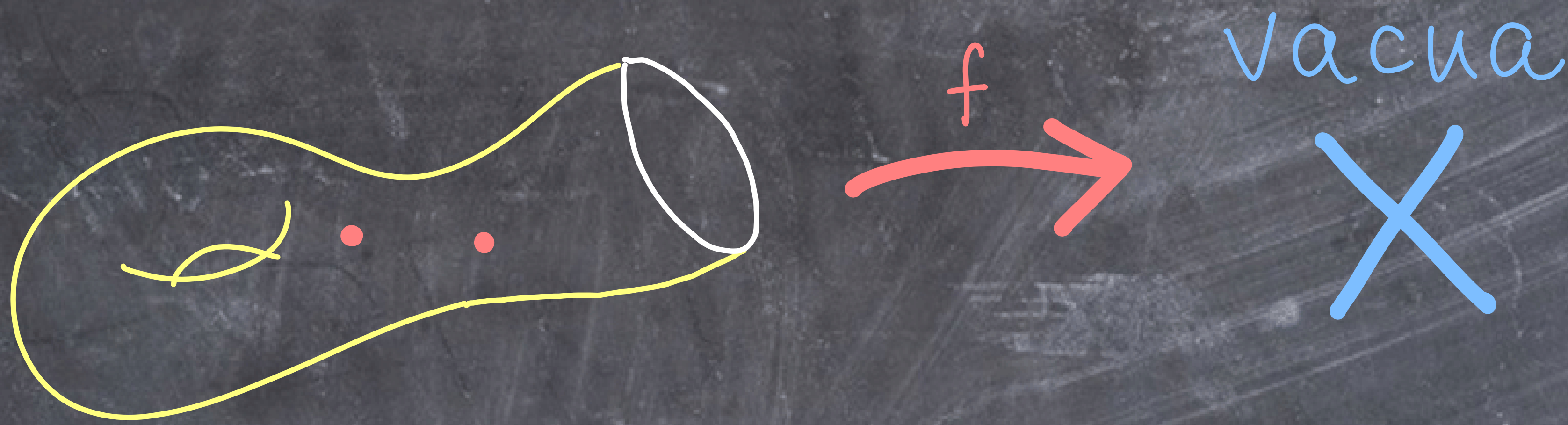
$$\mathrm{Aut} \to \mathrm{Aut}(X, \omega)$$

and also graded by the topological charge

$$\cap$$

$$H_2(X, \mathbb{Z})$$

actual computations take place in K-theory of the moduli spaces of holomorphic "maps" (with sing)



lots of foundational as well as actual geometric issues

$$\mathrm{Index} = \chi\left(\text{moduli}\,,\ z^{\deg f}\ \text{virtual } \hat{A}\text{-genus}\right)$$

$\cap$

$K_{Aut}(pt)[[z]]$

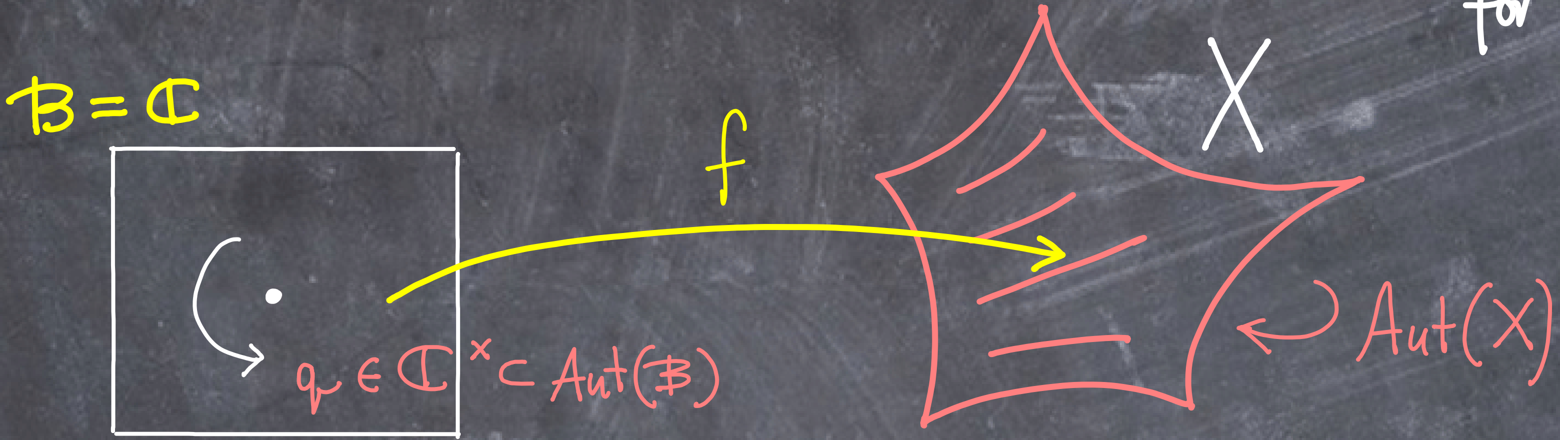
Aut-equivariant K-theory class

These indices, which are functions of $A \subset \mathrm{Aut}(X, \omega)$

and $Z = H^2(X, \mathbb{Z}) \otimes \mathbb{C}^*$

deformation torus for X^*

are by a certain yoga reduced to the case



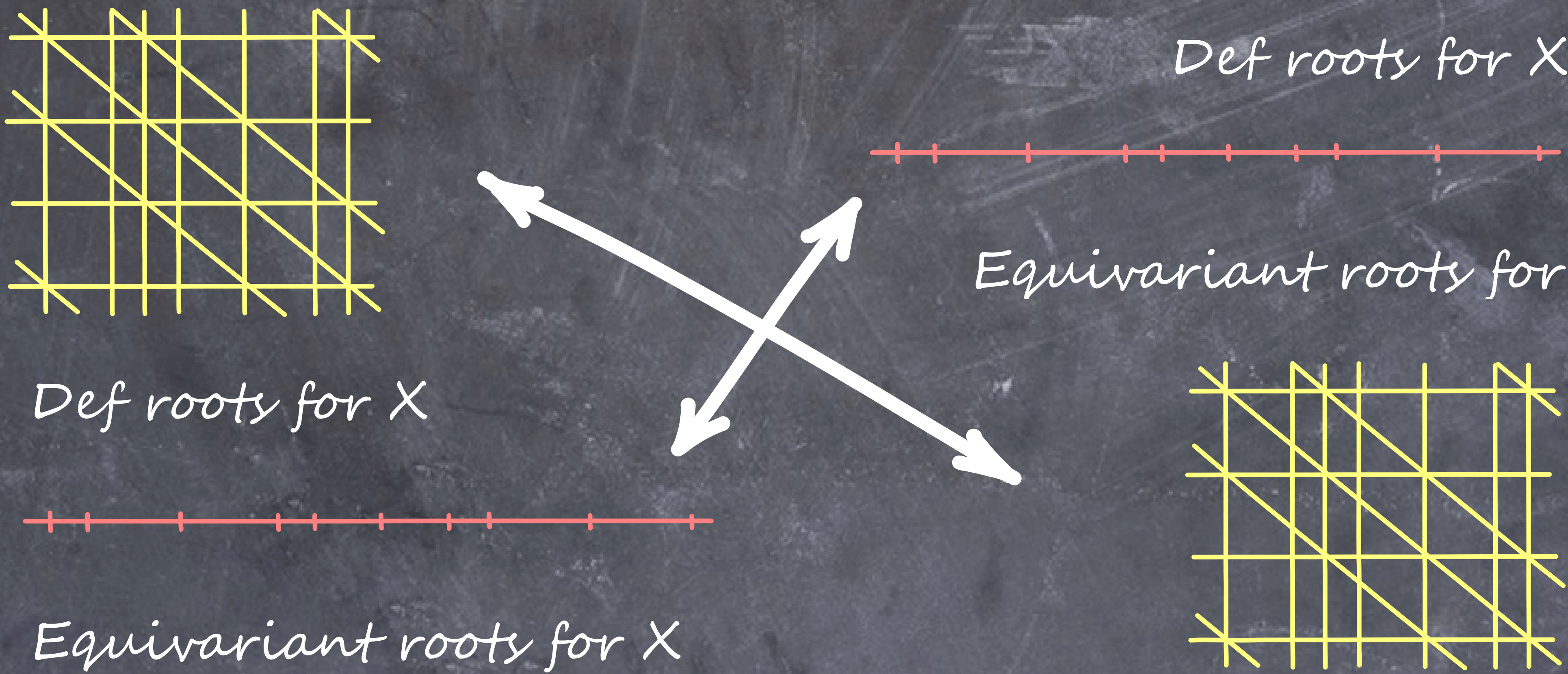
where they can be shown to satisfy certain linear q-difference equations in both A and Z

compatible, i.e. flat

These difference equations are of pronounced Lie-theoretic flavor (generalize, in particular q-Knizhnik-Zamolodchikov equations, Macdonald equatios, ... [O., Smirnov-O.])

their singularities in A and Z are regular and detemined by the roots of two kinds as defined above

There is an amazing duality, known as 3d mirror symmetry, symplectic duality etc. that goes back to some insights of Intrilligator and Seiberg. On roots as defined by us above, it should act as follows:



Most basic example: $X = T^*\mathbb{P}^{n-1} = T^* Gr(1,n)$

orbit of $\begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in GL(n,\mathbb{C})$

action $z \in Z = \mathbb{C}^*$, root $z = 1$

$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$, roots $a_i = a_j$

Dual $X^\vee$ = resolution of $xy = w^n$

deformation $xy = \prod (w - a_i)$

action $(x, y, w) \to (zx, z^{-1}y, w)$

The first really new example $\quad (a x_1, a^{-1} x_2)$

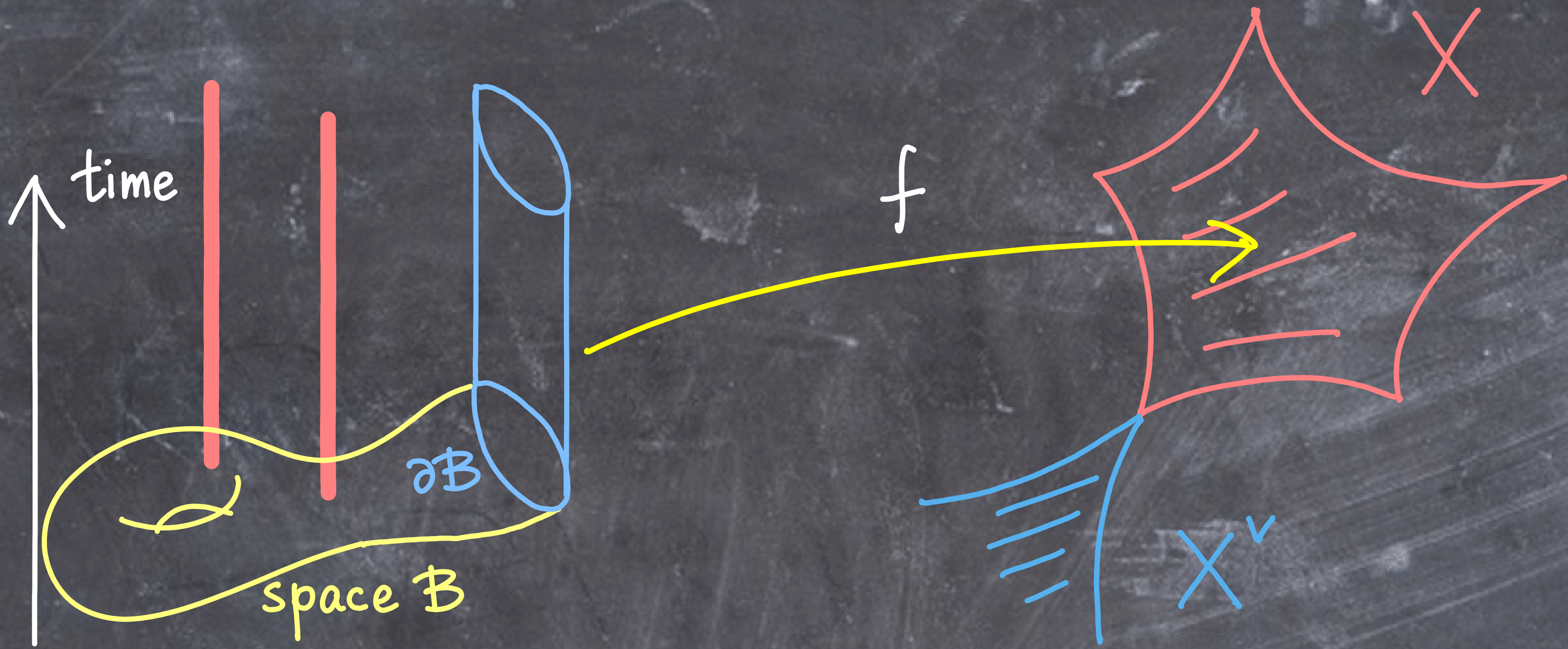
$$X = \mathrm{Hilb}(\mathbb{C}^2, n) \xrightarrow{a}$$

$$= \{ \text{ideals } I \subset \mathbb{C}[x_1, x_2] \text{ of codim} = n \}$$

self-dual!

roots $\quad z^k = 1, \quad k = 1, \dots, n$

$$a^k = 1$$



Duality connects different theories that have equivalent low-energy description. In particular, their indices are the same, with a change of variables

$A^\vee = Z$, $Z^\vee = A$, and a correspondence between boundary conditions on both sides

This correspondence is a certain canonical uniquely defined $A \times A^\vee$- equivariant elliptic cohomology class on $X \times X^\vee$

with Mina Aganagic, we have shown:

- its existence, in some generality
- it determines the monodromy of the q-difference equations, on both sides
- it makes the indices equal

of course, roots on both sides come into the very middle of this