Problem set 8

Due Wednesday, March 29

Recall from class that the Kac-Moody Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}_n}$ corresponding to the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & 0 & -1 & 2 & -1 \\ -1 & \vdots & 0 & -1 & 2 \end{pmatrix}$$

is a central extension of the algebra

$$\mathbb{C}t\frac{d}{dt} + \mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 1}]\,.$$

1. Show that the commutation relations in this algebra may be written as

$$[f(t), g(t)]_{\mathfrak{g}} = [f(t), g(t)]_{\mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 1}]} + K \frac{1}{2\pi i} \int \operatorname{tr} g(t) df(t)$$

where the integral extracts the residue of the differential form $\operatorname{tr} g(t)df(t)$ at $t = 0$ and

$$K = \sum h_i$$

is the canonical central element.

2. Show that via the map

$$\mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 1}] \to \mathfrak{gl}_\infty$$

constructed in class, the algebra $\mathfrak{g}$ acts in the Fock (projective) representation of $\mathfrak{gl}_\infty$ so that

$$K \mapsto 1\,.$$

3. Show that the image $\widehat{\mathfrak{gl}_n}$ of the similar map

$$\mathfrak{gl}_n \otimes \mathbb{C}[t^{\pm 1}] \to \mathfrak{gl}_\infty$$

acts irreducibly on subspaces of fixed charge in the Fock representation.

4. What is the commutant of $\widehat{\mathfrak{sl}_n}$ inside $\widehat{\mathfrak{gl}_n}$?