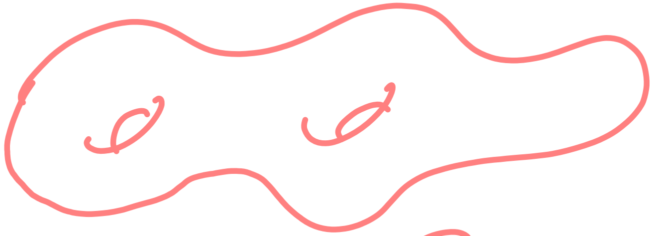
enumerative symplectic duality

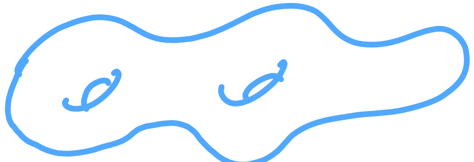
M. Aganagic & A. Okounkov

this work is about indices in SUSY theories in dim $= 2+1$

Space  $\times$ time S^1

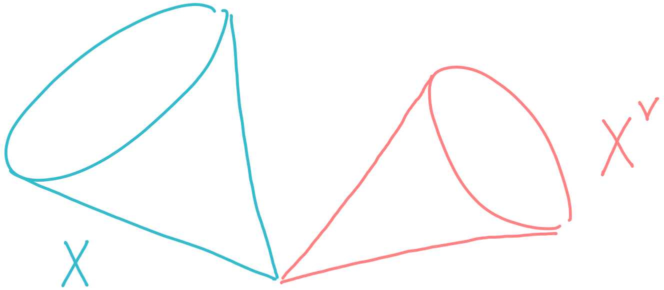
B

indices may be computed in IR (i.e for B very large) where we can sort of replace our original theory by a

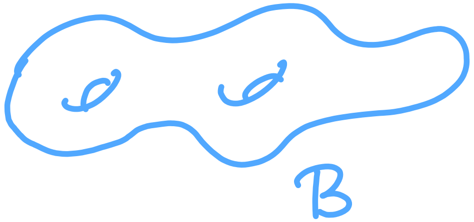
SUSY σ-model:  $\to X =$ moduli of vacua of the theory

(holo maps) B

a singular holo sympl variety, and at singularity some UV degrees of freedom will not go away



this is one "Higgs" branch of moduli of vacua

holo maps : 

$\to X$ = moduli of vacua of the theory

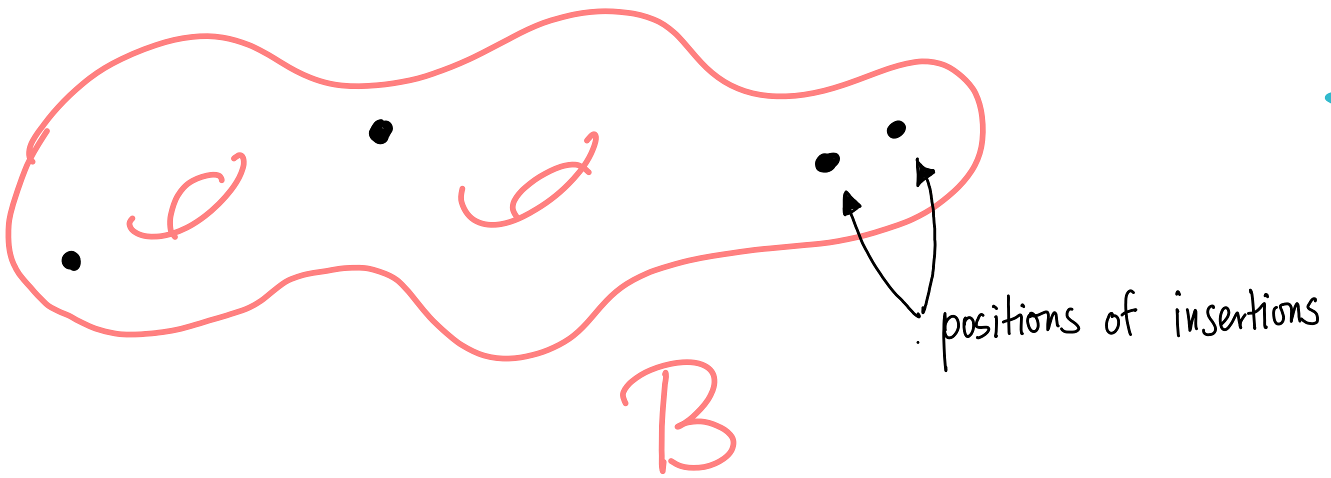
a <u>singular</u> holo sympl variety, and at singularity some UV degrees of freedom will not go away

the extra information that is required at singularities is described by $X^\vee$ = the Coulomb branch

Basic expectation: the roles of X and $X^\vee$ are, in principle, symmetric, and the curve counts (= indices) in two may be the same, with the right identification of insertions

[Intrilligator-Seiberg] + ...

amazing math statement whenever defined



and this identification is natural in B, i.e. the two kinds of counts define the same K-theory class on the moduli of

With current technology, this may be studied for gauge theories for which

[Nekrasov + Shatashvili]

$$X = M /\!\!/\!\!/\!\!/ G$$

Symplectic reprentation of G

algebraic symplectic reduction

complexified gauge group

$$\text{index} = \chi\left(\text{quasimaps } B \xrightarrow{f} X \,,\; z^{\deg f} \; \hat{\mathcal{O}}_{\text{vir}} \otimes \text{tautological} \right)$$

virtual representation of G_F = centralizer of G in M

$\cup$

A = max torus of $G_F \cap Sp(M)$

the Kähler variable

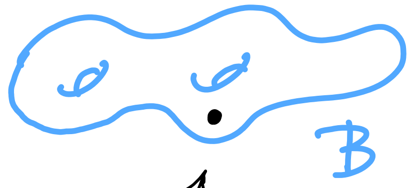
$$z \in \mathrm{Pic}(X) \otimes \mathbb{C}^*$$

$\parallel$

characters of G

index of virtual Dirac operator

the algebra $\mathbb{C}[X^\vee]$ has been defined by Nakajima as a certain algebra acting by Hecke modifications on

$$K_{crit}\Big(\text{quasimaps } \quad$$  $$B \longrightarrow X \Big)$$

adds & removes singularities at a given point $b \in B$

"monopole operators" in physics lit

followed by [BFN] and many others....

Gives a singular affine alg variety, with a partial resolution if $A \neq \{1\}$. Doesn't come, in general, from any gauge theory

Not clear how to count curves in $X^\vee$

Well-defined curve-counting theories on both sides currently exist only if $\dim X^A = 0$, in which case we can make the following provisional

<u>Def.</u> a pair $(X, X^\vee)$ is a 3D-mirror, or symplectic dual pair if

$$(X^A)^\vee = \text{tangent spaces to } (X^\vee)^{A^\vee}$$

in Nakajima, [BFN], or any other sense

these are mirrors of 0-dimensional varieties

their mirrors are vector spaces

includes the identification

$$Z = A^\vee, \quad A = Z^\vee$$

not obvious, but true that this relation is symmetric

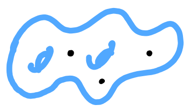
Theorem ☆ [Aganagic-O.]

Curve counts in X = Curve counts in $X^\vee$

really, a map

Kähler variables

$$K_{G_F}(X)^{\otimes n} \longrightarrow K(\overline{\mathcal{M}}_{g,n})[[z]]$$

$B =$ 

equivariant variables live here, in particular $A = Z^\vee$

with the right identification

$$K_{eq}(X)\otimes \mathbb{C}(z) \simeq K_{eq}(X^\vee)\otimes \mathbb{C}[z^\vee]$$

Main example:

$$\mathrm{Hilb}(\mathbb{C}^2, k)^\vee = \mathrm{Hilb}(\mathbb{C}^2, k)$$

and more generally

$r=1$
$n=1$

$$\left(\begin{array}{c}\text{sheaves of rank } r \\ \text{on an } A_{n-1}\text{-surface} \\ c_2 = k\end{array}\right)^\vee = \begin{array}{c}\text{Same with} \\ r \leftrightarrow n\end{array}$$

This is a double loop version of the $U_\hbar(\widehat{\mathfrak{gl}}(n)) \times U_\hbar(\widehat{\mathfrak{gl}}(r))$ dualities studied by Varchenko & collaborators, ...

The corresponding 3-dimensional theory has a conjectural manifestly $X \longleftrightarrow X^\vee$ symmetric expression as the theory that lives on the k-fold M2 brane that wrapps the zero section in

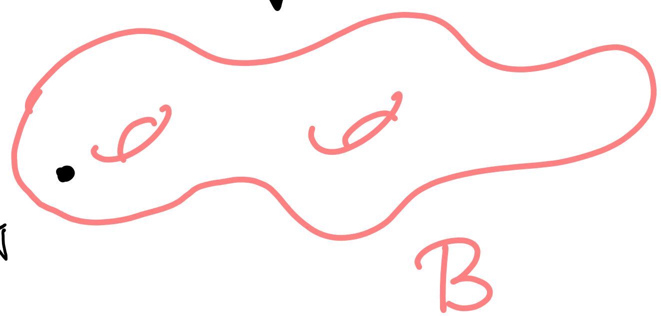
$$
\begin{array}{ccc}
A_{n-1} \times A_{r-1} \hookrightarrow & CY^{11} & \\
 & \downarrow & \\
 & S^1 \times B &
\end{array}
$$

the space-time of M-theory

This particular geometry is, basically, the old geometric engineering of gauge theories and fits into general math conjectures made in [Nekrasov-O.]

Similarly to the above, the right generality to consider are twisted quasimaps to X, that is, quasimap sections

bundle with fiber X and structure group $G_F \curvearrowright X$



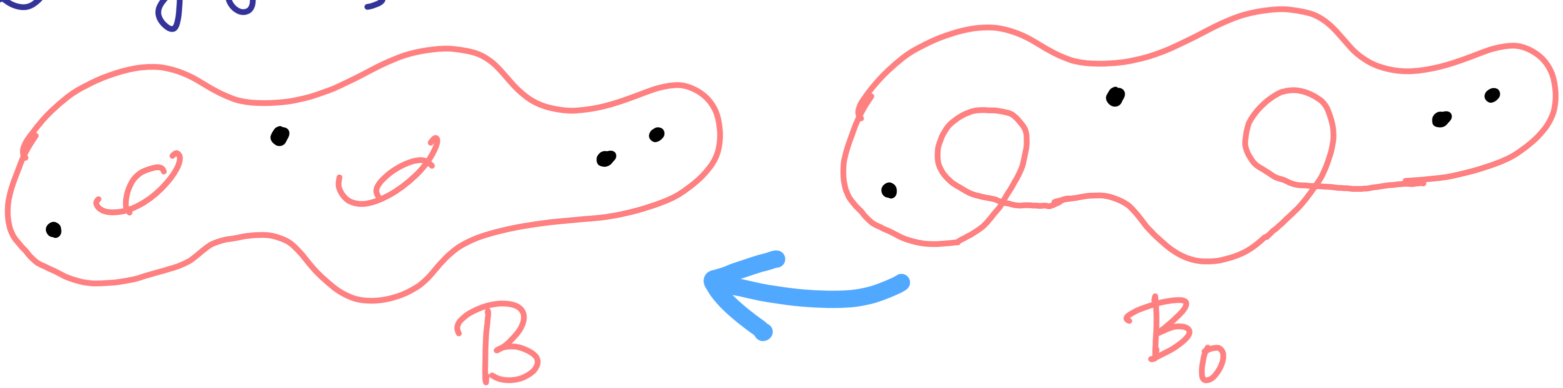
one can modify such bundles by cocharacters of A = characters of $(Z^\vee)^{\text{dual}}$

$\cap$

$G^\vee$

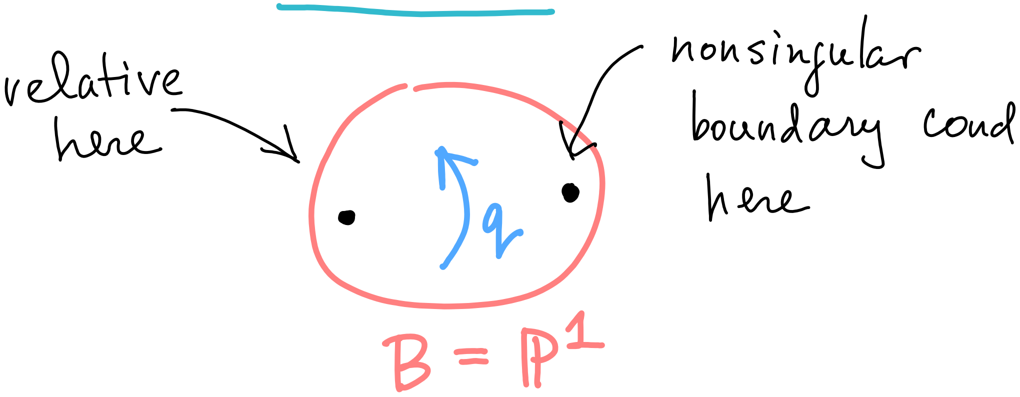
twists by line bundles on the other side

Main steps ① by gluing

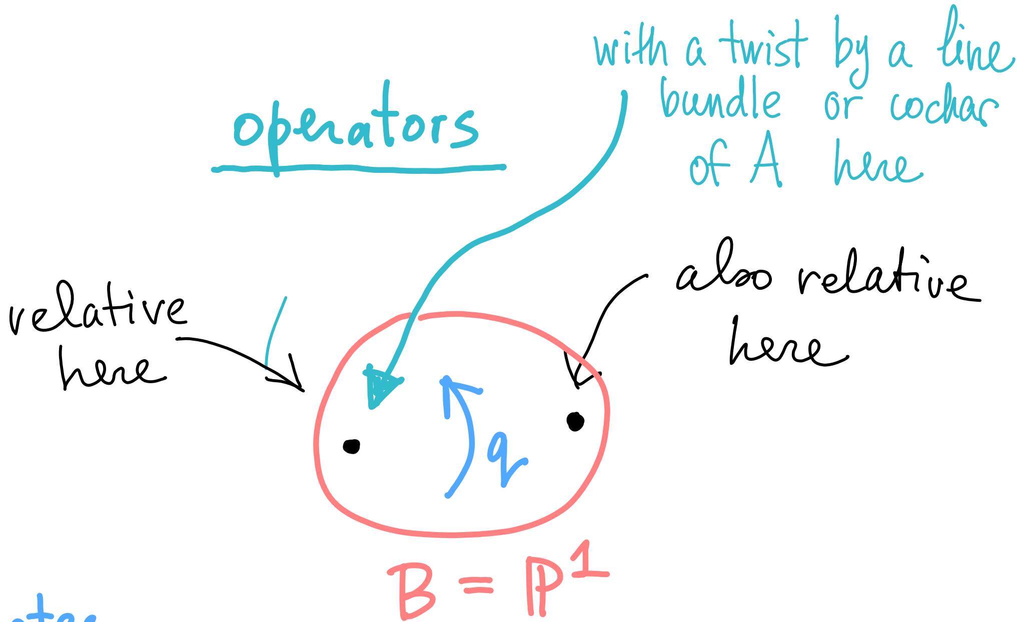


and general reconstruction theorems of Givental-Teleman-type everything is determined by a certain flat q-difference connection with

Solutions



operators



see PCMI notes

(2) (the crucial step) show that these q-connections on two sides are gauge equivalent with

$$A = Z^{\vee} \quad \text{and} \quad Z = A^{\vee}$$

When X and $X^{\vee}$ are Nakajima varieties, these are given "explicitly" in terms of certain quantum groups $\mathcal{U}_\hbar(\hat{\mathfrak{g}}_Q)$ [Smirnov-O.]

where

$$\mathfrak{g}_Q = \mathfrak{f} \oplus \bigoplus_{\alpha} \mathfrak{g}_{Q,\alpha}$$

Lie algebra associated to a quiver

graded dimensions given by Kac polynomials ?!

but the equality of connections seems to be highly nonobvious even for $\mathfrak{g}_Q = \widehat{\mathfrak{gl}(r)}$, our main example

Instead, we identify the monodromy of the two equations geometrically, using our theory of elliptic stable envelopes

③ chase the gauge equivalence

not today

will be explained in the rest of the talk

Important point #1

The z-equations and the a-equations while consistent and regular separately, are not regular jointly.

This manifests itself by:

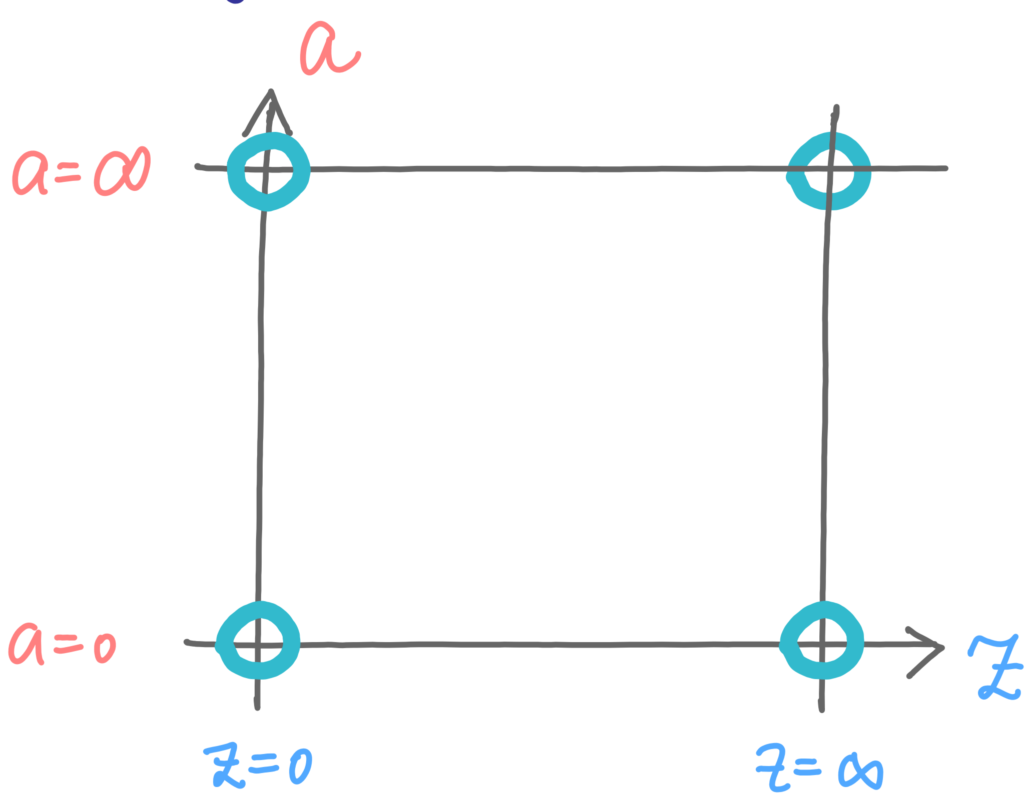
- solutions grow superpolynomially as $(z, a) \to (0,0)$
- there is no basis of solutions holomorphic for for $0 < |z| < \varepsilon$ and $0 < |a| < \varepsilon$

This can never happen for differential equations by a deep theorem of Deligne, but is commonplace for q-difference equations

For a silly 1×1 example, one can take

$$\begin{cases} F(qz, a) = aF \\ F(z, qa) = zF \end{cases} \rightsquigarrow e^{\frac{\ln z \ln a}{\ln q}}$$

which is not regular separately but <u>not</u> at $(z,a) = (0,0)$, $(0,\infty)$, $(\infty,0)$, (∞,∞)



for a 2×2 example take the q-hypergeometric equation

Feature, not a bug!

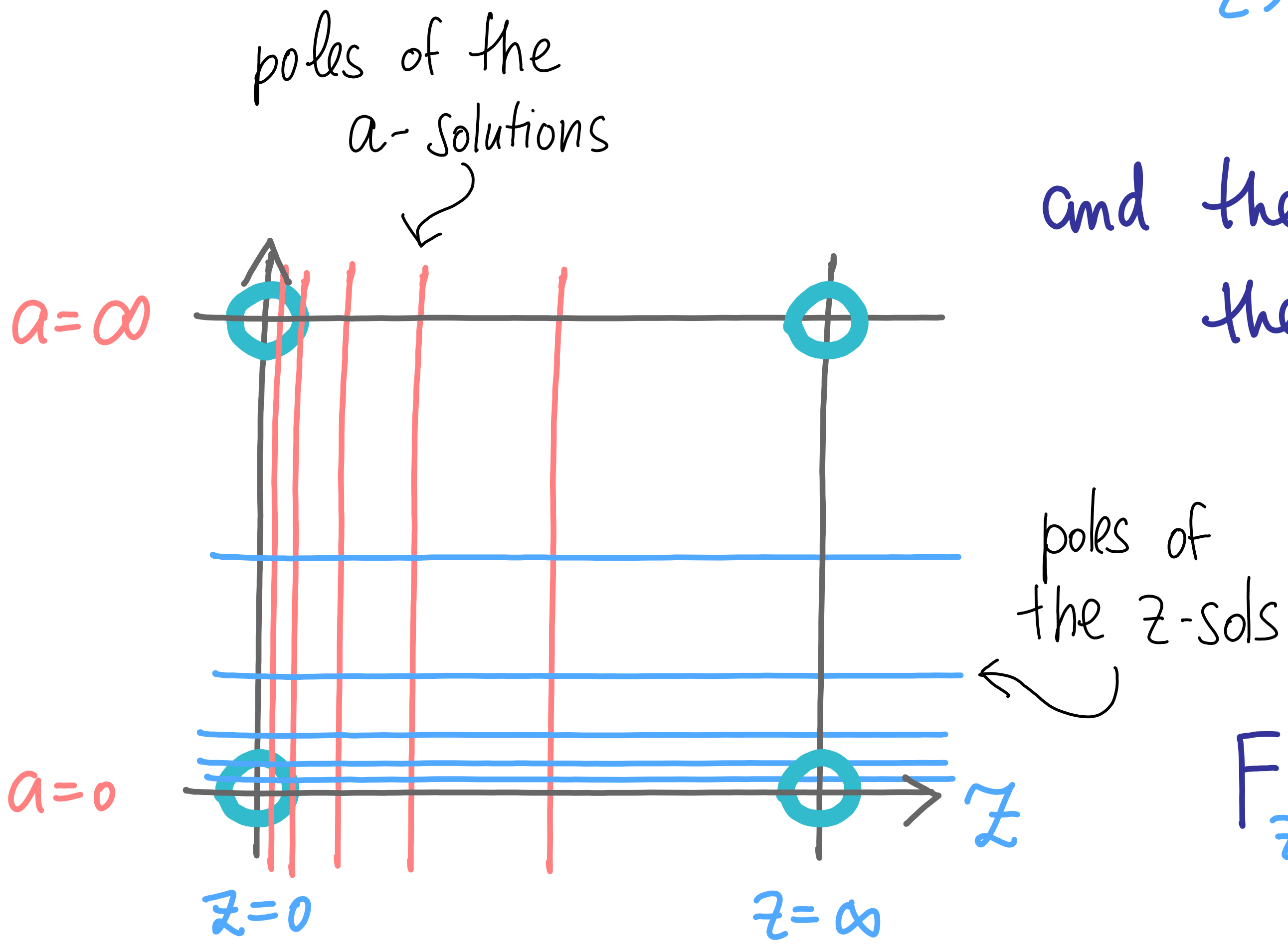
In a nbhd of a point like this, we have 2 kinds of solutions:

the z-solutions

$$F_{z\to 0} = \sum_n r_n(a)\, z^n$$

meromorphic

and the a-solutions, with the opposite properties



There is an elliptic transition matrix

$$F_{z\to 0} \xleftrightarrow{\quad} F_{a\to 0}$$

pole subtraction matrix $\mathcal{P}$

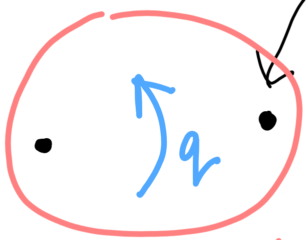
NB Once we identify the q-connections, the two kind of solutions

$$F_{z\to 0} \longleftrightarrow F_{a \to 0}$$

correspond to the counts

noncompact count, takes values in $K_{eq}(X)_{\text{localized}}$

relative here



nonsingular boundary cond here

$B = \mathbb{P}^1$

by construction, is a series in $z^{\deg}$ with coefficients in rational functions of a

in the two dual geometries

Monodromy vs. pole subtraction

- both elliptic
- one global / analytic, another local / algorithmic
- pole subtraction constraints monodromy:

$$
\begin{tikzcd}
F_{z\to 0} \arrow[r, "\text{monodromy}"] \arrow[d, "\mathcal{P}_{0,0}"'] & F_{z\to\infty} \arrow[d, "\mathcal{P}_{\infty,0}"] \\
F_{a\to 0} \arrow[r, dash, "\substack{\text{monodromy}\\ \text{for } a=0\text{ !!!}}"] & F_{a\to 0}
\end{tikzcd}
$$

$$F_{a\to 0} = \sum_n f_n(z)\, a^n, \quad \text{where } f_0(z) \text{ solves the } a=0 \text{ equation}$$

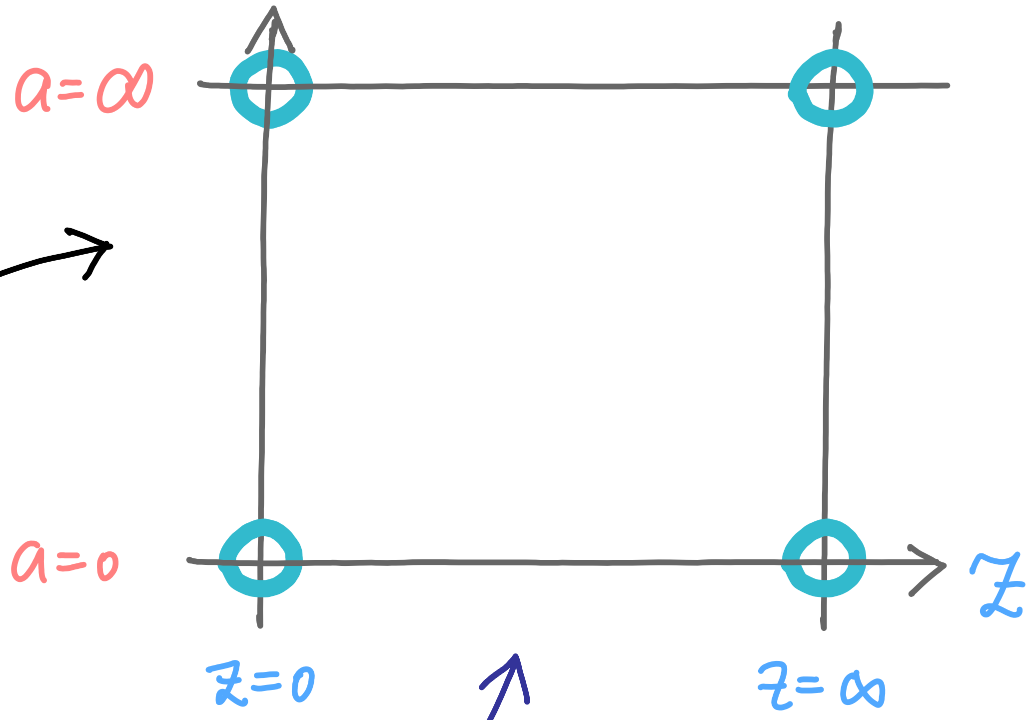
In the geometric context

at $z=0$,

there are no curves,

an easy computation

in $K(X)$



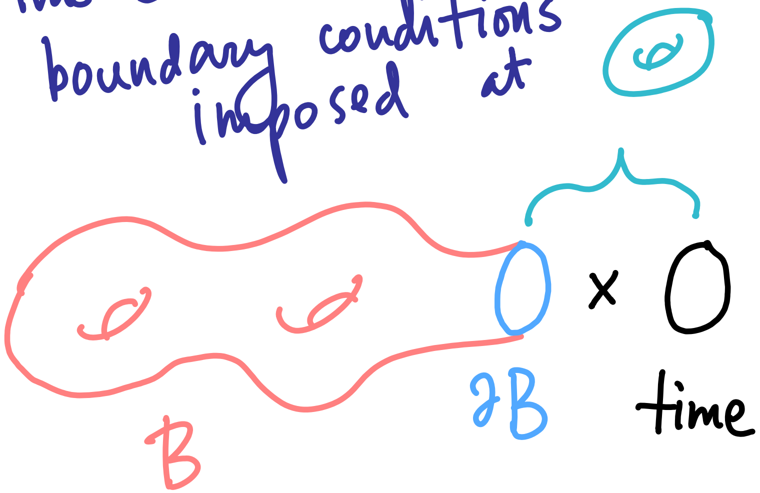
this is where our definition of $X^\vee$ comes in

at $a=0$ we have the quantum q-difference equation for the fixed points $X^a \subset X$

Important point #2

Pole subtraction matrices are geometric, come from a certain correspondence between X^a and X in equivariant elliptic cohomology

This is about boundary conditions imposed at



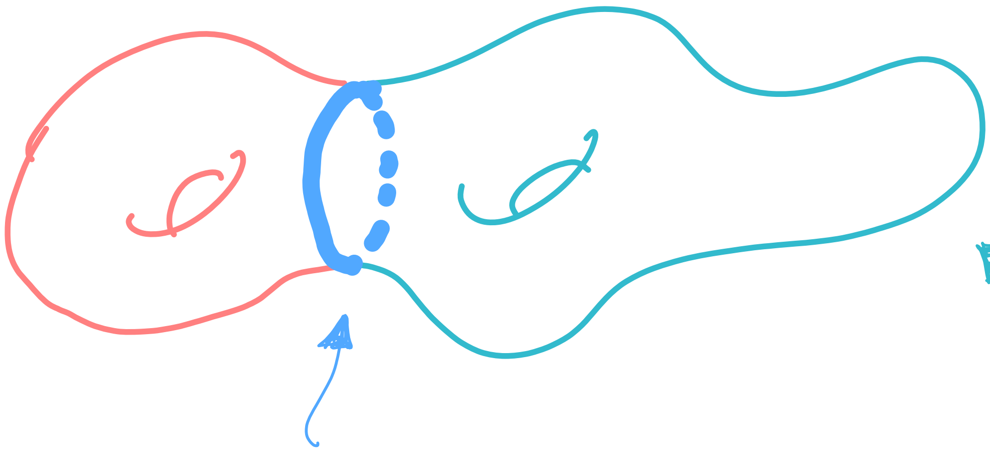
another manifestation of the same category of boundary conditions are coherent sheaves on loops to X

Theorem [Aganagic-O.] Pole subtraction matrices are the elliptic stable envelopes (with the right normaliz).

The duality interface

exists tautologically, once we have two different descriptions of the same theory

describe using X here



and in terms of $X^\vee$ here

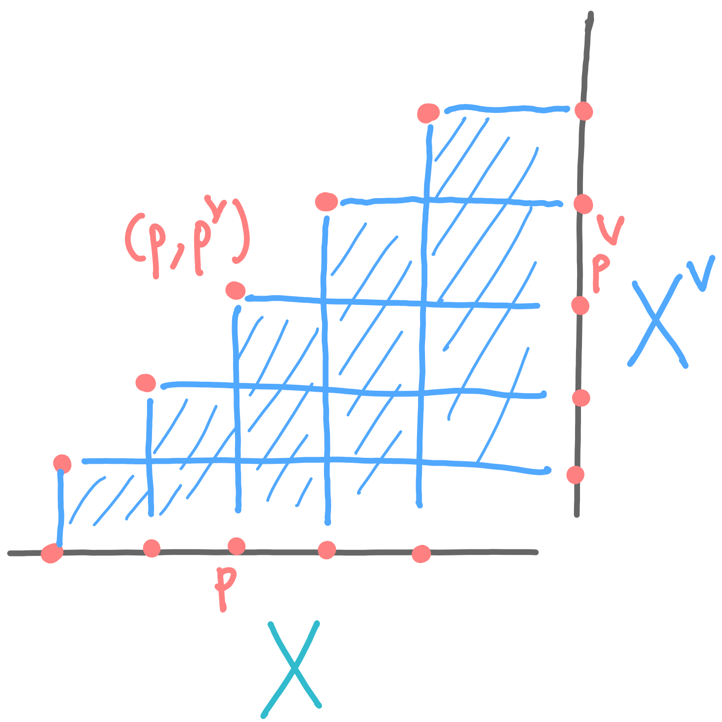
there is really no boundary here, just a change of description

should be an equivalence of the category of boundary cond.

The duality interface II

Thm [Agamagic-O.] $\exists!$ section Δ of a specific line bundle on $\mathrm{Ell}_T X \times \mathrm{Ell}_{T^\vee} X^\vee$ with support on

$$\bigcup_{p \in X^A = (X^\vee)^{A^\vee}} \mathrm{Attr}_X(p) \times \mathrm{Attr}_{X^\vee}(p^\vee)$$



the line bundle is such that

$$\Delta\Big|_{p \in X} = \mathrm{Stab}_{X^\vee}(p^\vee)$$

and also with $X \longleftrightarrow X^\vee$

the matching of the variables:

$$\begin{tikzcd}
\mathrm{Ell}_T(X) \arrow[d] & \times & \mathrm{Ell}_{T^\vee}(X^\vee) \arrow[d] & & \\
A/q^{\mathbb{Z}} & \times & Z/q^{\mathbb{Z}} & = & \mathrm{Pic}(X) \otimes \mathbb{C}^*/q^{\mathbb{Z}} \arrow[d, "c_1"'] \\
& & & & \mathrm{Pic}(\mathrm{Ell}_T(X))
\end{tikzcd}$$

this side parametrizes line bundles here

and vice versa

<u>Corollary:</u> $\mathrm{Stab}_X(p_1)\big|_{p_2} \propto \mathrm{Stab}_{X^\vee}(p_2^\vee)\big|_{p_1^\vee}$

and the monodromy of the two q-connections is the same