Problem set 5

Due Wednesday, March 1

For an arbitrary matrix $C = (a_{ij}) \in \mathrm{Mat}(n, \Bbbk)$ we can define a Lie algebra $\mathfrak{g}$ as the quotient

$$0 \to \mathfrak{r} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

by the ideal $\mathfrak{r}$ that is maximal among all ideals intersecting $\mathfrak{h}$ trivially.

1. What are the generators of $\mathfrak{r}$ if $C = 0$? The Lie algebra $\mathfrak{g}$ corresponding to $C = 0$ is called the Heisenberg Lie algebra.

2. Describe irreducible module in category $\mathcal{O}$ for the Heisenberg Lie algebra.

3. Let $\mathfrak{g}$ be arbitrary and let $M(\lambda) \in \mathcal{O}$ be a Verma module. Show that

$$\mathrm{Hom}_{\mathcal{O}}(M(\lambda), M(\lambda)) = \Bbbk\,.$$

4. Show that any nonzero map

$$M(\mu) \to M(\lambda)$$

is injective.

5. Let $\mathcal{O}_{\leq\lambda} \subset \mathcal{O}$ be the subcategory of modules of the form

$$M = \bigoplus_{\mu = \lambda - \sum \alpha_{i_k}} M_\mu\,.$$

Show that $M(\lambda)$ is projective in $\mathcal{O}_{\leq\lambda}$. Give an example (the case $\mathfrak{g} = \mathfrak{sl}_2$ will suffice) showing that $M(\lambda)$ may not be projective in $\mathcal{O}$.

6. As in the proof of the Gabber-Kac theorem, construct an exact sequence of the form

$$0 \to \mathfrak{r}_-\big/[\mathfrak{r}_-, \mathfrak{r}_-] \to \bigoplus_{i=1}^{n} M(-\alpha_i) \to M(0) \to \Bbbk \to 0$$

where $\Bbbk$ is a trivial 1-dimensional $\mathfrak{g}$-module.