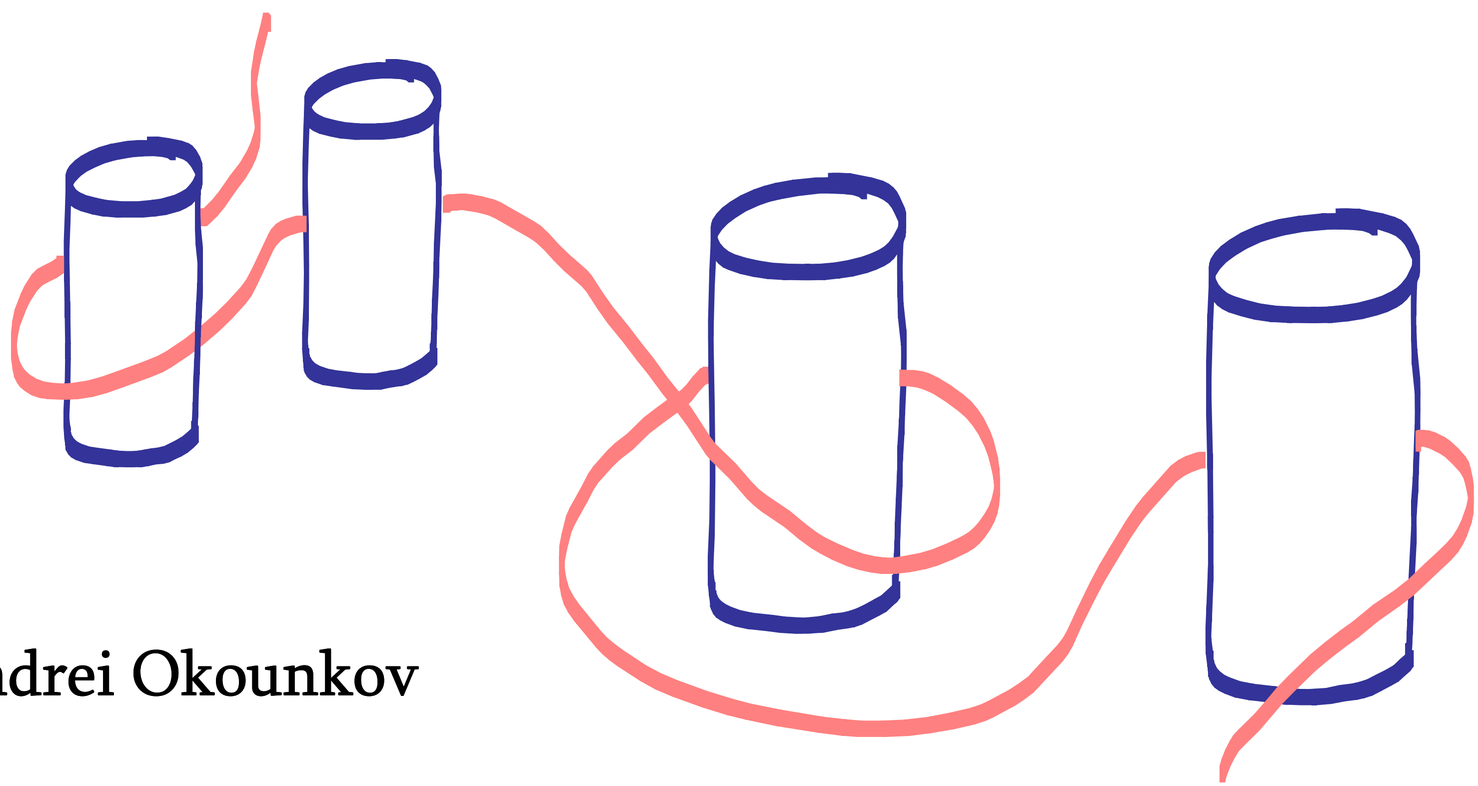
Monodromy: yesterday, today, and tomorrow



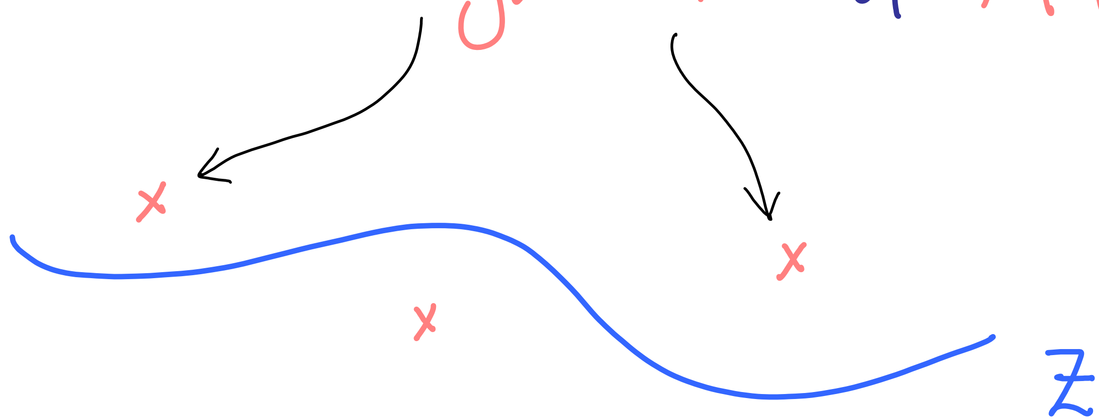
Andrei Okounkov

A linear differential equation

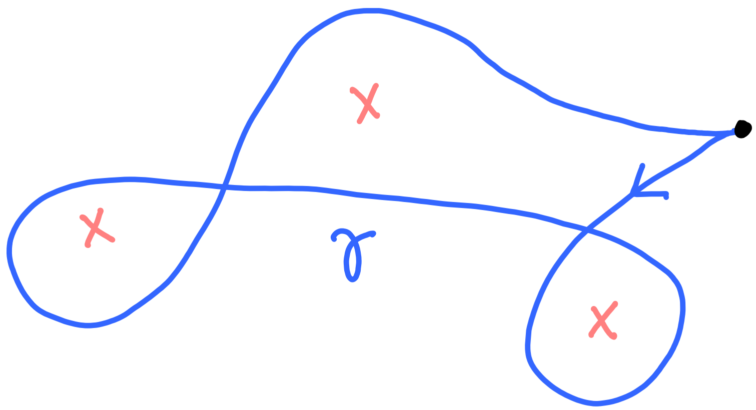
$$\frac{d}{dz} F(z) = A(z)\, F(z)$$

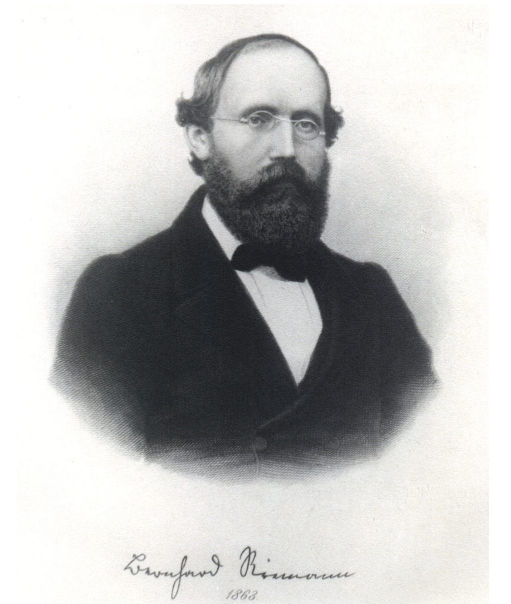
$n \times n$ matrices

may be solved along any path $\gamma \subset \mathbb{C} \ni z$ that avoids the singularities of $A(z)$



As the path γ comes back to the starting point





we get the monodromy matrix

$$M_\gamma = F(\text{end})^{-1}\, F(\text{start})$$

that defines a representation

$$\pi_1(\mathbb{CP}^1 \setminus \text{sing}) \ni \gamma \longmapsto M_\gamma \in GL(n, \mathbb{C})$$

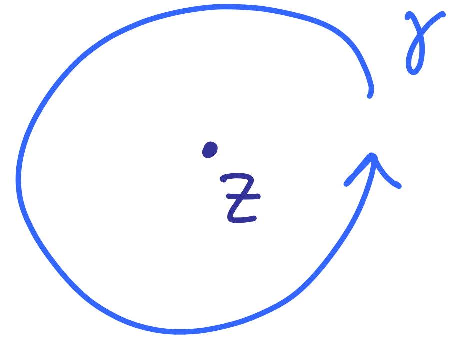
For example, the equation

constant matrix

$$\frac{d}{dz} F(z) = \frac{A_0}{z} F(z)$$

has solution

$$F(z) = e^{A_0 \ln z}$$



and monodromy

$$M_\gamma = e^{2\pi i A_0}$$

Monodromy is a noncommutative generalization of

exp: Lie algebra $\rightarrow$ Lie group

More generally, one can study compatible systems

$$\nabla_i F(z_1, z_2, \dots) = 0 \qquad \nabla_i = \frac{d}{dz_i} - A_i(z), \ [\nabla_i, \nabla_j] = 0$$

that is, flat meromorphic connections ∇ on a bundle $\mathcal{E}$ over some complex manifold B and, again, we get

$$\pi_1(B \setminus \text{sing}, b) \ni \gamma \longrightarrow M_\gamma \in \operatorname{Aut} \mathcal{E}_b$$

base point

fiber over b

from
monodromy.com

Does monodromy relate to your life? It does to nearly everyone's, but surprisingly few realize it.

Do you feel that you are going around in circles and not getting anywhere? Things may not be as bad as they seem. You might be getting somewhere, but not realizing it because you aren't aware of your personal monodromy.

Do you think you are exactly the same person you were half your lifetime ago? If not, it is almost certainly because you are aware, at some level, of your personal monodromy. Think how much richer and more fulfilling life would be if you were completely aware of all the monodromy which surrounds you.

General properties of the map

transcendental map between alg. varieties

$$\text{flat connections } \nabla \xrightarrow{\text{monodromy}} \text{Representations of } \pi_1(B \setminus \text{Sing})$$

are studied as generalizations of Hilbert's 21st problem

In this talk, we are interested in the monodromy of certain very special equations, a bit like

$$\exp(2\pi i \mathbb{Q}) = \sqrt{1} \quad \text{or} \quad j(\text{CM curve } E) \subset \text{algebraic integers}$$

Long gone are the times when mathematical physicists were interested in actual phenomena described by linear ODE

E.T. Whittaker

H.F. Weber

F.W. Bessel

G.B. Airy

. . .

Differential equations of interest to us come from enumerative geometry and representation theory

They include some of the most important linear equations of math physics, such as the Knizhnik-Zamolodchikov equations of conformal field theory

Have natural q-difference generalizations ...

$|q| < 1$

A regular q-difference equation

$$F(qz) = A(z)\,F(z), \qquad A(0) \in GL(n)$$

has a solution holomorphic in a punctured neighbhd of the regular singular point $z = 0$

$$\text{e.g.} \quad F(z) = \exp\left(A_0 \frac{\ln z}{\ln q}\right) \quad \text{if} \quad A = \underset{\text{const}}{A_0}$$

extends meromorphically to $\mathbb{C}$ if

$$A(z) = \text{rational function of } z$$

For instance, the function

$$\frac{1}{\Gamma_q(z)} = (z)_\infty = \prod_{n=0}^{\infty} (1 - q^n z)$$

solves

$$(qz)_\infty = \frac{1}{1-z} (z)_\infty$$

and the general solution of a 1×1 equation has the form

$$f(z) = e^{a_0 \frac{\ln z}{\ln q}} \prod \frac{(b_i z)_\infty}{(c_i z)_\infty}$$

Following Birkhoff, the q-difference monodromy is defined as

$$M_{0\to\infty} = F_\infty^{-1}(z)\, F_0(z)$$

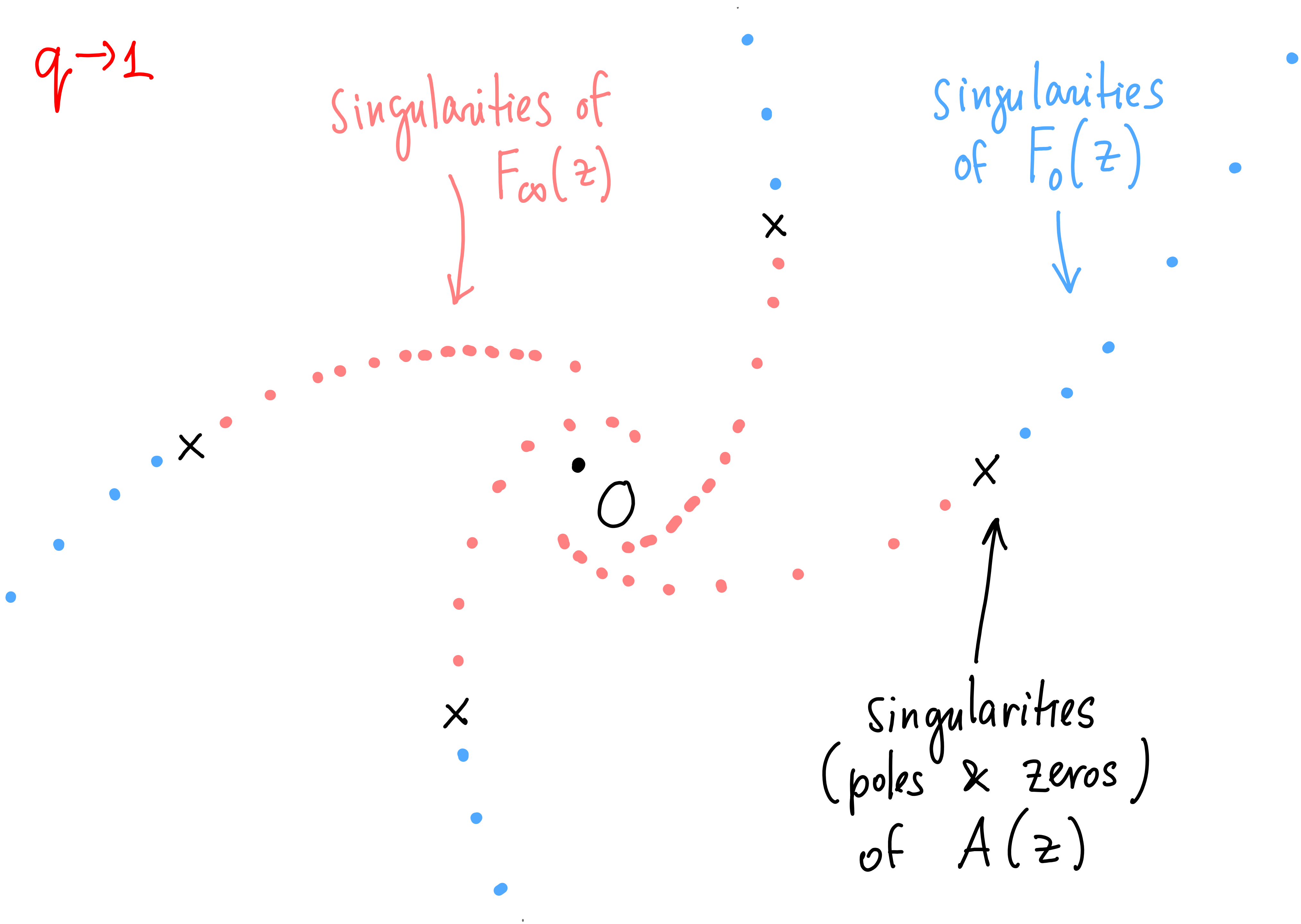
solution at $z=\infty$

solution at $z=0$

One function of $z \in E = \mathbb{C}^\times/q^{\mathbb{Z}}$ that contains as much information as a representation of π_1

In the 1×1 case is an explicit product of

$$\Theta(z,q) = \left(z^{1/2} - z^{-1/2}\right)(qz)_\infty (q/z)_\infty$$


$q \to 1$
Singularities of $F_\infty(z)$
Singularities of $F_0(z)$
0
Singularities (poles & zeros) of $A(z)$

The differential and difference equations of interest to us are all relatives of

"Quantum" differential equations

also known as the Dubrovin connection

a linear ODE, really,
"quantum" refers to the origin of its coefficients and solutions

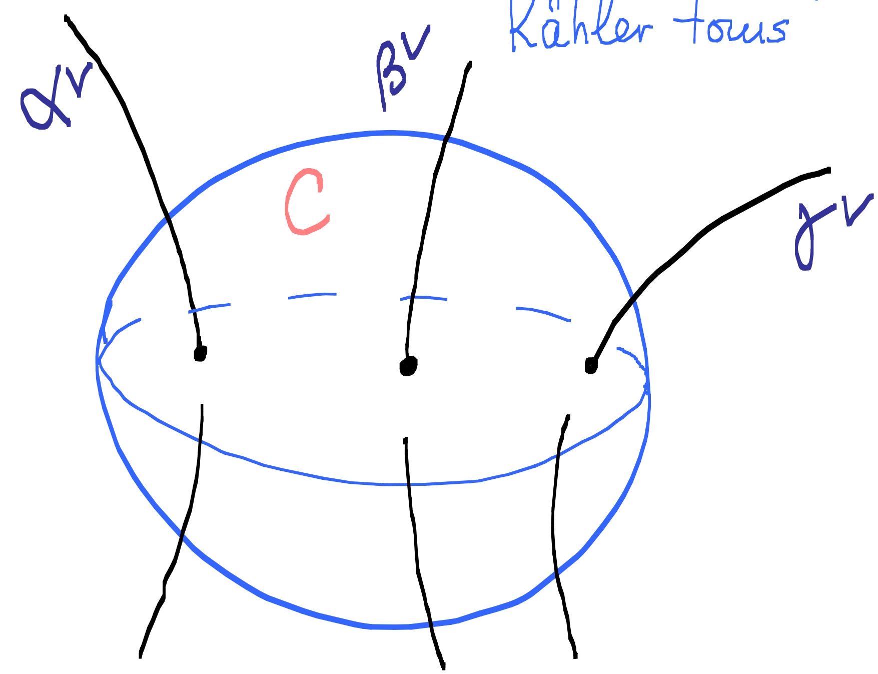
If X is an algebraic variety or a symplectic manifold then the cup product in $H^\bullet(X)$ has a quantum deformation

$$(\alpha \star \beta, \gamma) = \sum_{C} z^{[C]}$$

class of C in $H_2(X,\mathbb{Z})_{\mathrm{eff}}$

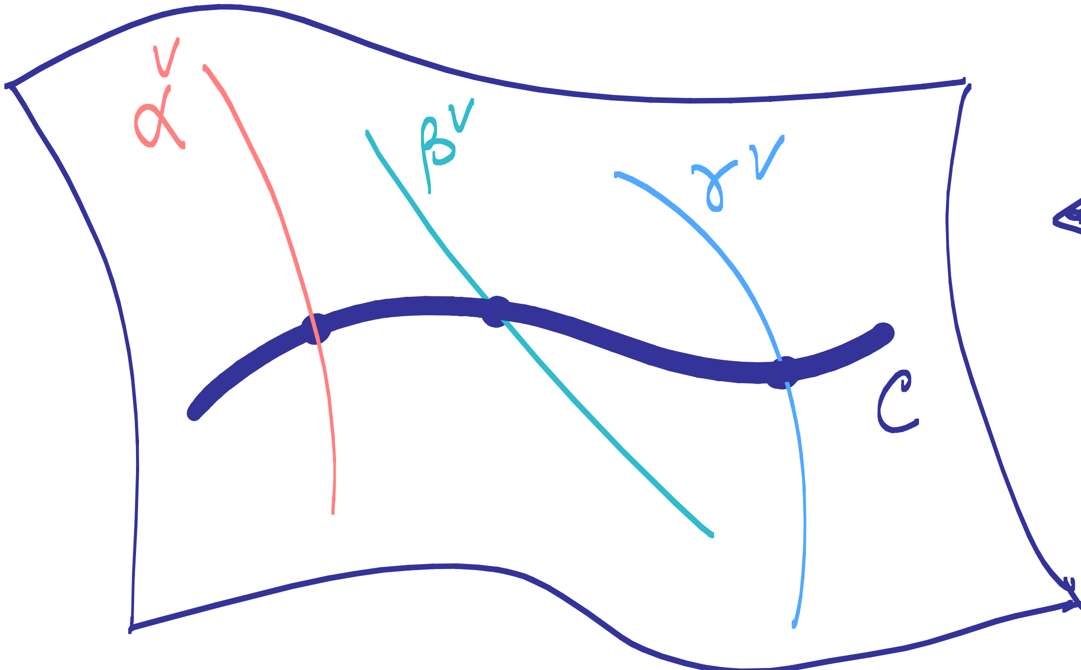
variable in $H^2(X,\mathbb{Z}) \otimes \mathbb{C}^\times$

"Kähler torus"



where the sum (an $\int$, really) is over all rational curves / pseudoholomorphic spheres C that meet Poincaré dual cycles $\alpha^\vee, \beta^\vee, \gamma^\vee$

Remarkably: associative !



what we show perspective grad students

in reality

Very important enumerative information is stored by the solution of the QDE

$$\frac{d}{d\lambda} F(z) = \lambda \star F(z), \qquad \frac{d}{d\lambda} z^{[C]} = (\lambda, C)\, z^{[C]}$$

$\rotatebox[origin=c]{90}{$\in$}$

$H^2(X,\mathbb{C})$ function with values in $H^{\bullet}(X,\mathbb{C})$

$\|$

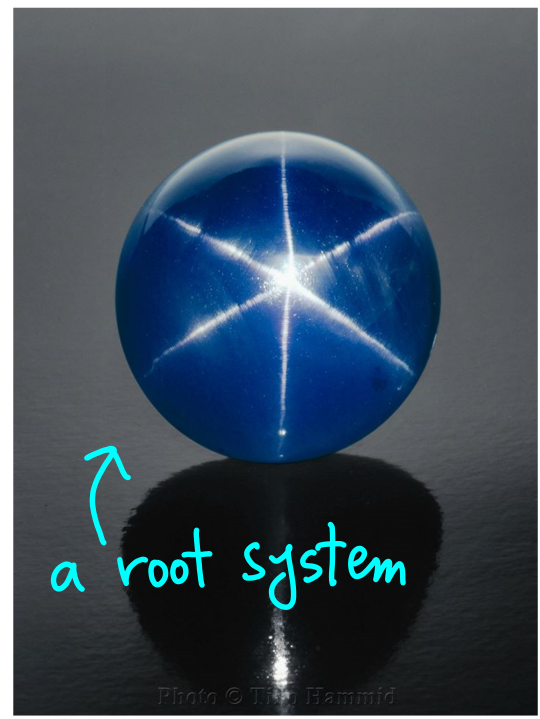
tangent to the Kähler torus $H_2(X,\mathbb{Z}) \otimes \mathbb{C}^* \ni z$

Both the equation and $F(z)$ start out as formal power series in z, analytic continuation in general a very hard chestnut

However, for certain special X, including

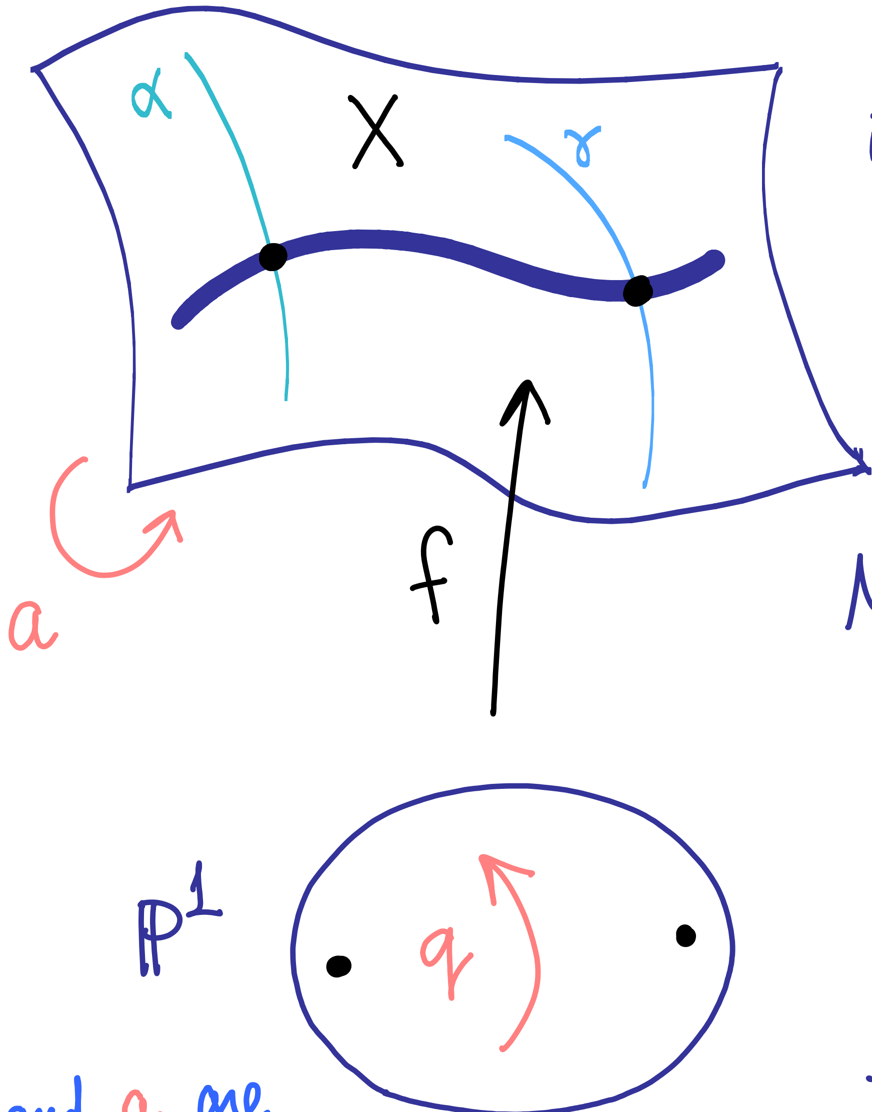
equivariant symplectic resolutions ← "Lie algebras of the XXI century"

- the QDEs are regular (in part., rational coeff) equations that generalize many important equations of rep. theory and math. phys
- their monodromy may be discussed from many different angles (including quantization of X in char $= p \gg 0$ a la Bezrukavnikov & Kaledin)



My goal in this talk is to explain the essential analytic and geometric handle that one has on the monodromy of these equations.

It is clearer in the q-difference situation.



z and a are exchanged by 3d mirror symmetry

in enumerative K-theory, the QDE is replaced by a q-difference equation, where $q \in \mathbb{C}^\times$ acts in the source of f

More parameters come from the action of $a \in \mathrm{Aut}(X)$ in the target of f

There are commuting q-difference equations in the variables a, which are regular if $a \in \mathrm{Aut}(X, \omega)$

alg. symplectic form

In a sense, all these equations are very fancy generalizations of the q-hypergeometric equation satisfied by

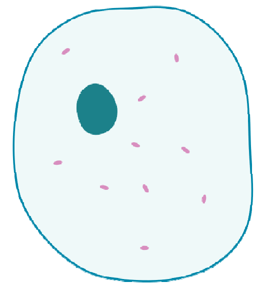
$$F\left(\begin{matrix} a_1 \ \ a_2 \\ a_3 \end{matrix} \middle| \, z \right) = \sum_{n \geq 0} \frac{(a_1)_n \, (a_2)_n}{(q)_n \, (a_3)_n} z^n$$

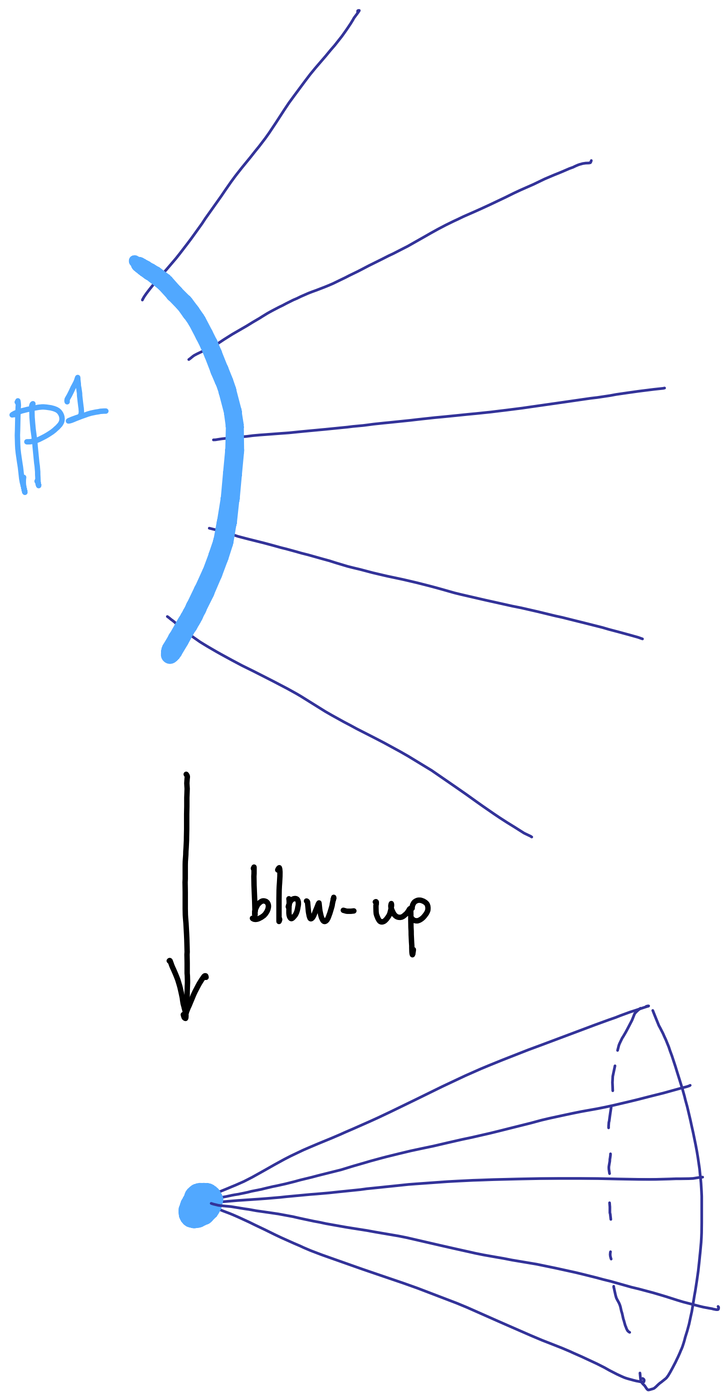
$$(a)_n = \prod_{i=0}^{n-1} (1 - q^i a)$$

solves a 2×2 diff eq in <u>all</u> variables, which we will call the z-equations and the a-equations for brevity

in geometry, corresponds basically to $X = T^* \mathbb{P}^1$

$$X = T^*\mathbb{P}^1 \circlearrowleft a \in PGL(2)$$

the  of the fauna of symplectic resolutions



nilpotent cone in $\mathfrak{sl}_2$

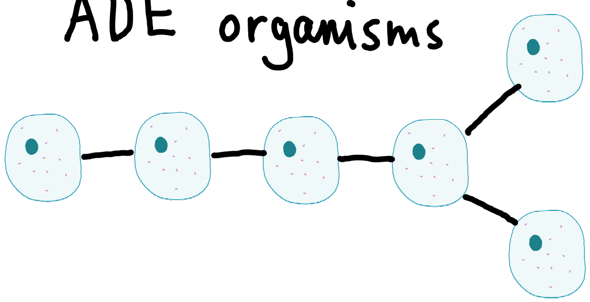
$$x_1^2 + x_2 x_3 = -\det \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} = 0$$

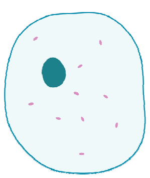
The tree of life of symplectic resolutions*....

a jungle of other possibilities

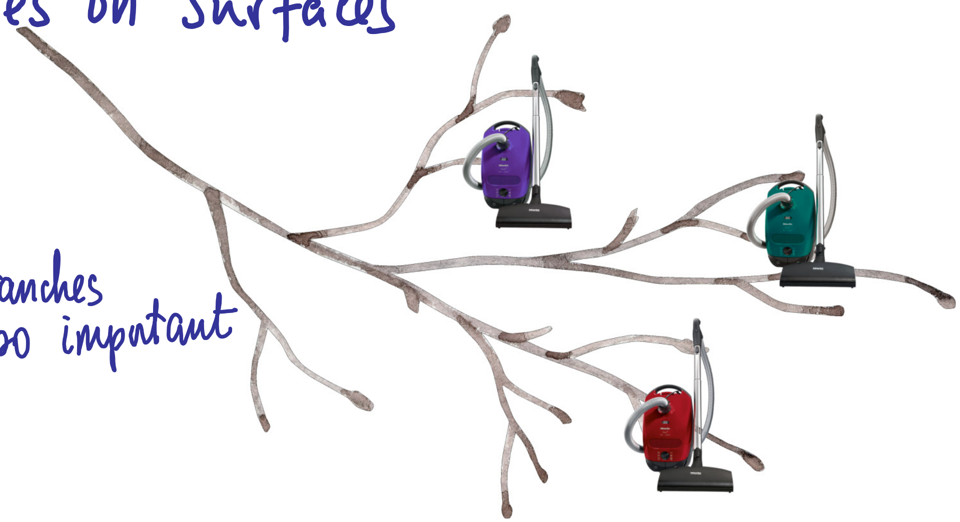
Sheaves on surfaces

ADE organisms





$T^* \mathbb{P}^1$



Other branches also important

★ QDE interpretation gives a precise geometric and representation-theoretic control over

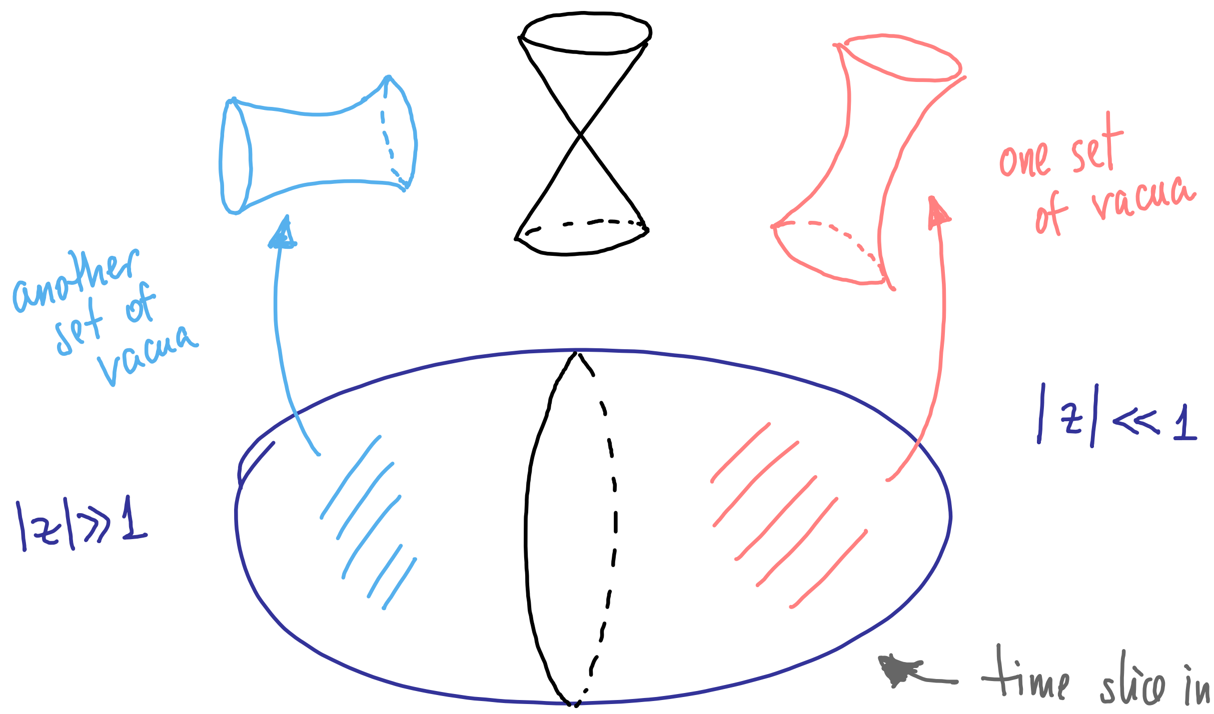
1) pole subtraction matrices ← important analytic concept, too

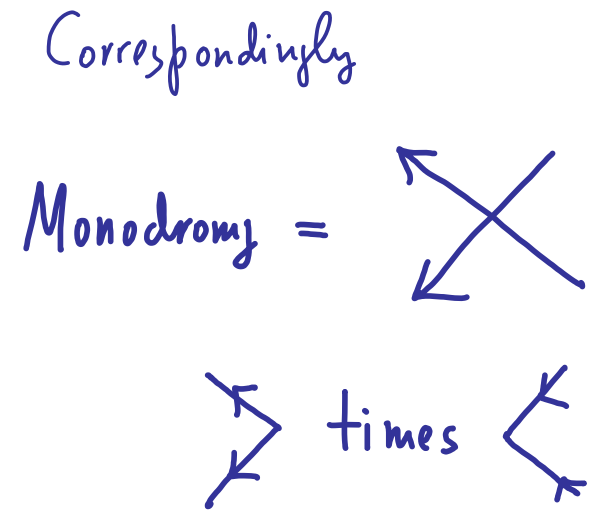
2) monodromy in both equivariant and Kähler variables

3) Categorification of monodromy

4) . . . .

The general physical/geometric idea behind monogromy is that of an interface in a theory that depends on parameters





< I am teaching an online course about it ... >

Pole subtraction

The z-equations and the a-equations while consistent and regular separately, are not regular jointly.

This manifests itself by:

- solutions grow superpolynomially as $(z, a) \to (0, 0)$
- there is no basis of solutions holomorphic for for $0 < |z| < \varepsilon$ and $0 < |a| < \varepsilon$

This can never happen for differential equations by a deep theorem of Deligne, but is commonplace for q-difference equations

For a silly 1×1 example, one can take

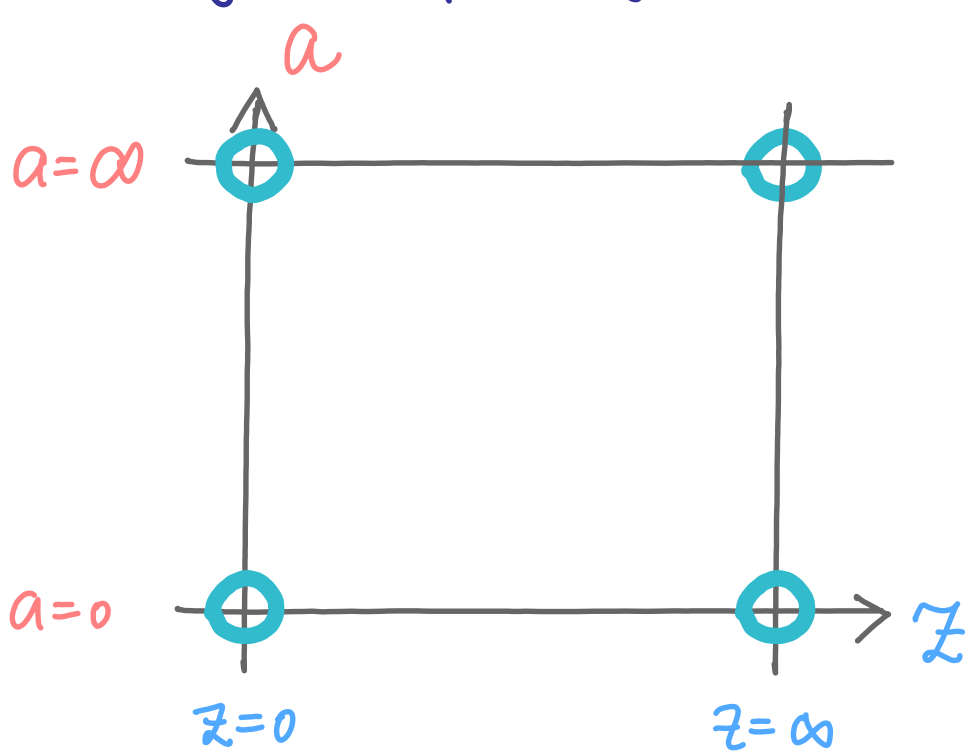
$$\begin{cases} F(qz, a) = aF \\ F(z, qa) = zF \end{cases} \rightsquigarrow e^{\frac{\ln z \ln a}{\ln q}}$$

which is not regular separately but not at $(z,a) = (0,0)$

$(0,\infty)$

$(\infty,0)$

(∞,∞)



for a 2×2 example take the q-hypergeometric equation

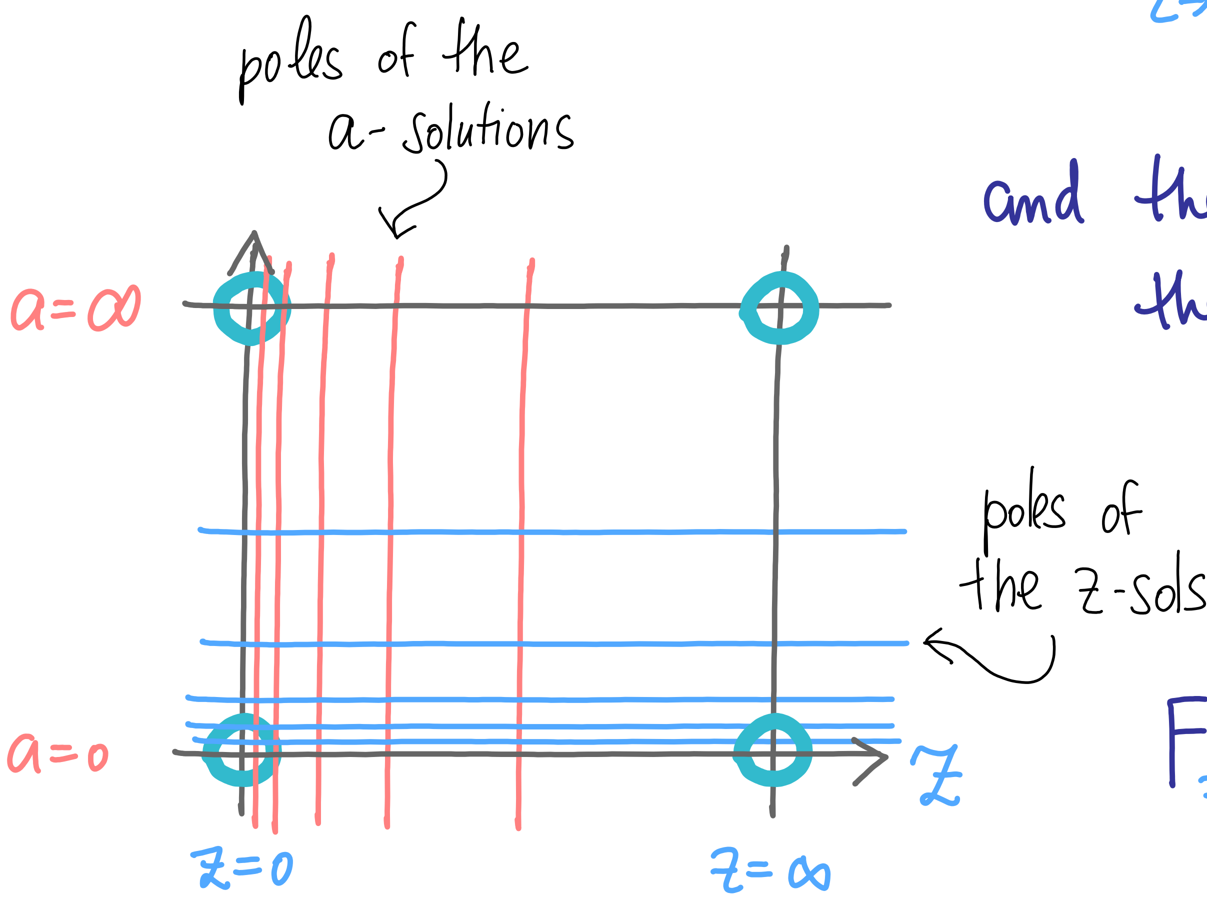
Feature, not a bug!

In a nbhd of a point like this, we have 2 kinds of solutions:

the z-solutions

$$F_{z\to 0} = \sum_n r_n(a)\, z^n$$

meromorphic



and the a-solutions, with the opposite properties

There is an elliptic transition matrix

$$F_{z\to 0} \longleftrightarrow F_{a\to 0}$$

pole subtraction matrix $\mathcal{P}$

Monodromy vs. pole subtraction

- both elliptic
- one global / analytic, another local / algorithmic
- pole subtraction constraints monodromy:

$$
\begin{tikzcd}
F_{z\to 0} \arrow[r, "\text{monodromy}"] \arrow[d, "\mathcal{P}_{0,0}"'] & F_{z\to\infty} \arrow[d, "\mathcal{P}_{\infty,0}"] \\
F_{a\to 0} \arrow[r, dash, "\text{monodromy for } a=0\ !!!"] & F_{a\to 0}
\end{tikzcd}
$$

$$F_{a\to 0} = \sum_n f_n(z)\, a^n,$$ where $f_0(z)$ solves the $a=0$ equation

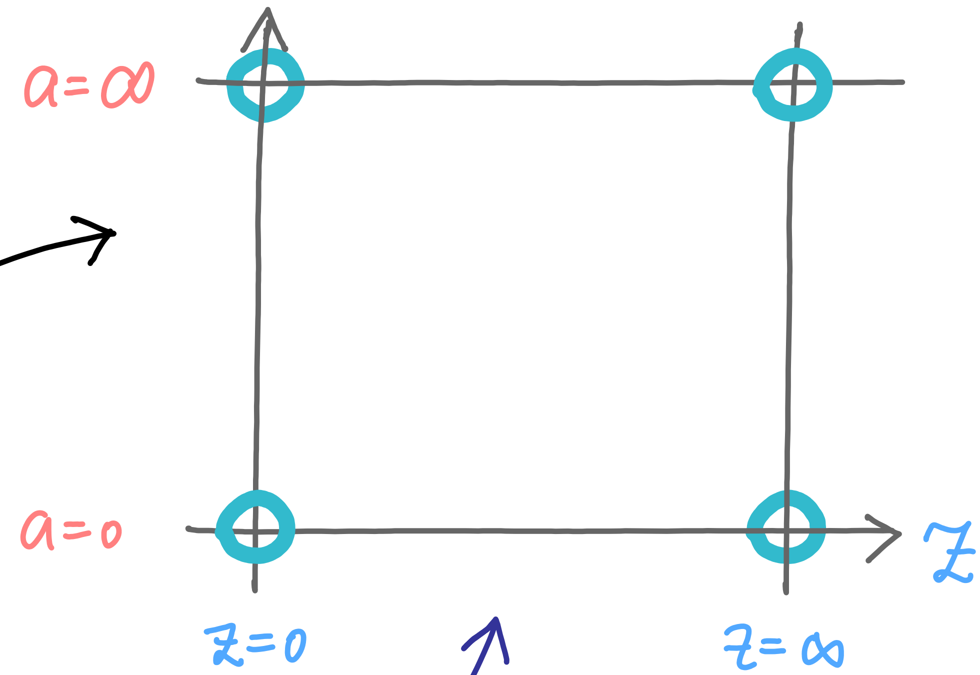
In the geometric context

at $z=0$,

there are no curves,

an easy computation

in $K(X)$



at $a=0$ we have the

quantum q-difference equation for the

fixed points $X^a \subset X$

Pole subtraction matrices are geometric, come from a certain correspondence between X^a and X in equivariant elliptic cohomology

indeed, $\mathcal{P}$ is about cancelling poles, and there isn't a more effective way to do it than localization formulas in equivariant cohomology/ K-theory/....

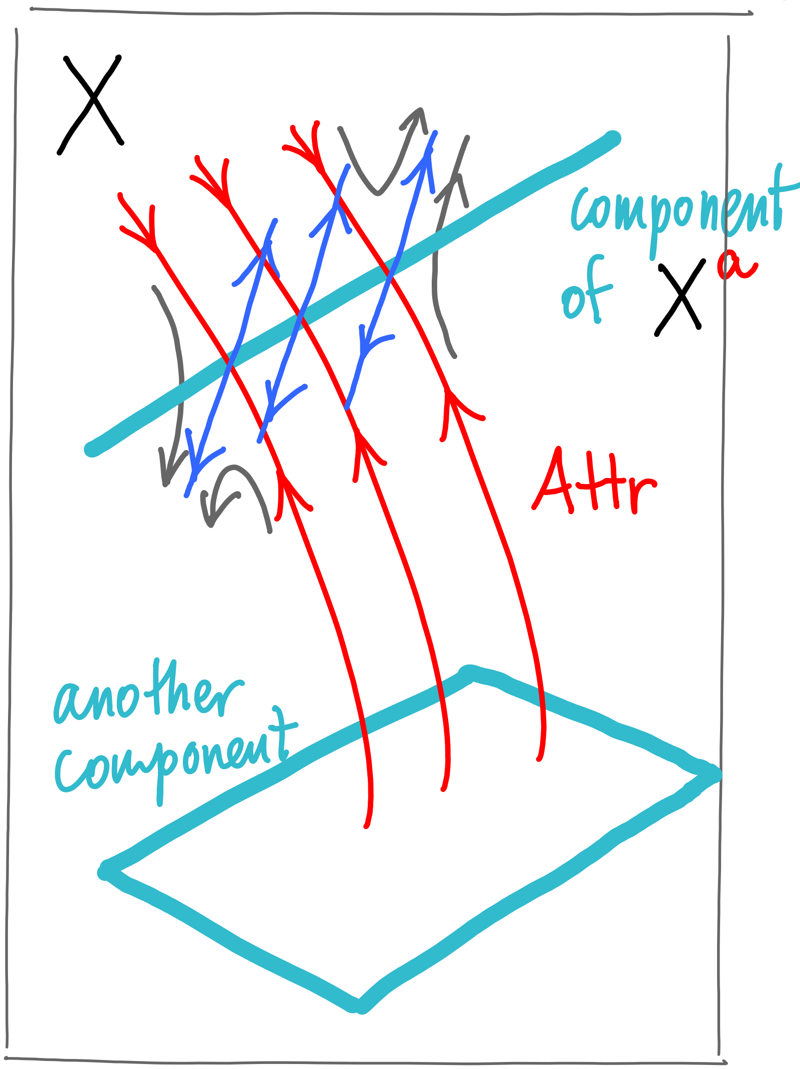
E.g. Weyl character formula

$$\sum_{w \in W} w \cdot \frac{a^\lambda}{\prod_{\alpha > 0} (1 - a^{-\alpha})} = \operatorname{tr} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{pmatrix} \chi(G/B, \mathcal{O}(\lambda))$$

fixed pts $(G/B)^a$

polynomial in a

flag manifold



there is a very important Lagrangian corrsp.

$$\text{Attr} = \left\{ (x,y),\ \lim_{a\to 0} a\cdot x = y \right\}$$

$$\cap$$

$$X \times X^a$$

not closed

e.g. conormals to Schubert for $X = T^*G/B$

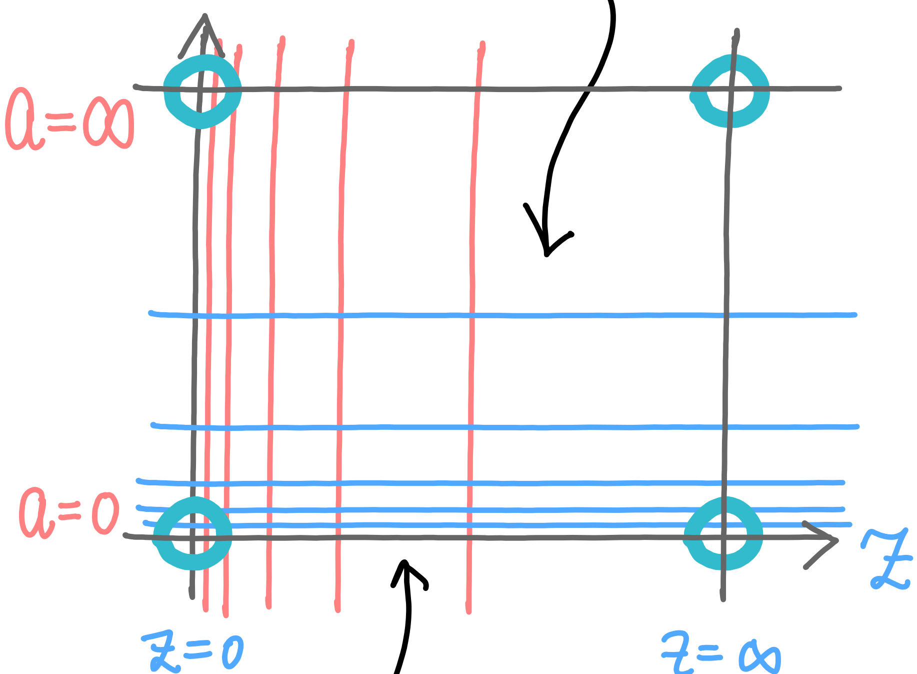
Very interesting theory to extend to a good correspondence

$\Rightarrow$ Stable envelopes [Maulik - O.] (2010)

In equivariant elliptic cohomology [Aganagic - O.] (2016)

Recall:

counts of curves in X
give the z-Solutions



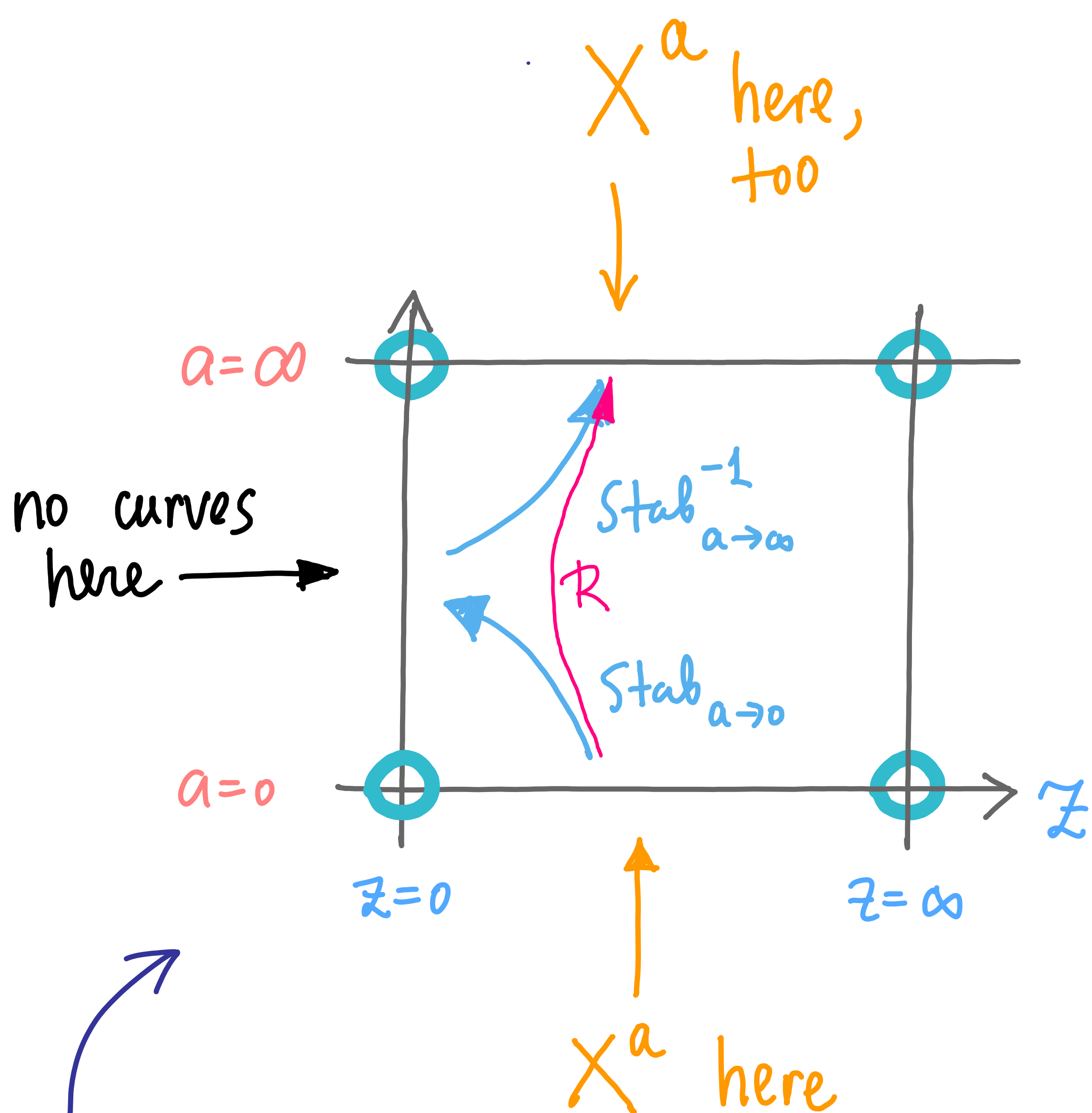
at $a=0$ we get the
quantum difference eq. for X^a

<u>Thm</u> [Aganagic - O.] (2016)

the pole subtraction matrix is
the elliptic stable envelope

also the main input
into the geometric theory
of elliptic quantum groups

hence the description of the
monodromy in terms of
these



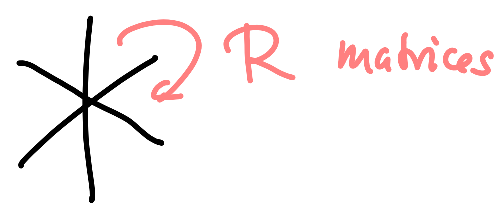
Corollary:

the monodromy of the equation in the equivariant variables is given by the elliptic R-matrix (as constructed, in general, by Aganagic – O.)

Generalizes a lot of previous work on qKZ etc ....

In general

$a \in$ torus $A \subset \overline{A}$ toric compactification by a fan



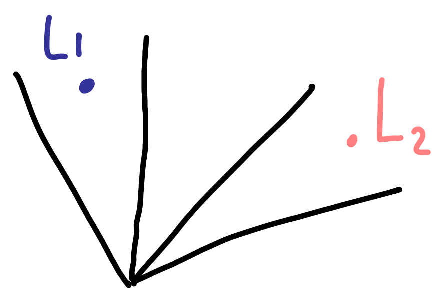
The 3d-mirror symmetry exchanges a = equivariant variables
with z = Kähler torus

e.g. in the following setup

$$X_1 = \text{stable locus for } L_1 \subset \mathcal{X} = \left[\frac{\text{affine}}{\text{reductive } G}\right] \supset X_2 = \text{stable locus for } L_2$$

$$0_1 \in \overline{Z} = \text{toric compactif of } Z \ni 0_2$$

Monodromy from 0_1 to 0_2 = ?



fan of cones for $\mathrm{Pic}_G(\ldots)$

Theorem (A.O., 2020) The monodromy is given by

$$\mathrm{Ell}(X_1) \xrightarrow{\mathrm{Stab}} \mathrm{Ell}(\mathfrak{X}) \xrightarrow{\mathrm{Stab}^{-1}} \mathrm{Ell}(X_2)$$

this is interpolation for sections of a certain line bundle on $\mathrm{Ell}(\mathfrak{X}) \times Z/q^{\mathrm{cochar}}$

May be interpreted as an explicit integral formula for solutions of quantum difference eq.

Categorification. at the dawn of the subject

M. Kontsevich $\Longrightarrow$ Monodromy (of Dubrovin connection) should categorify to

$$\begin{tikzcd} & \mathrm{Aut}\, D^b\mathrm{Coh}\, X \arrow[d] \\ \pi_1(Z \setminus \mathrm{sing}) \arrow[ur] \arrow[r] & GL(H^\bullet(X)) \end{tikzcd}$$

Studied by Horja, Borisov, ... for toric varieties

Theorem (R. Bezrukavnikov - A.O., conj in 2008, proven 2017, published ???)

This is indeed the case, moreover intermediate categories equivalences may be interpreted as the categories of representations of quantization $\hat{X}_\lambda$ of X in char $p >> 0$

parameter of the quantization $\lambda \in Pic(X)$

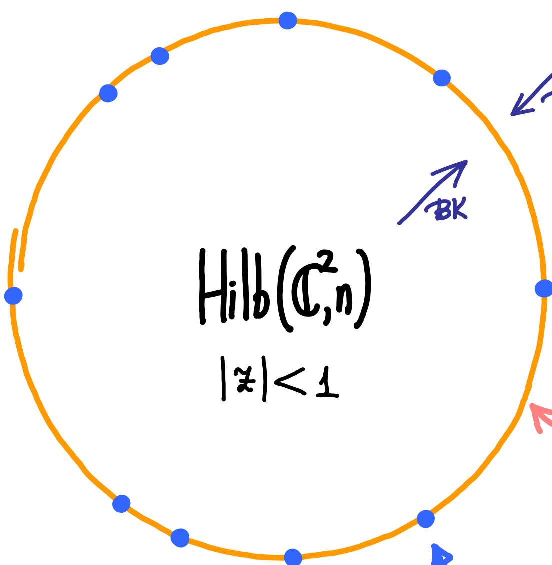
and the equivalences $D^b \, Coh \, X \longrightarrow D^b \, \hat{X}_\lambda\text{-mod} \longrightarrow D^b \, Coh \, X_{flop}$ constructed by Bezrukavnikov and Kaledin

The Hecke algebra of KL theory $\Longrightarrow$ Monodromy group!

Example:

flop $\mathrm{Hilb}(\mathbb{C}^2, n)$

$|z| > 1$



Cherednik$\left(n,\ c \sim \frac{\ln z}{2\pi i}\, p\right)$

$|z| = 1$

singularities of the ∇ =

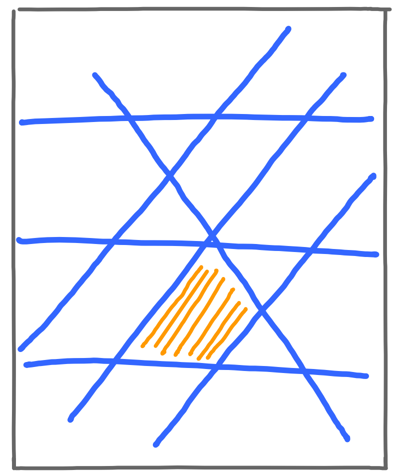
= bad quantization parameters

$(\lambda, \underline{\underline{\text{root}}}) \in \mathbb{Z}$

In general

cpt torus $\subset Z$

$$\Big\downarrow \ \frac{\ln(\cdot)}{2\pi i}\, p$$



$\mathrm{Pic}(X)$

Main ingredient of the proof:

$$\begin{tikzcd} D^b \mathrm{Coh}\, X \arrow[rr, leftrightarrow, "BK"] & & \hat{X}\text{-modules} \\ D^b \mathrm{Coh}\, X^A \arrow[u, "\mathrm{Stab}"] \arrow[rr, leftrightarrow] & & \widehat{(X^A)}\text{-modules} \arrow[u, "\text{Parabolic induction.}"'] \end{tikzcd}$$

fixed locus of
$A \subset \mathrm{Aut}(X, \omega)$

"Tomorrow" More natural to quantize

- to quantum groups at $\hbar \in \sqrt{1}$,

  see e.g. my talk at CM120

in particular, for roots of unity there is a direct link between enumeration and quantization

- also quantize $\mathrm{loop}(X)$ to a q-vertex algebra,

  categorify elliptic stable envelopes,

  see e.g. my talk at String Math 2019