The Shuffle Algebra

Today, we will study 4 incarnations of an algebra $\mathcal{A}$, with emphasis on the last one. In the order we will be presenting them, they are:

- The **current algebra** $\mathcal{W}_{1+\infty}$, doubly deformed
- The **Hall algebra** of coherent sheaves on an elliptic curve
- The spherical part of Cherednik's **double affine Hecke algebra**
- The Drinfeld double of the **shuffle algebra**

By definition, the current algebra $\mathcal{W}_{1+\infty}$ is generated over $\mathbb{C}$ by elements x_v for all $v \in \mathbb{Z}^2$, satisfying the relations:

$$[x_v, x_{v'}] = \det(v, v') \cdot x_{v+v'} + \delta^0_{v+v'} \cdot \text{ central}$$

Call $\mathcal{A}^1$ the algebra obtained by deforming this commutation relation over the field $\mathbb{K} = \mathbb{C}(q_1, q_2)$:

$$[x_v, x_{v'}] = \sum x_{w_n} ... x_{w_1} \cdot \text{ coefficient} \tag{1}$$

where the sum goes over all broken lines $0, w_1, ..., w_1 + ... + w_n = v + v'$ with $w_i \in \mathbb{Z}^2 \backslash 0$, lying inside the parallelogram spanned by v and v'. Note that $SL_2(\mathbb{Z})$ acts by automorphisms on $\mathcal{A}^1$, by changing v.

Consider an elliptic curve X over a finite field $\mathbb{F}_q$. Its **Hall algebra** is generated by the isomorphism classes of coherent sheaves $\mathcal{F}$ on X, with the multiplication defined by:

$$[\mathcal{F}] \cdot [\mathcal{G}] = q^{\frac{1}{2}\langle \mathcal{F}, \mathcal{G} \rangle} \sum_{\mathcal{H}} [\mathcal{H}] \cdot \frac{\#\{0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{F} \to 0\}}{\text{Aut } \mathcal{F} \cdot \text{Aut } \mathcal{G}}$$

A coherent sheaf $\mathcal{F}$ is called semistable if $\frac{\deg \mathcal{G}}{\text{rank } \mathcal{G}} \leq \frac{\deg \mathcal{F}}{\text{rank } \mathcal{F}}$ for all subsheaves $\mathcal{G} \subset \mathcal{F}$. Then we can consider the averages:

$$1^{ss}_{(r,d)} = \sum_{\mathcal{F} \text{ semistable}}^{\text{rank } r, \text{ degree } d} [\mathcal{F}]$$

We consider the subalgebra of the Hall algebra generated by these averages, and then let $\mathcal{A}^2$ denote its Drinfeld double. We have $\mathcal{A}^2 \cong \mathcal{A}^1$, where the parameters q_1, q_2 are connected to the Frobenius eigenvalues of the curve X.

Cherednik's **double affine Hecke algebra** for GL_n has generators:

$$X_1, ..., X_n, Y_1, ..., Y_n, T_1, ..., T_{n-1}$$

which satisfy a lot of relations, for example:

$$[X_i, X_j] = [Y_i, Y_j] = 0, \qquad Y_1 X_1 ... X_n = q X_1 ... X_n Y_1$$

but also more exciting ones, such as:

$$T_i X_i T_i = X_{i+1}, \qquad T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$$

The **spherical part** of the DAHA is generated certain expressions in the $X_i^{\pm 1}$ and $Y_i^{\pm 1}$; in particular, symmetric polynomials in the X's and Y's are among them. One can construct a certain stabilization as $n \longrightarrow \infty$, and the resulting algebra $\mathcal{A}^3$ will be isomorphic to $\mathcal{A}^1 \cong \mathcal{A}^2$.

Consider symmetric rational functions in indeterminates $z_1, z_2, \ldots$:

$$\bigoplus_{n>0} \mathbb{K}(z_1, \ldots, z_n)^{S(n)}$$

We can define a new multiplication on the above space:

$$P(z_1, \ldots, z_n) * Q(z_1, \ldots, z_m) =$$

$$= \mathrm{Sym}\left[P(z_1, \ldots, z_n) Q(z_{n+1} \ldots z_{n+m}) \prod_{i \leq n < j} \lambda\left(\frac{z_j}{z_i}\right) \right]$$

by using the function $\lambda(x) = \frac{(x-1)(x-q_1 q_2)}{(x-q_1)(x-q_2)}$ and the averaging operator:

$$\mathrm{Sym}(P(z_1, \ldots, z_k)) = \frac{1}{k!} \sum_{\sigma \in S(k)} P(z_{\sigma(1)}, \ldots, z_{\sigma(k)})$$

We define the **shuffle algebra** $\mathcal{A}^+$ as the subalgebra generated by the first summand $\mathbb{K}(z_1)$. It is bigraded by the number of variables n and the total degree d, where the elements of $\mathcal{A}^+_{n,d}$ are all of the form:

$$\frac{P(z_1, ..., z_n) \cdot \prod_{1 \leq i < j \leq n} (z_i - z_j)^2}{\prod_{i,j=1}^{n} (z_i - q_1 z_j)(z_1 - q_2 z_j)}$$

for some degree $d + n(n-1)$ homogenous symmetric polynomial [1] P. There is a particular such polynomial U_n of degree $n(n-1)$, called the **ubiquitous polynomial**, with the property that:

$$U_n\left(\frac{1}{z_1}, ..., \frac{1}{z_n}\right) = \frac{U_n(z_1, ..., z_n)}{(z_1 ... z_n)^{2(n-1)}}$$

$$U_n(z_1, q_1 z_1, q_1 q_2 z_1, z_4, ...) = U_n(z_1, q_2 z_1, q_1 q_2 z_1, z_4, ...) = 0$$

[1] It may have poles at $z_i = 0$

We can use the ubiquitous polynomials to construct elements of $\mathcal{A}^+$:

$$y_{n,d} = \frac{U_n(z_1, ..., z_n)e_d(z_1, ..., z_n) \cdot \prod_{1\leq i<j\leq n}(z_i - z_j)^2}{\prod_{i,j=1}^n (z_i - q_1 z_j)(z_1 - q_2 z_j)} \in \mathcal{A}^+_{n,d}$$

where $e_d = e_{d \bmod n}(z_1, ..., z_n) \cdot (z_1...z_n)^{\lfloor \frac{d}{n} \rfloor}$.

Conjecture 1. *The assignment $x_{(n,d)} \longrightarrow y_{n,d}$ (for $n > 0$, $d \in \mathbb{Z}$) induces an isomorphism between the positive half of the algebras $\mathcal{A}^1 \cong \mathcal{A}^2 \cong \mathcal{A}^3$ and our shuffle algebra $\mathcal{A}^+$.*

The next step is to "double" the shuffle algebra $\mathcal{A}^+$ in such a way that it will be isomorphic to the whole algebras defined above.

For that, we define two copies of the shuffle algebra,

$$\mathcal{A}^+ \subset \bigoplus_{n>0} \mathbb{K}(z_1^+, ..., z_n^+)^{S(n)}, \qquad \mathcal{A}^- \subset \bigoplus_{n>0} \mathbb{K}(z_1^-, ..., z_n^-)^{S(n)}$$

To complete the picture, we define $\mathcal{A}^0$ to be a copy of $\widehat{\mathfrak{gl}}_1$ with generators $\{\alpha_d, d \in \mathbb{Z}\backslash 0\}$, and set:

$$\mathcal{A} = \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+$$

with certain relations. Then we claim that the assignment:

$$x_{(-n,d)} \longrightarrow y^-_{n,d}, \qquad x_{(0,d)} \longrightarrow \alpha_d, \qquad x_{(n,d)} \longrightarrow y^+_{n,d}$$

gives rise to an isomorphism between $\mathcal{A}^1 \cong \mathcal{A}^2 \cong \mathcal{A}^3$ and $\mathcal{A}$.

Let us elaborate on the relations within the **double shuffle algebra** $\mathcal{A}$. The most interesting one is between the positive and negative halves:

$$P(z_1^+, ..., z_n^+) \in \mathcal{A}_n^+ \qquad \text{and} \qquad Q(z_1^-, ..., z_m^-) \in \mathcal{A}_m^-$$

and takes the form:

$$P(z_1^+, ..., z_n^+) * Q(z_1^-, ..., z_m^-) = \sum_{i=0}^{\min(m,n)} \frac{(q_1-1)^i (q_2-1)^i}{(q_1q_2-1)^i (2\pi\sqrt{-1})^i} \binom{n}{i} \binom{m}{i} i!$$

$$\int ... \int \left[\prod_{j=1}^{i} \frac{dw_j}{w_j} f(w_j) \right] \frac{Q(w_1, ..., w_i, z_1^-, ..., z_{m-i}^-) * P(w_1, ..., w_i, z_1^+, ..., z_{n-i}^+)}{\text{some boring product of } \lambda\text{'s}}$$

where f is a certain $\mathcal{A}^0$ valued series. Each integral in a w variable refers to the residue at 0 minus the residue at ∞.

So one can ask: why do we care about this shuffle algebra picture? For one thing, it's the only presentation of our algebra which is given explicitly, not just by generators and relations. So we can construct a lot of very interesting elements.

Secondly, all these mysterious polynomials are kind of cool (see "*A commutative algebra on degenerate* $\mathbb{P}^1$ *and Macdonald polynomials*")

Perhaps more importantly, we care about the shuffle algebra because it arises naturally in the study of the **moduli space of sheaves** on $\mathbb{P}^2$.

Consider the moduli space $\mathcal{M} = \mathcal{M}_d$ of rank r torsion free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ of second Chern class $-d$, together with a **framing**:

$$\mathcal{F}|_\infty \cong \mathcal{O}_\infty^{\oplus r} \tag{2}$$

on a distinguished line $\infty \subset \mathbb{P}^2$. This moduli space is smooth and quasi-projective of dimension $2rd$, and is acted on by the maximal torus:

$$T \subset GL_r(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$$

where the first factor acts on the framing (2), and the last two factors act on the base $\mathbb{P}^2$. We can define the **equivariant** $K-$theory of $\mathcal{M}$:

$$K = \bigoplus_{d \geq 0} K_T^*(\mathcal{M}_d)$$

The task in **geometric representation theory** is to make something act meaningfully on K. This has been first done by Nakajima-Grojnovsky in the related case of cohomology, who showed that $\widehat{\mathfrak{gl}}_1$ acts.

Then, Maulik and Okounkov constructed a Yangian action on the cohomology, which they realized as a tensor product of r Fock spacec via a construction known as the **stable basis**.

As for the $K-$theory, Feigin-Tsymbaliuk and Schiffmann-Vasserot recently constructed an action of the algebra $\mathcal{A}$ on K, via generators and relations. I will show how to use a different, more explicit, viewpoint on K in order to get more out of this construction.

The above authors describe K via its torus fixed point basis, which corresponds to the celebrated Macdonald polynomials. This has the advantage of being a, well, basis. But unfortunately, the analogous construction is not known in the case of more complicated quiver varieties.

Instead, we will describe K via **tautological classes**. Concretely, there exists a tautological bundle $\mathcal{T}$ on $\mathcal{M}$, with fibers:

$$\mathcal{T}|_{\mathcal{F}} = H^1(\mathbb{P}^2, \mathcal{F}(-\infty))$$

To this vector bundle, we can associate the rational function:

$$\Lambda(-\mathcal{T}, u) \in K(u)$$

The functor Λ is additive in its first argument, given by $\Lambda(-\mathcal{L}, u) = \frac{1}{[\mathcal{L}]-u}$ for a line bundle $\mathcal{L}$.

When we expand $\Lambda(-\mathcal{T}, u)$ around $u = 0$ or ∞, we get a bunch of coefficients in K called **tautological classes**.

Proposition 2. *The ring K is generated by the tautological classes, in other words any element of K can be written as a linear combination of coefficients of rational functions of the form:*

$$\gamma = \prod_{u \in U} \Lambda(-\mathcal{T}, u) \tag{3}$$

These classes are not linearly independent, and indeed it would be very interesting to describe the relations between them (they should depend strongly on the rank r, as opposed from everything else in the theory). But it's very easy to do computations in terms of these classes.

The shuffle algebra action on K is given in this presentation by:

$$P(z_1^{\pm}, ..., z_n^{\pm}) \in \mathcal{A}^{\pm} \qquad \text{sends} \qquad \gamma \in K_T^*(\mathcal{M}) \qquad \text{to}$$

$$\gamma \cdot \int ... \int P(w_1, ..., w_n) \prod_{i=1}^{n} \left[\Lambda(\pm\mathcal{W}, w_i) \prod_{u \in U} (w_i - u)^{\pm 1} \frac{dw_i}{2\pi\sqrt{-1}w_i} \right]$$

Once again, all the integrals refer to the residue at 0 minus the residue at ∞, and the **universal sheaf** $\mathcal{W}$ is given by:

$$[\mathcal{W}] = t_1^{-1} + ... + t_r^{-1} - (q_1 - 1)(q_2 - 1)[\mathcal{T}]$$

where $t_1, ..., t_r, q_1, q_2$ are the equivariant parameters along the factors of our torus $T \subset GL_r(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$.

We can also construct some more geometric operators on K. For example, consider the flag variety:

$$\mathfrak{Z}_n = \{(\mathcal{F}_0 \supset \mathcal{F}_1 \supset ... \supset \mathcal{F}_n)\} \subset \bigsqcup_{d \geq 0} \mathcal{M}_d \times \mathcal{M}_{d+1} \times ... \times \mathcal{M}_{d+n}$$

where all the inclusions are supported at the same point of $\mathbb{P}^2$. There are k line bundles on this variety:

$$\mathcal{L}_i|_{\mathcal{F}_0 \supset \mathcal{F}_1 \supset ... \supset \mathcal{F}_n} = \Gamma(\mathbb{P}^2, \mathcal{F}_i/\mathcal{F}_{i-1})$$

and two natural projections $p^-/p^+ : \mathfrak{Z}_n \longrightarrow \mathcal{M}$ that only remember the first / last sheaf in the flag. Then we can define operators:

$$a^{\pm}_{\beta_1,...,\beta_n} : K \longrightarrow K, \qquad\qquad a^{\pm}_{\beta_1,...,\beta_n}(c) = p^{\pm}_* \left([\mathcal{L}_1]^{\beta_1}...[\mathcal{L}_n]^{\beta_n} \cdot p^{\mp *}(c)\right)$$

Proposition 3. *All of these operators lie in the shuffle algebra* $\mathcal{A}^{\pm}$ *action:*

$$a^{\pm}_{\beta_1,\dots,\beta_n} = \mathrm{Sym}\left[\frac{(z_1^{\pm})^{\beta_1}\dots(z_n^{\pm})^{\beta_n}}{\left(q_1q_2 - \frac{z_2^{\pm}}{z_1^{\pm}}\right)\dots\left(q_1q_2 - \frac{z_n^{\pm}}{z_{n-1}^{\pm}}\right)} \prod_{1\leq i<j\leq n} \lambda\left(\frac{z_j^{\pm}}{z_i^{\pm}}\right)\right] \in \mathcal{A}_n^{\pm}$$

I would really love to understand how the rational functions in the RHS relate to the ubiquitous polynomials U_n and the generators $y_{n,d}$ from 11 slides ago. Examples of these relations are:

$$\frac{a^{\pm}_{-1,0,...,0,1} + q_1 q_2 a^{\pm}_{-1,0,...,0,1,0} + ... + (q_1 q_2)^{n-1} a^{\pm}_{0,...,0}}{(1 + q_1 + ... + q_1^{n-1})(1 + q_2 + ... + q_2^{n-1})} = \frac{(q_1 q_2 - 1)^{n-1} \cdot y_{\pm n,0}}{n!}$$

and:

$$a^{\pm}_{0,...,0,1} = \frac{(q_1 q_2 - 1)^{n-1} \cdot y_{\pm n,1}}{n!} \qquad a^{\pm}_{-1,0,...,0} = \frac{(q_1 q_2 - 1)^{n-1} \cdot y_{\pm n,-1}}{n!}$$